The Role of Convexity in Data-Driven Decision-Making

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Delft University of Technology

Robust Model Predictive Control Workshop

University of Freiburg, Germany

February 2023

Outline

- Data-Driven Decision-Making
  - Three research questions
- Online Convex Optimization
  - Algorithms and regret bounds

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Data-Driven Decision-Making

Loss function: $\ell(x, \xi) \in \mathbb{R}$

$x \in \mathbb{X}$ decision $\quad \xi \in (\Xi, \mathbb{P})$ uncertainty

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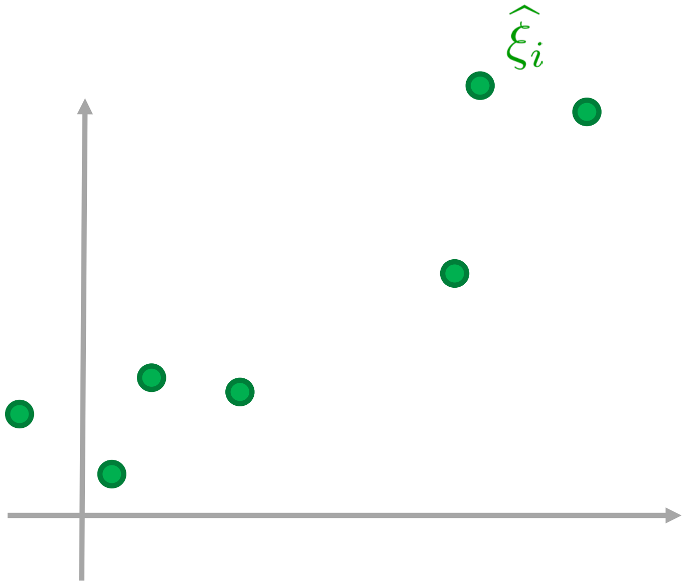

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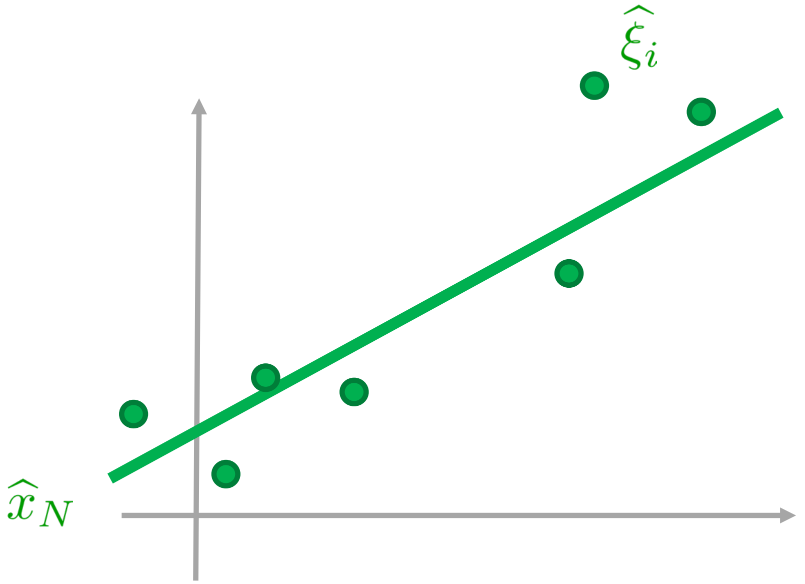

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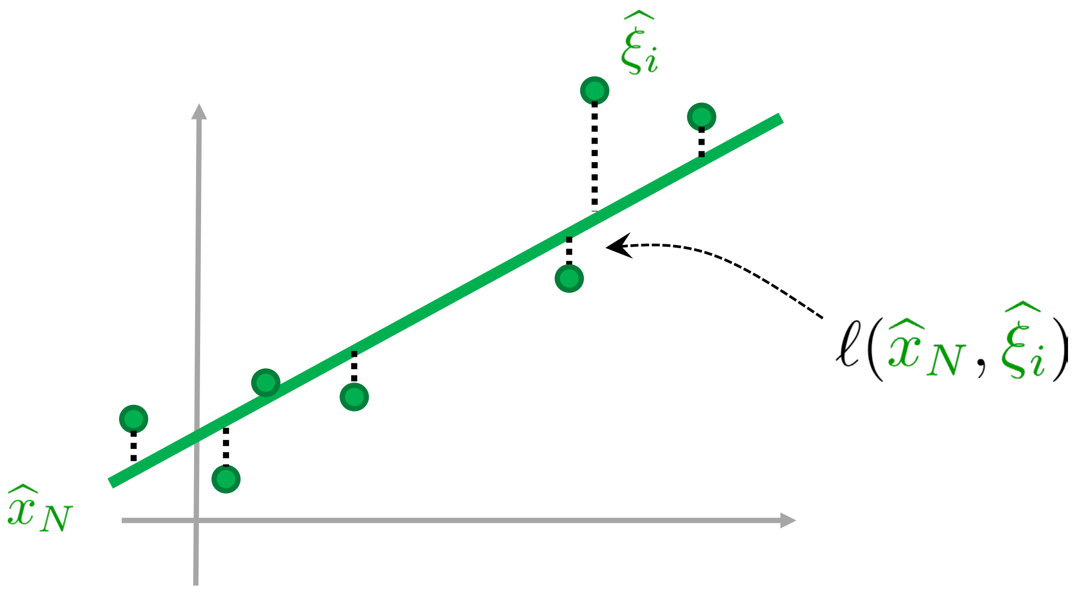

