Optimal ellipsoidal overapproximations for robust optimal control

Florian Messerer

Systems Control and Optimization Laboratory, University of Freiburg

syscop group retreat

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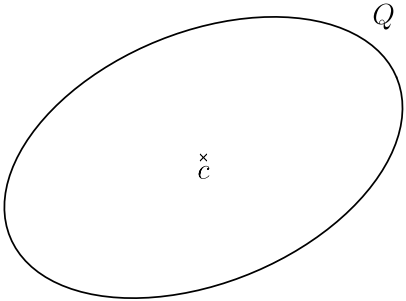
Ellipsoids 101 - Definition

- Define ellipsoid by center $c \in \mathbb{R}^n$ and shape matrix $Q \in \mathbb{S}^n_{++}$ (i.e.,$Q \succ 0$)

$$\mathcal{E}(Q,c) := \{x \in \mathbb{R}^n \mid \|x - c\|_{Q^{-1}} \leq 1\}$$



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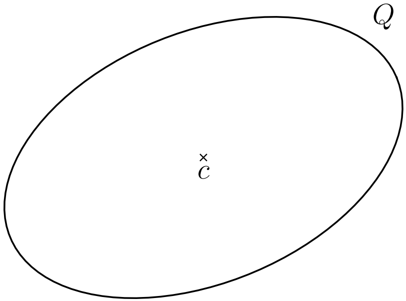
- Denote by λ_i, v_i, $i = 1, \ldots, n$, the eigenvalues / -vectors (normalized) of Q.
  - These correspond to the ellipsoid axes



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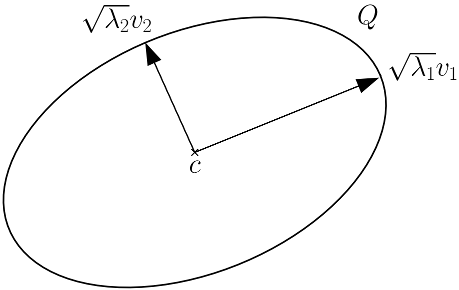

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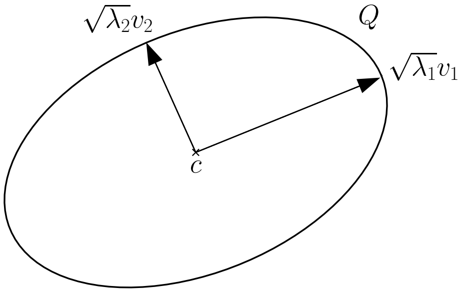
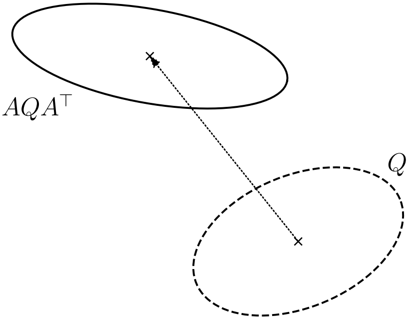
- Measuring the size of an ellipsoid

$$\operatorname{Tr} Q = \sum_{i=1}^{n} \lambda_i \qquad (\text{“least squares”})$$

$$\sqrt{\det Q} = \prod_{i=1}^{n} \sqrt{\lambda_i} \qquad \propto \text{Volume}$$



Ellipsoid 101 – Affine transformation



$$A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m,$$

$$A\mathcal{E}(Q, c) + b,$$
$$:= \{Ax + b \mid x \in \mathcal{E}(Q, c)\}$$

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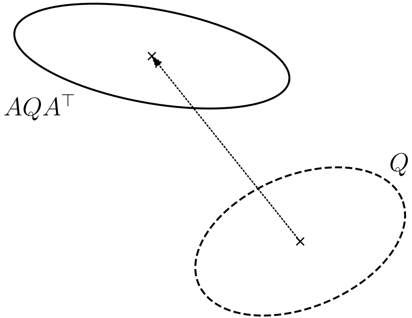

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$$\begin{aligned} & A\mathcal{E}(Q,c) + b, \\ :=\ & \{Ax + b \mid x \in \mathcal{E}(Q,c)\} \\ =\ & \mathcal{E}(AQA^\top, Ac + b) \end{aligned}$$

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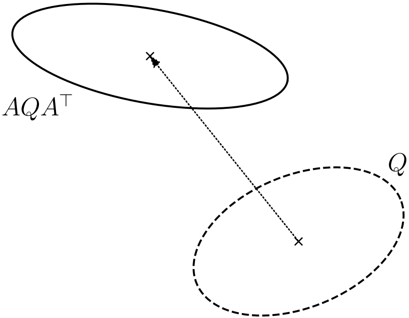

