Model Predictive Control for Renewable Energy Systems – Sample exam

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Page	0	1	2	3	4	5	6	7	8	9	Sum
Points on page (max)	4	9	9	6	6	4	4	0	0	0	42
Points obtained											
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Intermediate sum (max)	4	13	22	28	34	38	42	42	42	42	

Mark: Exam inspected on: Signature of examiner:

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Subject: Programme: Bachelor ☐ Master ☐ Lehramt ☐ others ☐ Signature:

Please fill in your name above. For the multiple choice questions, which give exactly one point, tick exactly one box for the right answer. For the text questions, give a short formula or text answer just below the question in the space provided, and, if necessary, write on the **backpage of the same sheet** where the question appears, and add a comment "see backpage". Do not add extra pages (for fast correction, all pages will be separated for parallelization). The exam is a closed book exam, i.e. no books or other material are allowed besides 1 sheet (with 2 pages) of notes and a non-programmable calculator. Some legal comments are found in a footnote[1].

1. Name two advantages and two limitations of using an MPC controller instead of traditional controller.

   Advantages of MPC Controller:
   - MPC can handle nonlinear systems better than PID.
   - MPC has predictive capabilities, allowing anticipatory behavior based on future behavior or disturbance predictions.
   - MPC can easily handle constraints on system variables/states.
   - MPC can easily handle systems with multiple in- and outputs.

   Limitations of MPC:
   - computational requirements
   - reasonably accurate model of the system needed (may not be effective if the model is incorrect or if the system characteristics change significantly over time)

2	

2. Which of the following functions $f(x)$ is NOT convex (c and A are of appropriate dimensions and fixed)?

(a) [x] $\|Ax\|_2^2+\log(c^\top x)$	(b) [ ] $\|Ax\|_2^2+\exp(c^\top x)$
(c) [ ] $c^\top x+\exp(\|Ax\|_2^2)$	(d) [ ] $-c^\top x+\|Ax\|_2^2$

1	

3. A point in the feasible set of an NLP that satisfies the KKT optimality conditions is

(a) [ ] the global minimum	(b) [ ] a local minimum
(c) [ ] a boundary point	(d) [x] a candidate for local minimum

1	

[1]WITHDRAWING FROM AN EXAMINATION: In case of illness, you must supply proof of your illness by submitting a medical report to the Examinations Office. Please note that the medical examination must be done at the latest on the same day of the missed exam. In case of illness while writing the exam please contact the supervisory staff, inform them about your illness and immidiatelly see your doctor. The medical certificate must be submitted latest 3 days after the medical examination. More informations: http://www.tf.uni-freiburg.de/studies/exams/withdrawing_exam.html

CHEATING/DISTRUBING IN EXAMINATIONS: A student who disrupts the orderly proceedings of an examination will be excluded from the remainder of the exam by the respective examiners or invigilators. In such a case, the written exam of the student in question will be graded as 'nicht bestanden' (5.0, fail) on the grounds of cheating. In severe cases, the Board of Examiners will exclude the student from further examinations.

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4. Apply the full step Newton method for optimization to the scalar optimization problem $\min_{x\in\mathbb{R}} f(x)$ with $f(x) = x + x^6$. Given the current iterate x_k, which exact formula determines the next iterate x_{k+1} ?

(a) [x] $x_k - \frac{1+6x_k^5}{30x_k^4}$	(b) [ ] $x_k - \frac{x+x_k^6}{1+6x_k^5}$
(c) [ ] $x_k + \frac{1+6x_k^5}{30x_k^4}$	(d) [ ] $x_k + \frac{x+x_k^6}{1+6x_k^5}$

1

5. A system is described by the differential equation $\log(a)\dot{y}(t) = e^{-b} \cdot u(t)$, where a and b are strictly positive constant parameters. Is the system *linear* and/or *time invariant*?

(a) [x] linear and time invariant	(b) [ ] only time invariant
(c) [ ] only linear	(d) [ ] neither of both

1

6. A hanging pendulum subjected to an external force is described by the linearized ODE $I\ddot{\theta} = -mgL\theta - c\dot{\theta}L + FL$. Take $x = \begin{bmatrix}\theta & \dot{\theta}\end{bmatrix}^\top$ as the state and $u = F$ as the input. Write the system in the form $\dot{x} = Ax + Bu$. Specify A and B.