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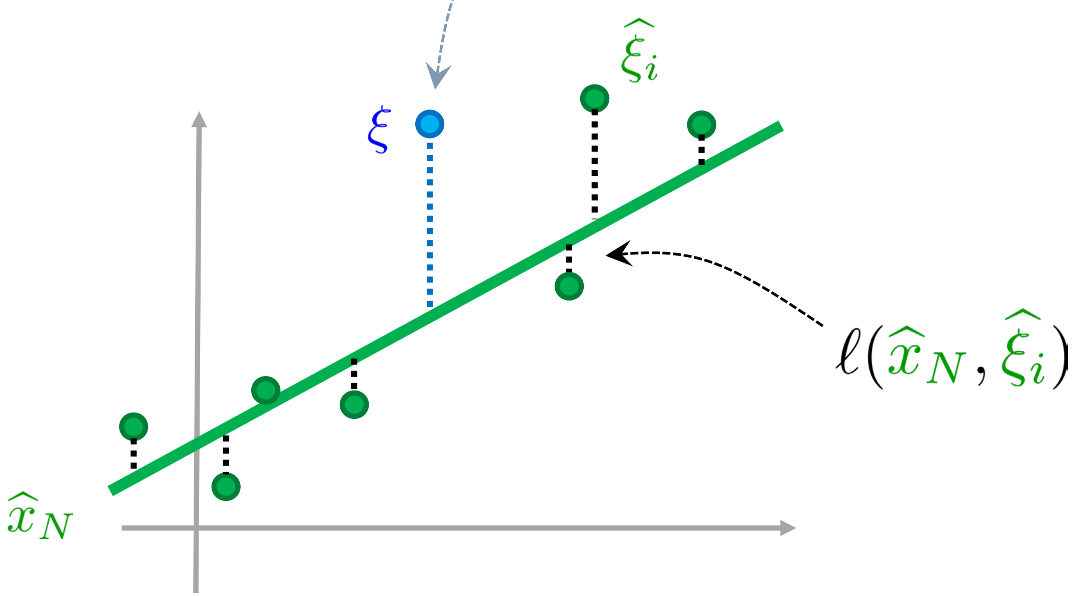
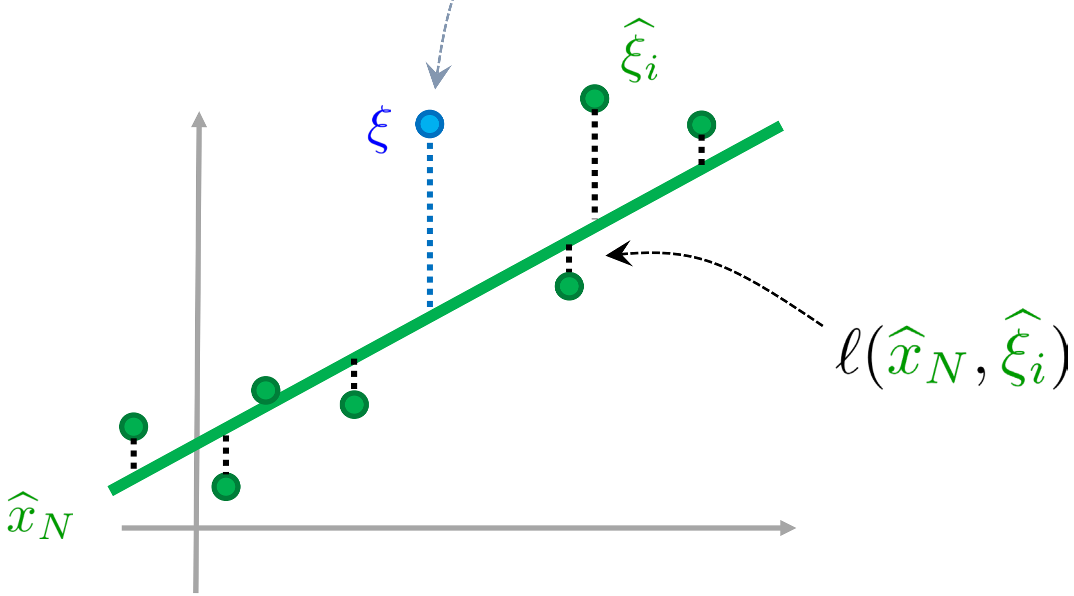