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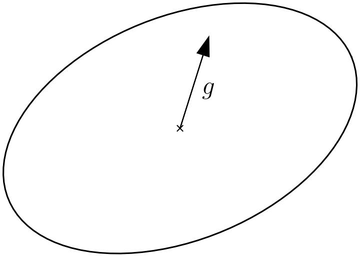
center position of the ellipsoid usually unspectular $\rightarrow$ We focus on ellipsoids centered around the origin

$$\mathcal{E}(Q) := \mathcal{E}(Q, 0)$$

Support function



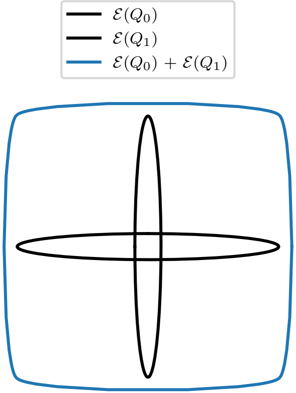
- Any non-empty compact convex set $\mathcal{S} \subset \mathbb{R}^n$ can be defined via its support function:

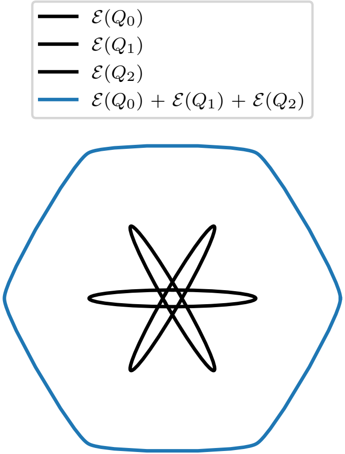
$$V(g) = \max_{x \in \mathbb{R}^n} \; g^\top x \quad \text{s.t.} \quad x \in \mathcal{S}$$

- Important tool for analysis of convex sets
- For ellipsoid:

$$\begin{aligned} V(g) &= \max_{x \in \mathbb{R}^n} \; g^\top x \quad \text{s.t.} \quad x \in \mathcal{E}(Q) \\ &= \sqrt{g^\top Q g} \end{aligned}$$

Sum of ellipsoids (Minkowski sum)





Overapproximating sum of ellipsoids by ellipsoid

- Aim: find Q such that $\mathcal{E}(Q) \supseteq \mathcal{E}(Q_1) + \mathcal{E}(Q_2)$

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- Construct family of outer approximations parametrized by $\alpha \in \mathbb{R}_{++}^{KN}$

$$Q(\alpha) = \sum_{k=1}^{N} \frac{1}{\alpha_k} Q_k \quad \Rightarrow \quad \mathcal{E}(Q(\alpha)) \supseteq \sum_{k=1}^{N} \mathcal{E}(Q_k) \quad \forall \alpha \in \mathbb{R}_{++}^{N} \quad \text{with} \quad \sum_{k=1}^{N} \alpha_k = 1$$

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- Denote set of feasible α by $\mathcal{A}^N$ (basically a simplex)
- Parametrized outer approximation is tight (but not complete)

$$\bigcap_{\alpha \in \mathcal{A}^N} \mathcal{E}(Q(\alpha)) = \sum_{k=1}^{N} \mathcal{E}(Q_k)$$

Overapproximating sum of ellipsoids by ellipsoid (cont.)

- In general: Choose α according to some criterion
  - e.g., such that $\mathcal{E}(Q(\alpha))$ has minimal size, e.g., $\min_{\alpha \in \mathcal{A}^N} \mathrm{Tr}(Q(\alpha))$
  - or $\mathcal{E}(Q(\alpha))$ tight in a given direction $g \in \mathbb{R}^n$ (approximation touches true sum)

$$\min_{\alpha \in \mathcal{A}^N} \left(\max_{x \in \mathbb{R}^n} \; g^\top x \quad \text{s.t.} \quad x \in \mathcal{E}(Q(\alpha)) \right) = \min_{\alpha \in \mathcal{A}^N} \sqrt{g^\top Q(\alpha) g} \;\hat{=}\; \min_{\alpha \in \mathcal{A}^N} \mathrm{Tr}(gg^\top Q(\alpha))$$

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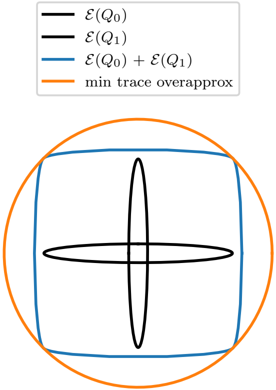
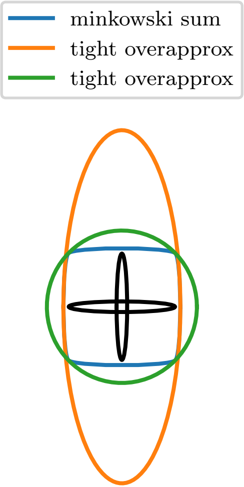
- Special case $N = 2$
  - $Q(\alpha) = \frac{1}{\alpha_1} Q_1 + \frac{1}{\alpha_2} Q_2$ with $\alpha_1 + \alpha_2 = 1$
  - Reparametrize: $\alpha_2 = 1 - \alpha_1$, $\beta = \frac{1}{1-\alpha_1} > 0$
  - $\tilde{Q}(\beta) = (1 + \frac{1}{\beta}) Q_1 + (1 + \beta) Q_2$
  - Inclusion-minimal (contains all the “best” overapproximations)