(a) [ ] $A = \begin{bmatrix} 0 & 1 \\ \frac{mgL}{I} & \frac{cL}{I} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{L}{I} \end{bmatrix}$	(b) [ ] $A = \begin{bmatrix} \frac{mgL}{I} & \frac{cL}{I} \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} L \\ 0 \end{bmatrix}$
(c) [x] $A = \begin{bmatrix} 0 & 1 \\ -\frac{mgL}{I} & -\frac{cL}{I} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{L}{I} \end{bmatrix}$	(d) [ ] $A = \begin{bmatrix} 0 & 1 \\ -mgL & -cL \end{bmatrix}, B = \begin{bmatrix} 0 \\ L \end{bmatrix}$

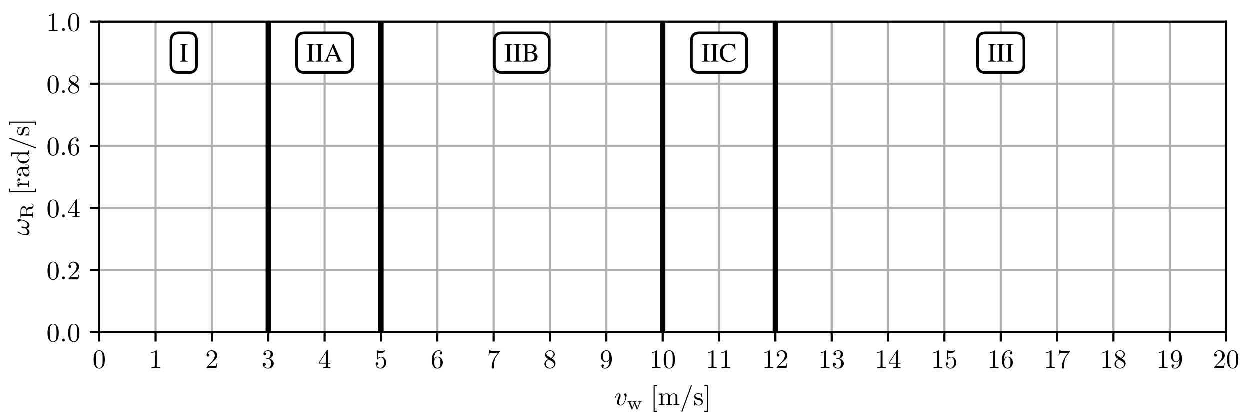
1

7. The nonlinear system $\dot{y}(t) = \sin(u(t)) + \cos(y(t))$ should be linearized at the steady state $u_{ss} = 0,\ \ y_{ss} = \frac{\pi}{2}$. What is the differential equation of the linearized system as a function of $\Delta u(t) = u(t) - u_{ss}$ und $\Delta y(t) = y(t) - y_{ss}$?

(a) [x] $\Delta\dot{y}(t) = \Delta u(t) - \Delta y(t)$	(b) [ ] $\Delta\dot{y}(t) = \Delta u(t)$
(c) [ ] $\Delta\dot{y}(t) = \Delta y(t)$	(d) [ ] $\Delta\dot{y}(t) = \Delta u(t) + \Delta y(t)$

2

8. A variable-speed wind turbine is characterized by the following limitations: $0.4\ \mathrm{rad/s} \leq \omega_\mathrm{R} \leq 0.8\ \mathrm{rad/s}$, with ω_R the rotor speed. Draw the optimal rotor speed curve as a function of wind speed v_w over the different operating regions in the plot below.



2

9. How does the choice of estimation horizon length influence the performance of an MHE observer?
The choice of estimation horizon should balance accuracy, robustness, and computational efficiency. A longer horizon can provide improved estimation accuracy and robustness to noise, as it integrates information over a larger set of past measurements. However, a longer horizon also increases computational complexity and may slow down the responsiveness of the estimator to abrupt changes. Conversely, a shorter horizon is computationally faster, but might be more susceptible to noise and less accurate.