Data-Driven Decision-Making

$$\underbrace{(\widehat{\xi}_1, \ldots, \widehat{\xi}_N)}_{\text{past}} \overset{\text{(Prescriptive)}}{\rightsquigarrow} \underbrace{\widehat{x}_N}_{\text{present}} \overset{\text{(Predictive)}}{\rightsquigarrow} \underbrace{\ell(\widehat{x}_N, \xi)}_{\text{future}}$$



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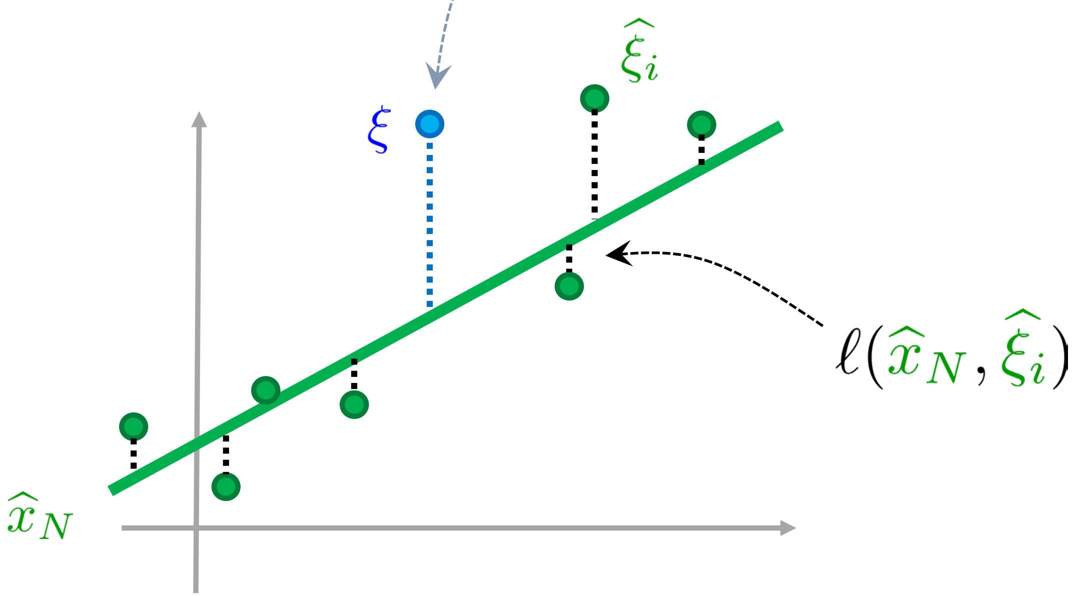
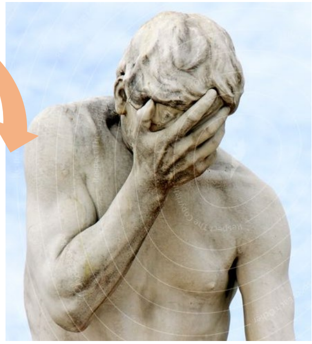


