Model Predictive Control for Renewable Energy Systems
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Exercise 4: Linear Model Predictive Control

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1. **Feasible set**: Consider again the inverted pendulum from Exercise 3 with linearized discrete dynamics:

$$x_{k+1} = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 0.99 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u_k \ . \tag{1}$$

with state vector $x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^\top \in \mathbb{R}^2$ and $u \in \mathbb{R}$. The variables $\theta, \dot{\theta}$ represent the angle deviation and speed w.r.t. the top position, while the control variable u is a horizontal force applied at the tip of the pendulum.

We consider a regulation MPC problem with weight matrices $Q = I$, $R = 1$, horizon $N = 10$ and a terminal point constraint $x_N = 0$ to guarantee stability and recursive feasibility.

Consider the following state and control constraints:

$$-\frac{\pi}{6} \leq \theta_k \leq \frac{\pi}{6} \ , \qquad -1.5 \leq u_k \leq 1.5 \ . \tag{2}$$

   (a) Implement the optimal control problem with help of the `Opti` class in `CasADi`. You can start with the Python template `ex4_lmpc_example.py`.

   (b) Investigate the feasible set $\mathcal{X}_\mathrm{N}$ of the MPC problem via sampling. Consider a grid of initial values $\theta_0, \dot{\theta}_0$ ranging between

$$-\frac{\pi}{6} \leq \theta_0 \leq \frac{\pi}{6} \ , \qquad -\frac{2\pi}{3} \leq \dot{\theta}_0 \leq \frac{2\pi}{3} \ . \tag{3}$$

   and plot the feasible initial points. Interpret the results. Do they make sense from a physical point of view?

   (c) The LQR controller from Exercise 3 in theory has a feasible set $\mathcal{X}_\mathrm{N} \in \mathbb{R}^{n_x}$. However, it will violate state and control constraints in the closed-loop response. Compute the practical feasible set of the LQR controller in similar fashion as for the MPC controller, plot and compare.

   (d) Plot the MPC feedback law as a function of θ_0, for different values of $\dot{\theta}_0$. Is the result as expected?

2. **Stability and terminal sets:** We now replace the terminal point constraint with a terminal set constraint, i.e. $x_N \in \mathcal{X}_\mathrm{f}$ and we add to the cost function the terminal cost $x_N^\top P_\infty x_N$, where P_∞ is the infinite-horizon cost-to-go obtained in Exercise 3. The terminal set is positive invariant under the control law $\kappa_\mathrm{f}(x) = K_\infty x$.

   (a) Prove, using Lyapunov theory, that the resulting MPC controller is stabilizing.

   (b) Let the terminal set be given by

$$\mathcal{X}_\mathrm{f} = \{x \in \mathbb{R}^{n_x} \mid A_\mathrm{f} x \leq b_\mathrm{f}\} \tag{4}$$

   with

$$A_\mathrm{f} = \begin{bmatrix} -0.0053 & -0.5294 \\ 0.0053 & 0.5294 \\ -0.5198 & -0.9400 \\ 0.5198 & 0.9400 \\ -1.0000 & 0 \\ 1.0000 & 0 \end{bmatrix} \qquad b_\mathrm{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} \ . \tag{5}$$

   Compute again the feasible set $\mathcal{X}_\mathrm{N}$ of the MPC problem via sampling and compare to the result with the terminal point constraint.