2

Points on page 9

10. Regard an MPC optimization problem for $N = 10$ steps of the discrete-time system $x_{k+1} = 2x_k + u_k$ with continuous state $x_k \in \mathbb{R}$ and continuous bounded control action $u_k \in [-1, 1]$. The stage cost is given by $l(x_k, u_k) = x_k^2$ and the terminal cost by $E(x_N) = 100x_N^2$. The initial state is $\bar{x}_0$. To which optimization problem class does the problem belong?

(a) [ ] Mixed Integer Programming (MIP) but not LP	(b) [x] Quadratic Programming (QP) but not LP
(c) [ ] Nonlinear Programming (NLP) but not QP	(d) [ ] Linear Programming (LP)

1

11. Regard an MPC optimization problem for $N = 10$ steps of the discrete time system $x_{k+1} = 2x_k + u_k$ with continuous state $x_k \in \mathbb{R}$ and continuous bounded control action $u_k \in [-1, 1]$. The stage cost is given by $l(x_k, u_k) = u_k^4$ and the terminal cost by $E(x_k) = 100x_k^2$. The initial state is $\bar{x}_0$. To which optimization problem class does the problem belong?

(a) [ ] Nonlinear Programming (NLP) but not convex	(b) [x] Convex Optimization but not QP
(c) [ ] Linear Programming (LP)	(d) [ ] Quadratic Programming (QP) but not LP

1

12. We want to apply the explicit Euler integration rule to the scalar differential equation $\dot{x} = \lambda x$ on a time grid $t_k = hk$ with time step $h > 0$. Which formula relates the state x_{k+1} at time t_{k+1} to the state x_k at t_k?

(a) [ ] $x_{k+1} - x_k = \frac{\lambda}{2}(x_k + x_{k+1})$	(b) [ ] $x_{k+1} - x_k = \lambda x_{k+1}$
(c) [x] $x_{k+1} - x_k = h\lambda x_k$	(d) [ ] $x_{k+1} - x_k = h\lambda x_{k+1}$

1

13. When is condensing most advantageous for solving the QP arising from linear MPC problems?

(a) [ ] for long horizons and many controls	(b) [ ] for unstable systems
(c) [x] for short horizons and more states than controls	(d) [ ] for non-convex QPs

1

14. How many optimization variables does the NLP arising in the direct multiple shooting method have, if the system has n_x states, n_u controls, the initial value is fixed, and the time horizon is divided into N control intervals (piecewise-constant)?

(a) [ ] $Nn_x^3 + Nn_u^2$	(b) [ ] $\frac{1}{3}N^3n_u^3$	(c) [x] $(N+1)n_x + Nn_u$	(d) [ ] Nn_u

1

15. Which of the following statements on the sequential (single shooting) vs the simultaneous (multiple shooting) formulation of an Optimal Control Problem is wrong?

(a) [ ] The simultaneous formulation has an exploitable sparsity structure.	(b) [ ] The sequential formulation has less optimization variables.
(c) [ ] The simultaneous formulation is usually better at handling unstable nonlinear systems.	(d) [x] The sequential formulation is always cheaper to solve.

1

16. Consider a discrete-time linear system with the state-space representation: $x_{k+1} = Ax_k + Bu_k$. The control goal is to minimize the cost function

$$J = \sum_{k=0}^{\infty} x_k^\top Q x_k + u_k^\top R u_k.$$

online for a given initial state $\bar{x}_0$ Give the control law $\kappa(\bar{x}_0)$ that minimizes the cost function and explain how it is computed. What is the name of this particular MPC problem formulation?

The problem is called (infinite-horizon) Linear Quadratic Regulator (LQR).
The optimal control law in the LQR framework is the constant, linear state-feedback law of the form: $\kappa(\bar{x}_0) = K\bar{x}_0$ where K is the optimal state-feedback gain, which is computed by solving the Discrete Algebraic Riccati Equation (DARE):

$$P = A^\top PA - A^\top PB(R + B^\top PB)^{-1}B^\top PA + Q$$

and K is given by

$$K = -(R + B^\top PB)^{-1}B^\top PA$$

Where P is the solution to the Algebraic Riccati Equation.

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17. Consider the following system matrices of a building heating system.

$$A = \begin{bmatrix} (-H_{\text{rad,con}} - H_{\text{ve,tr}})/C_{\text{bldg}} & H_{\text{rad,con}}/C_{\text{bldg}} \\ H_{\text{rad,con}}/C_{\text{water}} & -H_{\text{rad,con}}/C_{\text{water}} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{C_{\text{water}}} \end{bmatrix}$$

Is this system given by the matrices A and B controllable? Either make the necessary calculations or give an argumentation.
Is the system observable for $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$?
You can use the following values for the building parameters.