Post-decision disappointment



Data-Driven Decision-Making

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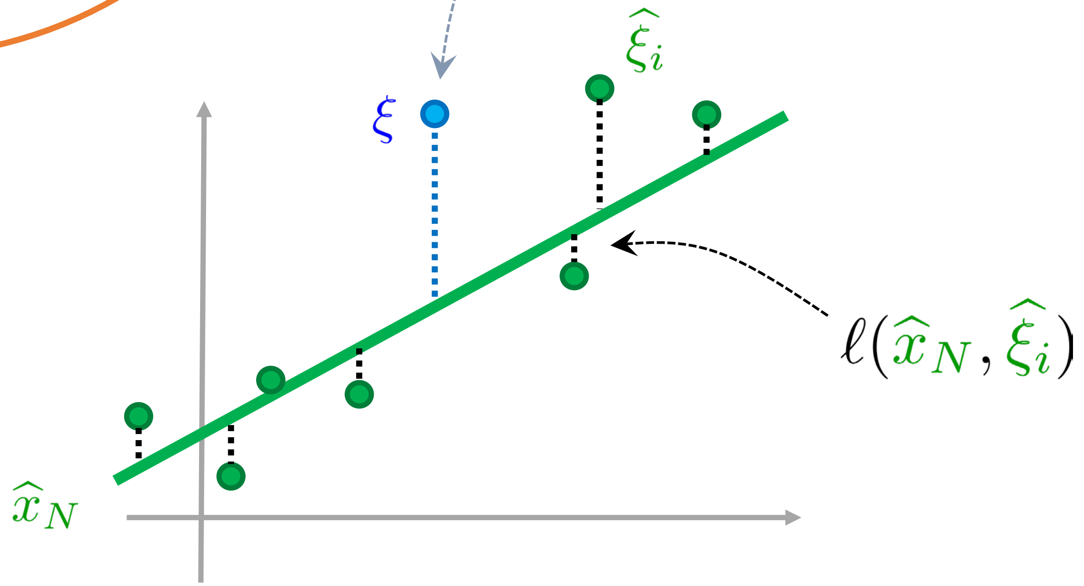
(1) Computation



Data-Driven Decision-Making

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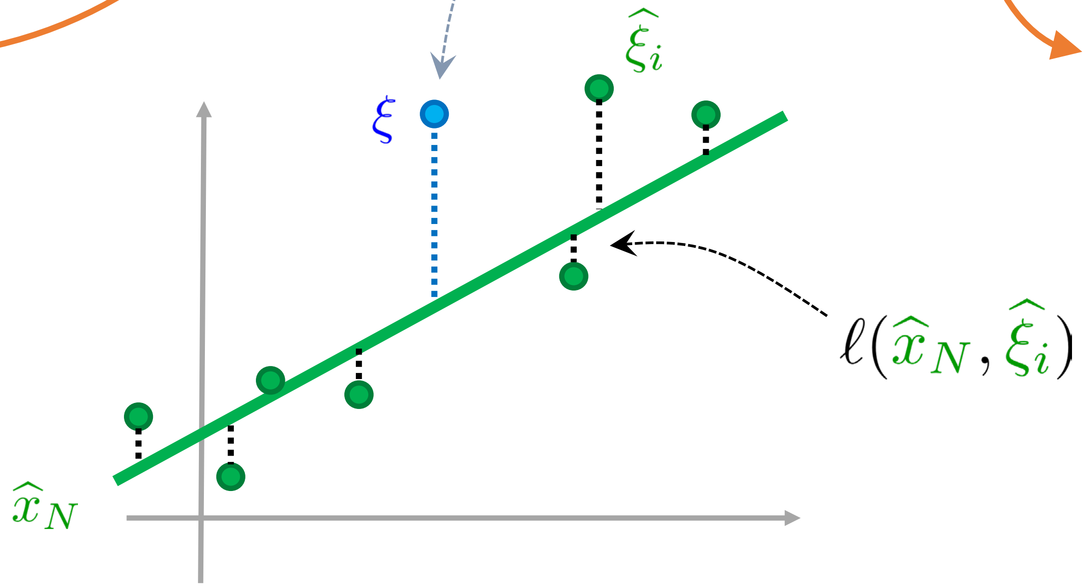
(1) Computation