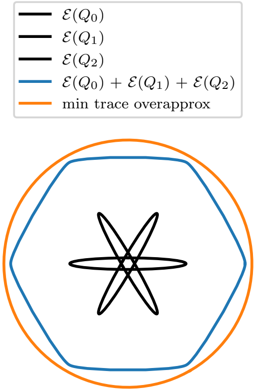
$$\arg\min_{\beta > 0} \operatorname{Tr} \tilde{Q}(\beta) = \arg\min_{\beta > 0} (1 + \frac{1}{\beta}) \operatorname{Tr} Q_1 + (1 + \beta) \operatorname{Tr} Q_2 = \sqrt{\frac{\operatorname{Tr} Q_1}{\operatorname{Tr} Q_2}}$$

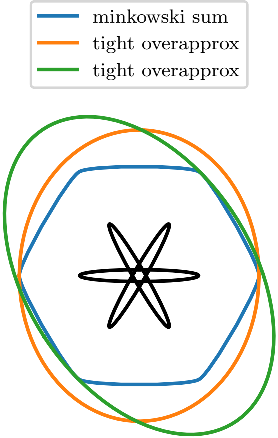
Overapproximations of sum of two ellipsoids





Overapproximations of sum of three ellipsoids





Uncertain linear dynamical systems

$$x_{k+1} = Ax_k + \Gamma w_k$$

- Reachable set

$$\begin{aligned} x_k &\in \mathcal{E}(P_k),\ w_k \in \mathcal{E}(W) \\ \Rightarrow x_{k+1} &\in \mathcal{E}(AP_kA^\top) + \mathcal{E}(\Gamma W \Gamma^\top) \end{aligned}$$

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  - Uncertainty set not ellipsoidal :(
  - Overapproximate by ellipsoid

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- Overapproximation of reachable set

$$x_k \in \mathcal{E}(P_k(\beta)),\ w_k \in \mathcal{E}(W)$$
$$P_{k+1}(\beta) = (1+\beta_k)AP_k(\beta)A^\top + (1+\tfrac{1}{\beta_k})\Gamma W \Gamma^\top$$
$$\Rightarrow x_{k+1} \in \mathcal{E}(P_{k+1}(\beta))$$

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Overapproximation OCP

$$\min_{\substack{\beta_0,\ldots,\beta_{N-1}\in\mathbb{R}_{++},\\ P_0,\ldots,P_N}} \quad \sum_{k=0}^{N} \operatorname{Tr} L_k P_k \tag{1a}$$

$$\text{s.t.} \quad P_0 = \bar{P}_0, \tag{1b}$$

$$P_{k+1} = (1+\beta_k) A P_k A^\top + (1+\frac{1}{\beta_k}) \Gamma W \Gamma^\top, \quad k = 0,\ldots,N. \tag{1c}$$

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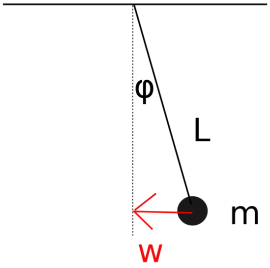
- Eliminating P_k and substituting $\exp\gamma \leftarrow \beta$ results in convex problem
- special case $A = I$, $L_0 = \cdots = L_{N-1} = 0$ or $A = I$, $L_0 = \cdots = L_N$
  - solve by forward recursion, $\beta_k = \sqrt{\frac{\operatorname{Tr}\Gamma W \Gamma^\top}{\operatorname{Tr} P_k}}$
- Recursive overapproximation of N sums of two ellipsoids vs. overapproximation of one sum of N ellipsoids?
  - No loss of expressiveness (I think)

Example – Linearized Pendulum

$$x = \begin{bmatrix} \varphi \\ \omega \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} \omega \\ -\frac{g}{L}\varphi + \frac{1}{mL}w \end{bmatrix}$$

- discretize in time with $T = 1.2$, $N = 10$
- uncertainty $w \in [-1, 1]$ piecewise constant
- Start at $x_0 = 0$.
- compute reachable set at final time and compare to overapproximations

$$\min_{\beta > 0} \operatorname{Tr} P_N(\beta), \qquad \min_{\beta > 0} \operatorname{Tr}(gg^\top P_N(\beta)),$$



