Model Predictive Control for Renewable Energy Systems
University of Freiburg – Summer semester 2023

Exercise 4: Linear Model Predictive Control

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1. **Feasible set**: Consider again the inverted pendulum from Exercise 3 with linearized discrete dynamics:

$$x_{k+1} = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 0.99 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u_k \,. \tag{1}$$

with state vector $x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^\top \in \mathbb{R}^2$ and $u \in \mathbb{R}$. The variables $\theta, \dot{\theta}$ represent the angle deviation and speed w.r.t. the top position, while the control variable u is a horizontal force applied at the tip of the pendulum.

We consider a regulation MPC problem with weight matrices $Q = I$, $R = 1$, horizon $N = 10$ and a terminal point constraint $x_N = 0$ to guarantee stability and recursive feasibility.

Consider the following state and control constraints:

$$-\frac{\pi}{6} \leq \theta_k \leq \frac{\pi}{6}\,, \qquad -1.5 \leq u_k \leq 1.5\,. \tag{2}$$

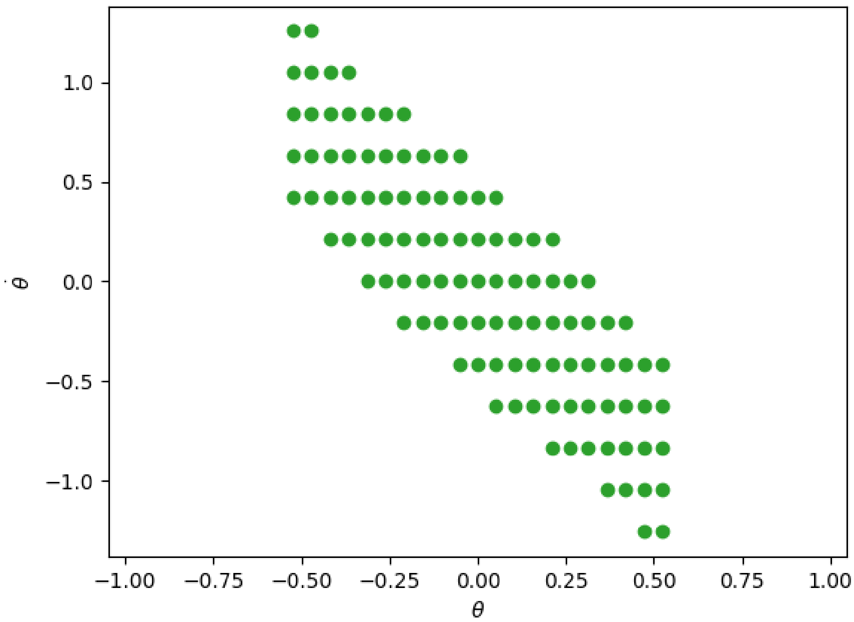
(a) Implement the optimal control problem with help of the `Opti` class in `CasADi`. You can start with the Python template `ex4_lmpc_example.py`.

(b) Investigate the feasible set $\mathcal{X}_\mathrm{N}$ of the MPC problem via sampling. Consider a grid of initial values $\theta_0, \dot{\theta}_0$ ranging between

$$-\frac{\pi}{6} \leq \theta_0 \leq \frac{\pi}{6}\,, \qquad -\frac{2\pi}{3} \leq \dot{\theta}_0 \leq \frac{2\pi}{3}\,. \tag{3}$$