(2) Statistics



Data-Driven Decision-Making

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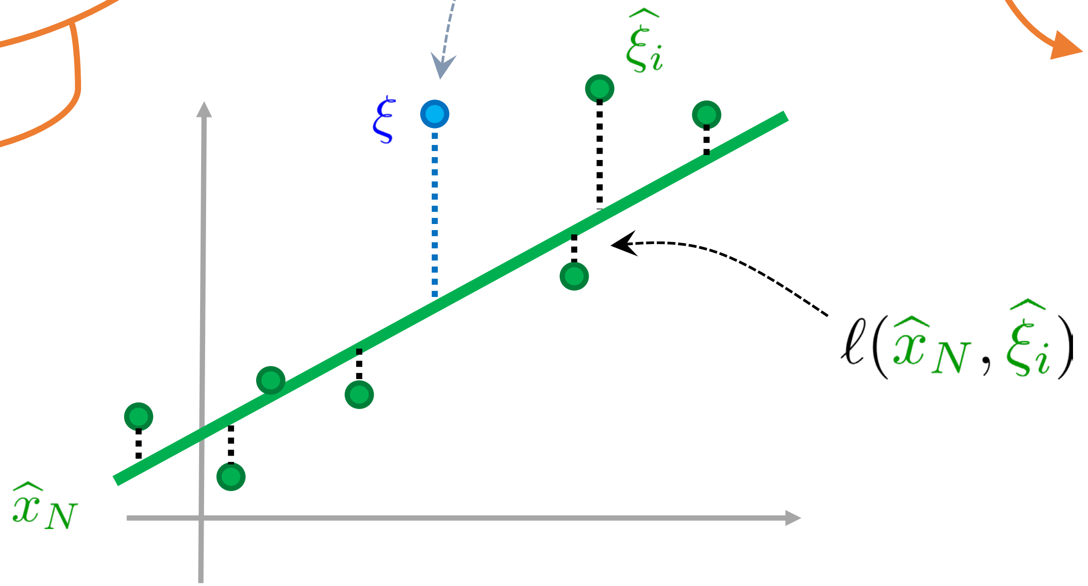
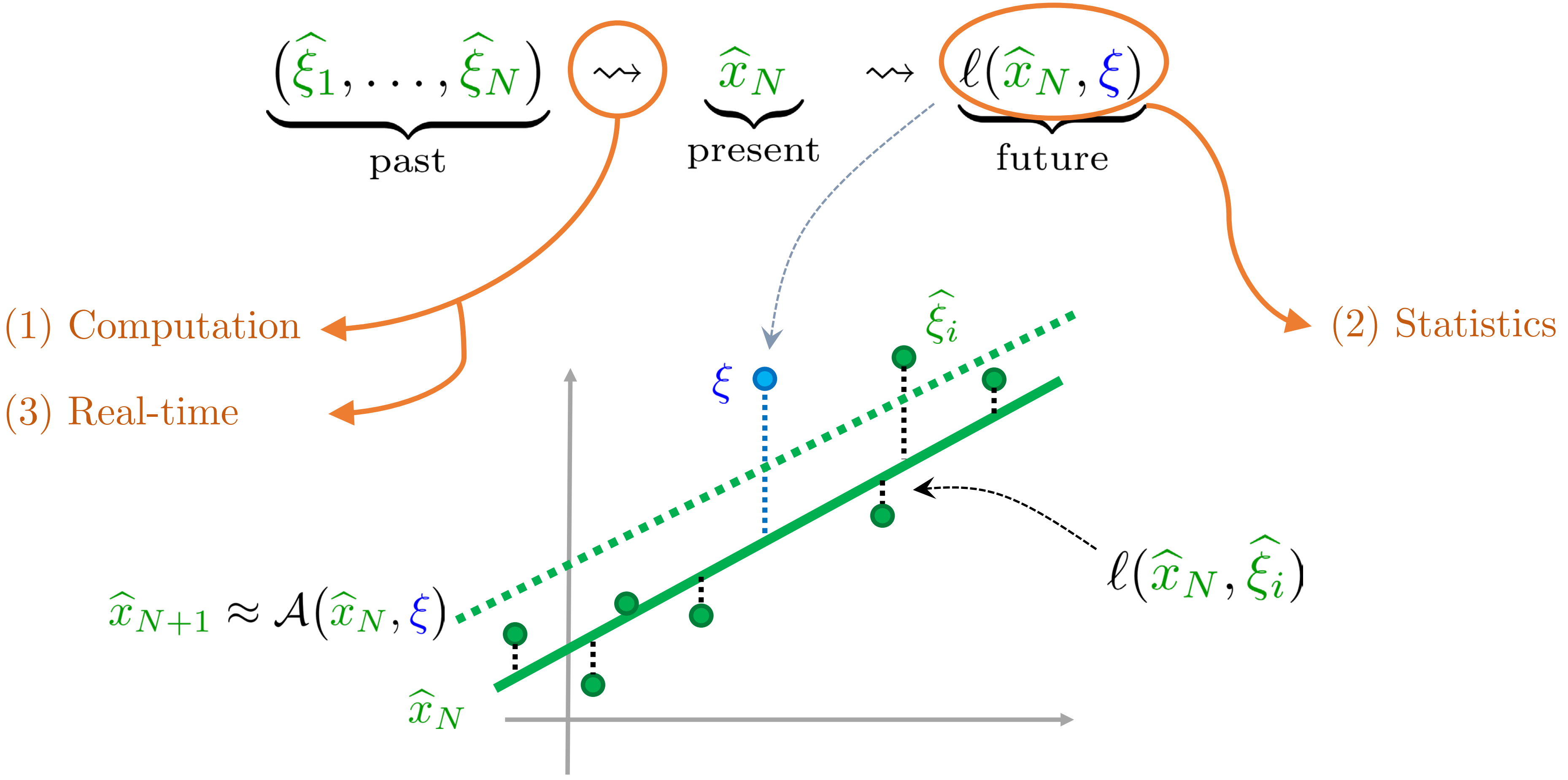
(1) Computation