(Houska 2011)

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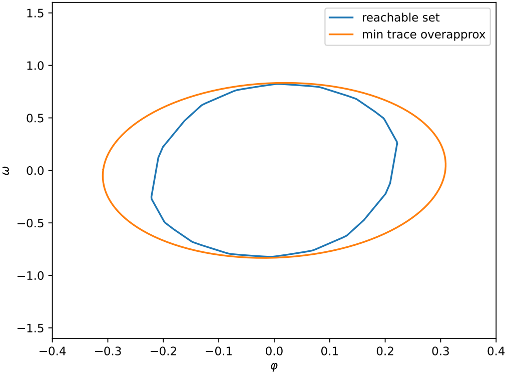
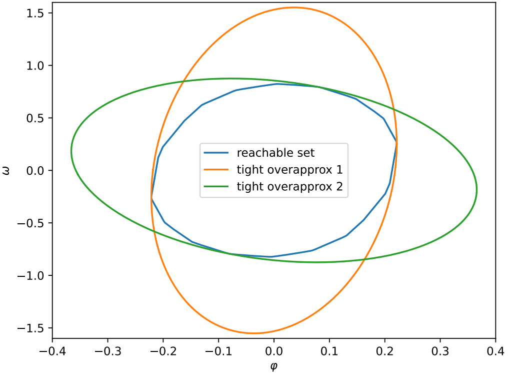




Uncertain linear dynamical systems

- Now consider controlled system

$$x_{k+1} = Ax_k + Bu_k + \Gamma w_k$$

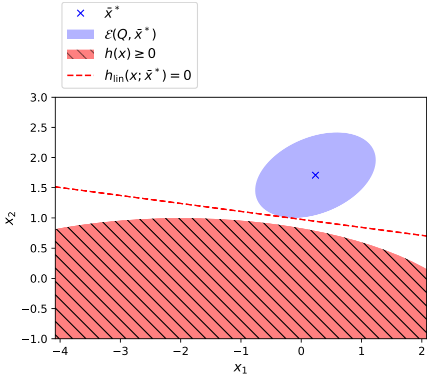
- plan over feedback law to manipulate the ellipsoids

$$u_k = \bar{u}_k + K_k(x - \bar{x}_k)$$
$$P_{k+1} = (1+\beta_k)(A+BK_k)P_k(A+BK_k)^\top + (1+\tfrac{1}{\beta_k})\Gamma W \Gamma^\top$$

- Linearize Constraints

$$h(x) \leq 0 \quad \forall x \in \mathcal{E}(P, \bar{x})$$
$$\rightarrow \quad h(\bar{x}) + \nabla h(\bar{x})^\top (x - \bar{x}) \leq 0 \quad \forall x \in \mathcal{E}(P, \bar{x})$$

$\rightarrow$ Conservative for concave $h(x)$



Where do I want to go with this?

Robust OCP with optimal overapproximation and feedback, for linear dynamics, and "concave constraints".

$$
\begin{aligned}
\min_{x,u,\beta,P,K} \quad & \sum_{k=0}^{N-1} l_k(\bar{x}_k, \bar{u}_k) + l_N(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \; P_0 = \bar{P}_0, \\
& \bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k, && k = 0, \ldots, N-1, \\
& P_{k+1} = (1+\beta_k)(A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + (1 + \tfrac{1}{\beta_k}) \tilde{W}_k^\top, && k = 0, \ldots, N-1, \\
& 0 \geq h_k(\bar{x}_k, \bar{u}_k) + \nabla h_k(\bar{x}_k, \bar{u}_k)^\top (x - \bar{x}) \quad \forall z \in \mathcal{E}(P_z(P_k, K_k), \bar{z}), && k = 0, \ldots, N-1, \\
& 0 \geq h_N(\bar{x}_N) + \nabla h_N(\bar{x}_N)^\top (x - \bar{x}_N) \quad \forall x \in \mathcal{E}(P_N),
\end{aligned}
$$

with $z = (x, u)$ and P_z the corresponding ellipsoid matrix

Further considerations

- Establish advantage of minimizing over β, K
  - vs. precomputing arbitrary LQR gain and choosing β as minimizing trace step-by-step
  - feedback $\rightarrow$ small uncertainty $\rightarrow$ choice of β less relevant
- Design taylored algorithm, ZORO / SIRO style (alternate Riccati recursion, "Trace OCP", and nominal OCP with fixed back-off?)
- Find a nice linear system as example
- Alternative: affine-in-state-and-disturbance

$$x_{k+1} = A(u_k)x_k + \Gamma(u_k)w_k + b(u_k)$$

  - in this case without feedback-law
  - Examples of this system class
    - systems with controlled mass-flow-rate (heating)
    - ... ?