Exercise 3: Linear Model Predictive Control

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1. **Discrete linear system models**: Consider an inverted pendulum with nonlinear dynamics

$$\dot{x} = f(x, u) = \begin{bmatrix} x_2 \\ \sin x_1 - cx_2 + u \cos x_2 \end{bmatrix} . \tag{1}$$

with state vector $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^\top \in \mathbb{R}^2$ and $u \in \mathbb{R}$. The variables $\theta, \dot{\theta}$ represent the angle deviation and speed w.r.t. the top position, while the control variable u is a horizontal force applied at the tip of the pendulum.

   (a) Linearize the system around $x_{\mathrm{ss}} = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ and $u_{\mathrm{ss}} = 0$ to get the linearized system

$$\dot{x}(t) = A_{\mathrm{c}} x(t) + B_{\mathrm{c}} u(t) \tag{2}$$

   What are the system matrices A_{c} and B_{c}? Assume a damping constant $c = 0.1$.

   (b) Now discretize the state space model with a sampling time $T_{\mathrm{s}} = 0.1s$. What are the discrete-time system matrices A and B? Use the analytic solution for continuous-time LTI systems based on the matrix exponential

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k . \tag{3}$$

   To evaluate the matrix exponential, cut off the sum at $k = 1$. Terms proportional to T_{s}^k, $k \geq 2$, can be neglected.

   (c) Is the resulting discrete-time system controllable?

2. **Linear quadratic regulator:** Let us now assume that the exact system dynamics are given by the discrete-time model

$$x_{k+1} = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 0.99 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u_k \tag{4}$$

We want to design an infinite-horizon LQR controller to control the system, using weight matrices $Q = I$, $R = 1$.

   (a) Compute the infinite horizon cost-to-go weight matrix P_∞ using the Ricatti recursion:

$$P_{k+1} = A^\top P_k A - (A^\top P_k B)(R + B^\top P_k B)^{-1}(B^\top P_\infty A) + Q , \tag{5}$$

   with initialization $P_0 = Q$. You can start with the Python template `ex3_lmpc_example.py`.

   (b) Compute the LQR feedback matrix K_∞.

   (c) Compute the closed-loop system matrix $A_{\mathrm{CL}} = A + BK_\infty$. Simulate and plot the closed-loop response from the initial condition $x_0 = \begin{bmatrix} \frac{\pi}{6} & 0 \end{bmatrix}^\top$ for $N_{\mathrm{sim}} = 100$ steps.

3. **Linear model predictive control:** We now introduce the input constraint $-1 \leq u \leq 1$.

   (a) Formulate the linear MPC controller for the discretized system with horizon $N = 10$. Choose as terminal weight matrix the infinite-horizon cost-to-go. We do not use a terminal region.

   (b) Implement the optimal control problem with help of the `Opti` class in `CasADi`.

   (c) Compute the closed-loop response for the same scenario as in Task (2c) and compare to the LQR response.