(3) Real-time

(2) Statistics



Data-Driven Decision-Making



Data-Driven Decision-Making

The Role of Convexity

(1) Computational aspects

(2) Statistical aspects

(3) Real-time aspects

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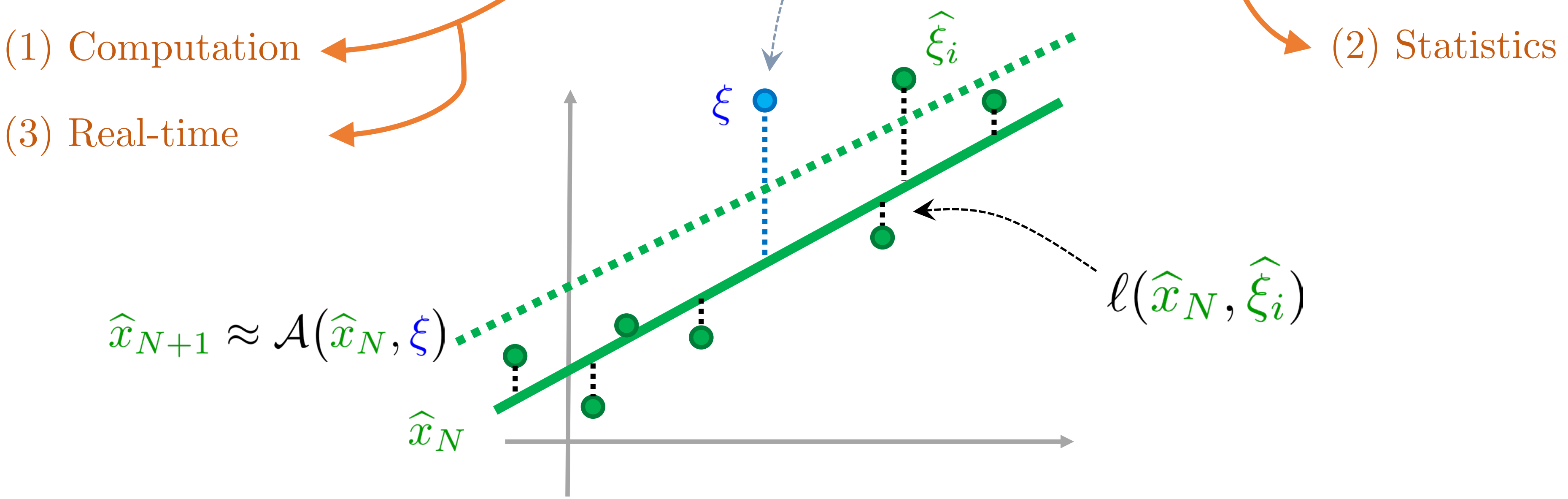