C_{water}	$H_{\text{ve,tr}}$	$H_{\text{rad,con}}$	C_{bldg}
80	396	661	1000

You can round the values to one decimal place.
Note: The definitions of the observability and controllability matrices are the same for continuous-time and discrete-time linear systems.

$$A = \begin{bmatrix} (-661 - 396)/1000 & 661/1000 \\ 661/80 & -661/80 \end{bmatrix} \approx \begin{bmatrix} -1.1 & 0.7 \\ 8.3 & -8.3 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 1/80 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

Calculate AB:

$$AB \approx \begin{bmatrix} -1.1 & 0.7 \\ 8.3 & -8.3 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \approx \begin{bmatrix} 0.1 \\ -0.8 \end{bmatrix}$$

Construct the controllability matrix:

$$S_C = \begin{bmatrix} B & AB \end{bmatrix} \approx \begin{bmatrix} 0 & 0.1 \\ 0.1 & -0.8 \end{bmatrix}$$

The system is controllable if and only if the controllability matrix is of rank $n_x = 2$ which is the case here. For instance you can check the determinant of the controllability matrix to be nonzero (in this case, -0.007).
The system is also observable. Calculations are similar using the observability matrix

$$S_O = \begin{bmatrix} C \\ CA \end{bmatrix}$$

.

3	

18. Terminal constraints are typical ingredients of MPC problems. Describe (a) how you would design a terminal constraint in linear MPC to maximize the feasible set $\mathcal{X}_{\text{N}}$ and (b) the contribution of the terminal constraint to recursive feasbility and closed-loop stability respectively.
(a) The terminal constraint that maximizes the feasible set uses a terminal set formulation, i.e. $x_N \in \mathcal{X}_{\text{f}}$. The terminal set $\mathcal{X}_{\text{f}}$ should be chosen so that it is the largest positive invariant set under a local (unconstrained) control law such that the state and input constraints are not violated.
(b) Such a terminal constraint guarantees recursive feasibility, but not closed-loop stability (for this, an appropriately chosen terminal penalty is necessary).

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19. The state dynamics of an LTI system are given by the state equation $\dot{x} = Ax + Bu$, with $x \in \mathbb{R}^2$, $u \in \mathbb{R}^2$, and

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}. \tag{1}$$

Compute the controllability matrix for this system. Is the system controllable?
Note: The definition of controllability matrix is the same for continuous-time and discrete-time linear systems.

$$S_{\mathrm{C}} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Yes, the system is controllable since the controllability matrix has rank 2.

2	

20. Rewrite the following MPC problem using soft constraints on the state constraint. Choose a penalty function that does not change the location of the solution.

$$\begin{aligned} \underset{x(\cdot),\, u(\cdot)}{\text{minimize}} \quad & \int_0^{t_{\mathrm{f}}} L(x(t), u(t)) \mathrm{d}t \\ \text{subject to} \quad & x(0) = \hat{x}_0, \\ & \dot{x} = f(x(t), u(t)) \quad \forall t \in [0, t_{\mathrm{f}}], \\ & u_{\min} \leq u(t) \leq u_{\max} \quad \forall t \in [0, t_{\mathrm{f}}], \\ & x_{\min} \leq x(t) \leq x_{\max} \quad \forall t \in [0, t_{\mathrm{f}}] \end{aligned}$$

$$\begin{aligned} \underset{x(\cdot),\, u(\cdot),\, s(\cdot)}{\text{minimize}} \quad & \int_0^{t_{\mathrm{f}}} (L(x(t), u(t)) + S^\top s(t)) \mathrm{d}t \\ \text{subject to} \quad & x(0) = \hat{x}_0, \\ & \dot{x} = f(x(t), u(t)) \quad \forall t \in [0, t_{\mathrm{f}}], \\ & u_{\min} \leq u(t) \leq u_{\max} \quad \forall t \in [0, t_{\mathrm{f}}], \\ & x_{\min} - s(t) \leq x(t) \quad \forall t \in [0, t_{\mathrm{f}}], \\ & x(t) \leq x_{\max} + s(t) \quad \forall t \in [0, t_{\mathrm{f}}], \\ & s(t) \geq 0 \quad \forall t \in [0, t_{\mathrm{f}}] \end{aligned}$$

with $S \in \mathbb{R}^{n_x \times 1}$ chosen positive and sufficiently high.