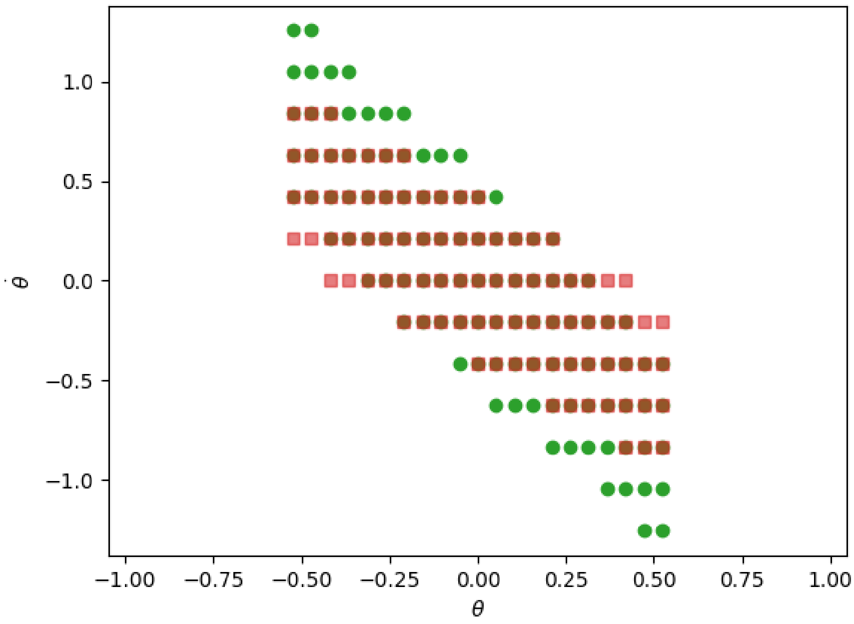
and plot the feasible initial points. Interpret the results. Do they make sense from a physical point of view?

The feasible set can be interpreted as follows: as the initial angle deviation θ_0 grows in the positive direction, the initial angular speed must become more and more negative in order to make the problem feasible. And vice versa. This is because the force on the pendulum is constrained and an initial speed towards the origin is necessary to be able to satisfy the terminal point constraint.

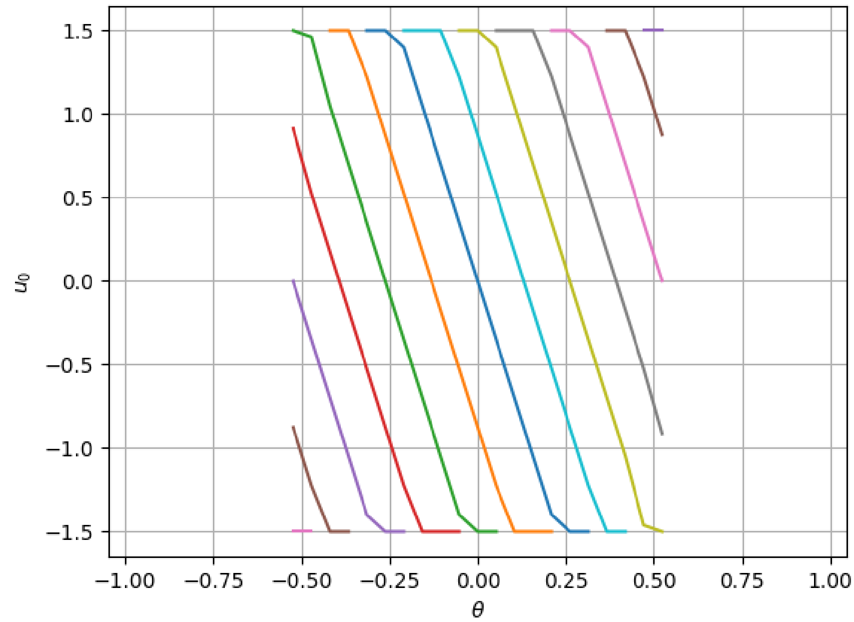


(c) The LQR controller from Exercise 3 in theory has a feasible set $\mathcal{X}_\mathrm{N} \in \mathbb{R}^{n_x}$. However, it will violate state and control constraints in the closed-loop response. Compute the practical feasible set of the LQR controller in similar fashion as for the MPC controller, plot and compare.

The MPC feasible set is larger than that of the LQR, since it directly takes into account the constraints. However, interestingly, the LQR feasible set partly extends beyond the MPC feasible set. This is because the terminal point constraint is very restrictive.