Data-Driven Decision-Making

The Role of Convexity

(1) Computational aspects

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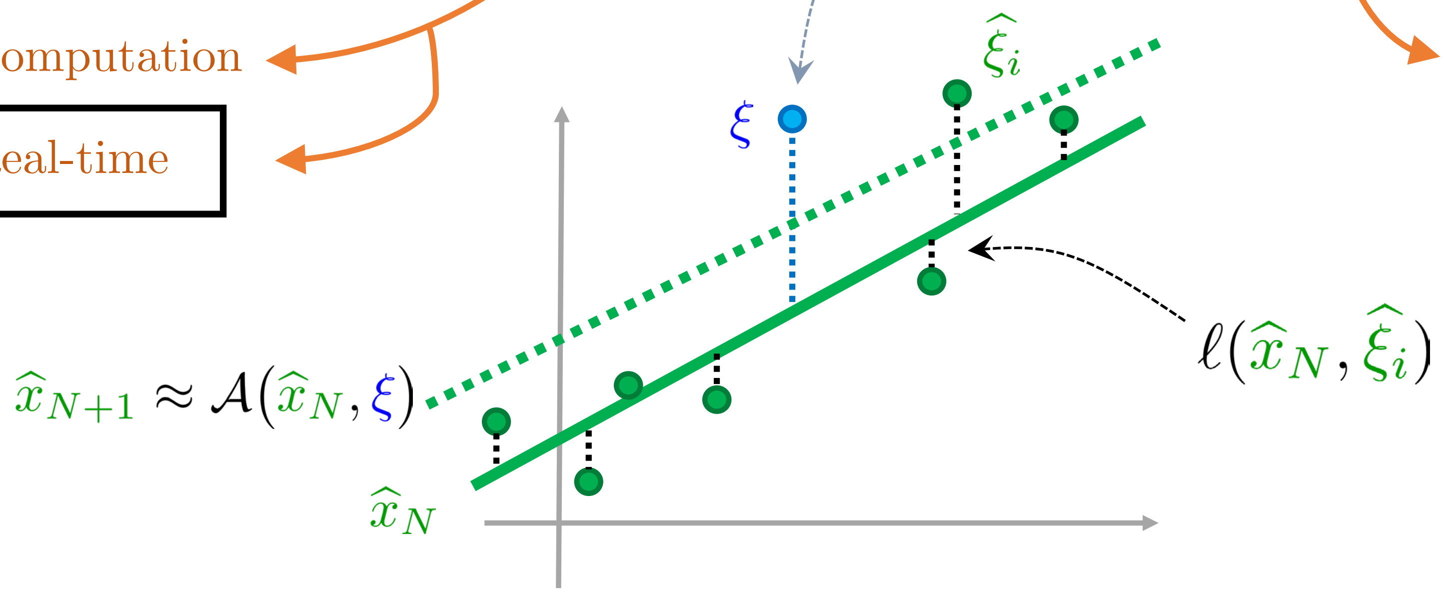
(1) Computation

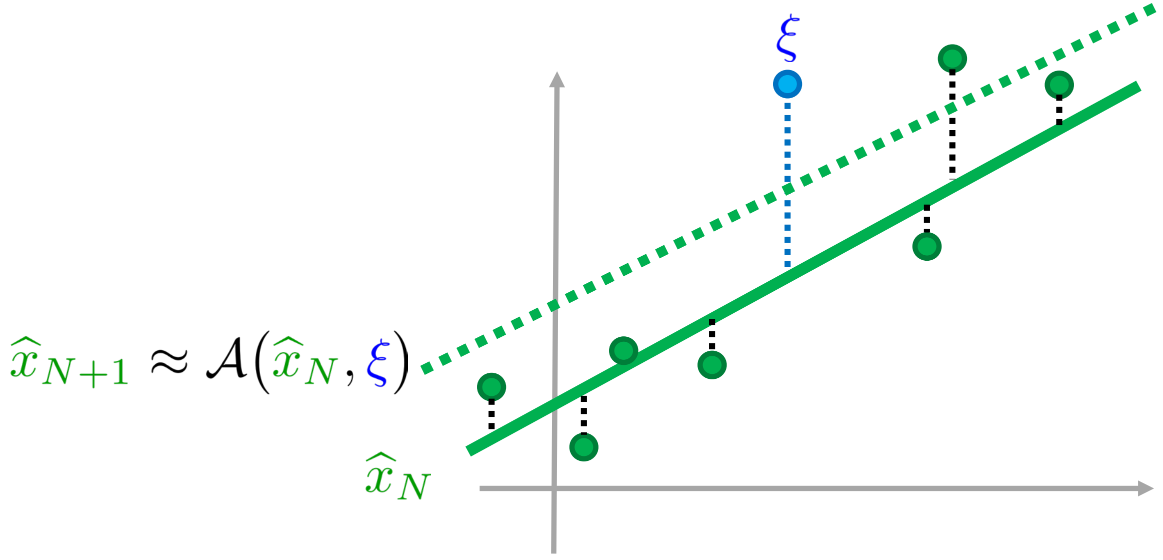
This talk $\Rightarrow$ $\boxed{\text{(3) Real-time}}$

(2) Statistics



Performance Assessment

$$(\widehat{\xi}_1, \ldots, \widehat{\xi}_N) \quad \rightsquigarrow \quad \widehat{x}_N \quad \rightsquigarrow \quad \ell(\widehat{x}_N, \xi)$$