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21. How does the Kalman Filter work, and what are its fundamental principles regarding the prediction and update step?
The Kalman Filter is an optimal recursive data processing algorithm. It uses a set of mathematical equations that provide an efficient recursive means to estimate the state of a process, in a way that minimizes the mean of the squared error between predicted state and measured state. In the prediction step, the Kalman filter produces estimates of the current state variables, along with their uncertainties, by using the system model. In the update step, it uses the actual measurement to refine the predictions, producing updated estimates.

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22. Define Moving Horizon Estimation (MHE) and compare it with the Kalman Filter. Does MHE have an advantage compared to Kalman filter for state estimation of linear systems? What advantages does MHE offer in state estimation for nonlinear systems?

MHE is an optimization-based estimation scheme that uses a finite time window of past measurements and input data to estimate the current state of a system. At each time step, MHE solves an optimization problem over a finite horizon in the past, considering the latest measurement and discarding the oldest one. It is often used when the system model is nonlinear, or when the noise or uncertainties are non-Gaussian. The Kalman filter is computationally efficient and provides optimal state estimates for linear systems with Gaussian noise. However, its performance can be suboptimal when these conditions are not met. MHE, on the other hand, can handle nonlinearities and non-Gaussian noise, but at the cost of higher computational complexity. Moreover, MHE is more flexible in handling constraints on states and inputs, which is not straightforward in the Kalman filter framework.

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23. Consider the following finite-horizon LQR problem with horizon $N = 1$:

$$\begin{aligned} \underset{x_0, u_0, x_1}{\text{minimize}} \quad & \frac{1}{2} x_1^\top P_N x_1 + \frac{1}{2} x_0^\top Q x_0 + u_0^\top R u_0 \\ \text{subject to} \quad & x_0 = \hat{x}_0, \\ & x_1 = A x_0 + B u_0, \end{aligned}$$

Eliminate the state variables x_0 and x_1 from the optimal control problem. What is the cost function of the resulting equivalent problem?

$$J = \frac{1}{2} u_0^\top (R + B^\top P_N B) u_0 + u_0^\top B^\top P_N A \hat{x}_0$$

1	

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24. Consider the following dense linear MPC problem with horizon $N = 1$. There is a single inequality constraint, i.e. $C \in \mathbb{R}^{1\times n_u}, D \in \mathbb{R}^{1\times n_x}, e \in \mathbb{R}$:

$$\begin{aligned} \underset{u_0}{\text{minimize}} \quad & \frac{1}{2} u_0^\top \hat{R} u_0 + u_0^\top \hat{Q} \hat{x}_0 \\ \text{subject to} \quad & C u_0 + D\hat{x}_0 + e \geq 0, \end{aligned}$$

Derive the optimal feedback law $\kappa(x) = Kx + v$ for the case that the inequality constraint is *strictly active*.

KKT conditions:

$$\hat{R}u_0 + \hat{Q}\hat{x}_0 + \lambda^\top C = 0 \tag{38}$$

$$Cu_0 + D\hat{x}_0 + +d = 0 \tag{39}$$

In matrix form:

$$\begin{bmatrix} \hat{R} & C^\top \\ C & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ \lambda \end{bmatrix} + \begin{bmatrix} \hat{Q} \\ D \end{bmatrix} \hat{x}_0 + \begin{bmatrix} 0 \\ d \end{bmatrix} = 0 \tag{40}$$

Solving this equation leads to:

$$\kappa(\hat{x}_0) = -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{R} & C^\top \\ C & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} \hat{Q} \\ D \end{bmatrix} \hat{x}_0 + \begin{bmatrix} 0 \\ d \end{bmatrix} \right), \tag{41}$$

so that

$$K = -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{R} & C^\top \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} \hat{Q} \\ D \end{bmatrix} \tag{42}$$

$$v = -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{R} & C^\top \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ d \end{bmatrix} \tag{43}$$

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