(d) Plot the MPC feedback law as a function of θ_0, for different values of $\dot{\theta}_0$. Is the result as expected?
As expected, we observe an affine feedback law as a function of the initial state x_0.



2. **Stability and terminal sets:** We now replace the terminal point constraint with a terminal set constraint, i.e. $x_N \in \mathcal{X}_\mathrm{f}$ and we add to the cost function the terminal cost $x_N^\top P_\infty x_N$, where P_∞ is the infinite-horizon cost-to-go obtained in Exercise 3. The terminal set is positive invariant under the control law $\kappa_\mathrm{f}(x) = K_\infty x$.

   (a) Prove, using Lyapunov theory, that the resulting MPC controller is stabilizing.
   We choose as a trial function the optimal MPC cost function

$$V(x_k) = J^*(x_k) = x_N^{*\top} P_\infty x_N^* + \sum_{i=0}^{N-1} x_i^{*\top} Q x_i^* + u_i^{*\top} R u_i^* \,, \tag{4}$$

   with x_i^*, u_i^* the optimal MPC solution for initial state x_k. To evaluate the trial function at x_{k+1}, we consider the following feasible (but possibly suboptimal) trajectory $(u_1^*, u_2^*, \ldots, u_{N-1}, K_\infty x_N^*)$, $(x_1^*, x_2^*, \ldots, x_N^*, \bar{x}_{N+1})$, with

$\bar{x}_{N+1} = Ax_N^* + BK_\infty x_N^*$. The trial function evaluates as:

$$V(x_{k+1}) \leq \bar{x}_{N+1}^\top P_\infty \bar{x}_{N+1} + x_N^{*\top} Q x_N^* + x_N^{*\top} K_\infty^\top R K_\infty x_N^* + \sum_{i=1}^{N-1} x_i^{*\top} Q x_i^* + u_i^{*\top} R u_i^* \quad (5)$$

$$\leq \bar{x}_{N+1}^\top P_\infty \bar{x}_{N+1} + x_N^{*\top} Q x_N^* + x_N^{*\top} K_\infty^\top R K_\infty x_N^* + V(x_k) - x_0^{*\top} Q x_0^* - u_0^{*\top} R u_0^* - x_N^{*\top} P_\infty x_N^* \quad (6)$$

The LQR controller within the terminal region is stable by design. Thus it holds (from Eq. (4.73) in the script), that:

$$\bar{x}_{N+1}^\top P_\infty \bar{x}_{N+1} + x_N^{*\top} Q x_N^* + x_N^{*\top} K_\infty^\top R K_\infty x_N^* - x_N^{*\top} P_\infty x_N^* = 0\,. \quad (7)$$

Therefore we can simplify (6) to:

$$V(x_{k+1} - V(x_k) \leq -x_0^{*\top} Q x_0^* - u_0^{*\top} R u_0^*\,, \quad (8)$$

So that, for $x_0^* \in \mathcal{X}_\mathrm{N}$, $Q \succeq 0$, $R \succ 0$, it holds that

$$V(x_{k+1}) - V(x_k) < 0\,, \quad (9)$$

such that the closed-loop system is stable.

(b) Let the terminal set be given by

$$\mathcal{X}_\mathrm{f} = \{x \in \mathbb{R}^{n_x} \mid A_\mathrm{f} x \leq b_\mathrm{f}\} \quad (10)$$

with

$$A_\mathrm{f} = \begin{bmatrix} -0.0053 & -0.5294 \\ 0.0053 & 0.5294 \\ -0.5198 & -0.9400 \\ 0.5198 & 0.9400 \\ -1.0000 & 0 \\ 1.0000 & 0 \end{bmatrix} \qquad b_\mathrm{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}\,. \quad (11)$$

Compute again the feasible set $\mathcal{X}_\mathrm{N}$ of the MPC problem via sampling and compare to the result with the terminal point constraint.

The feasible set of the reformulated problem significantly enlarges and completely includes the feasible set of the problem with the terminal point constraint.