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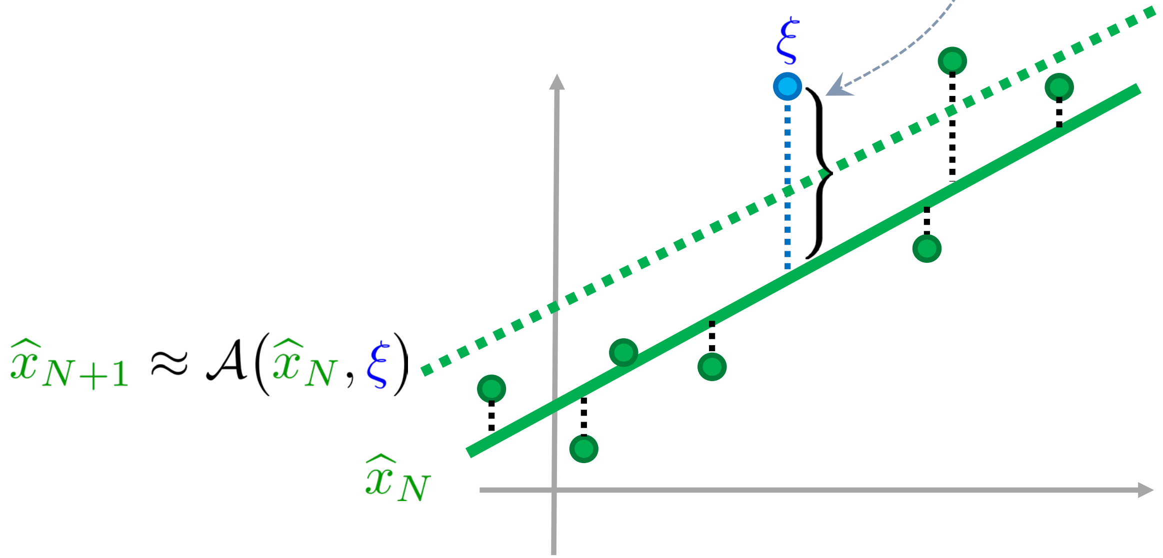

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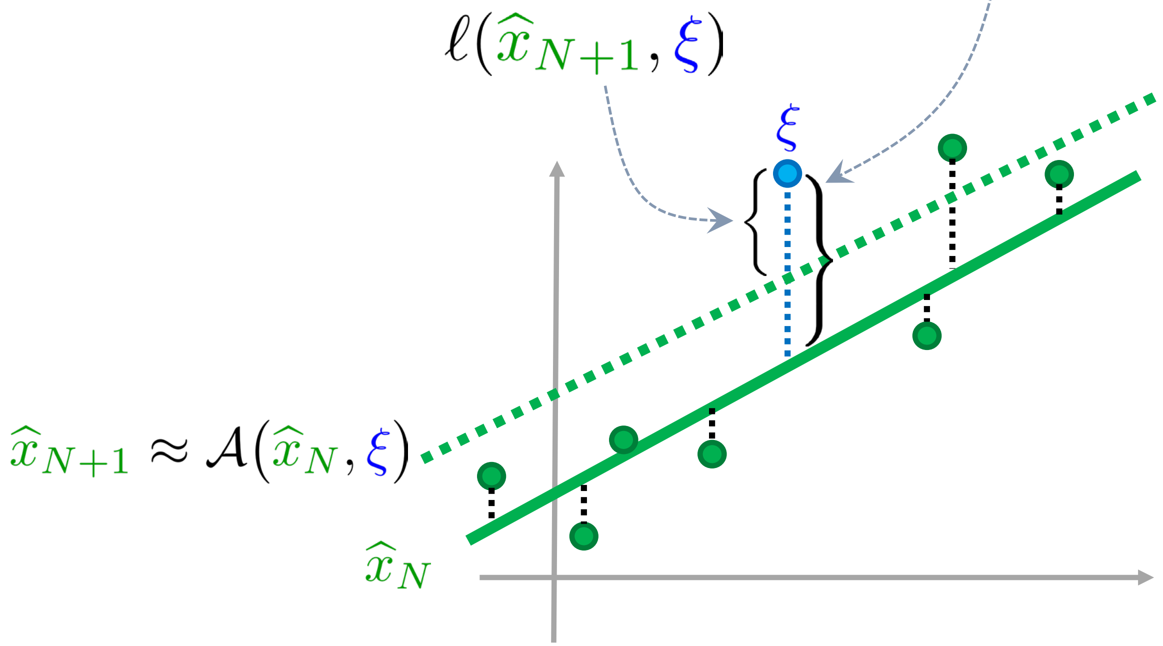

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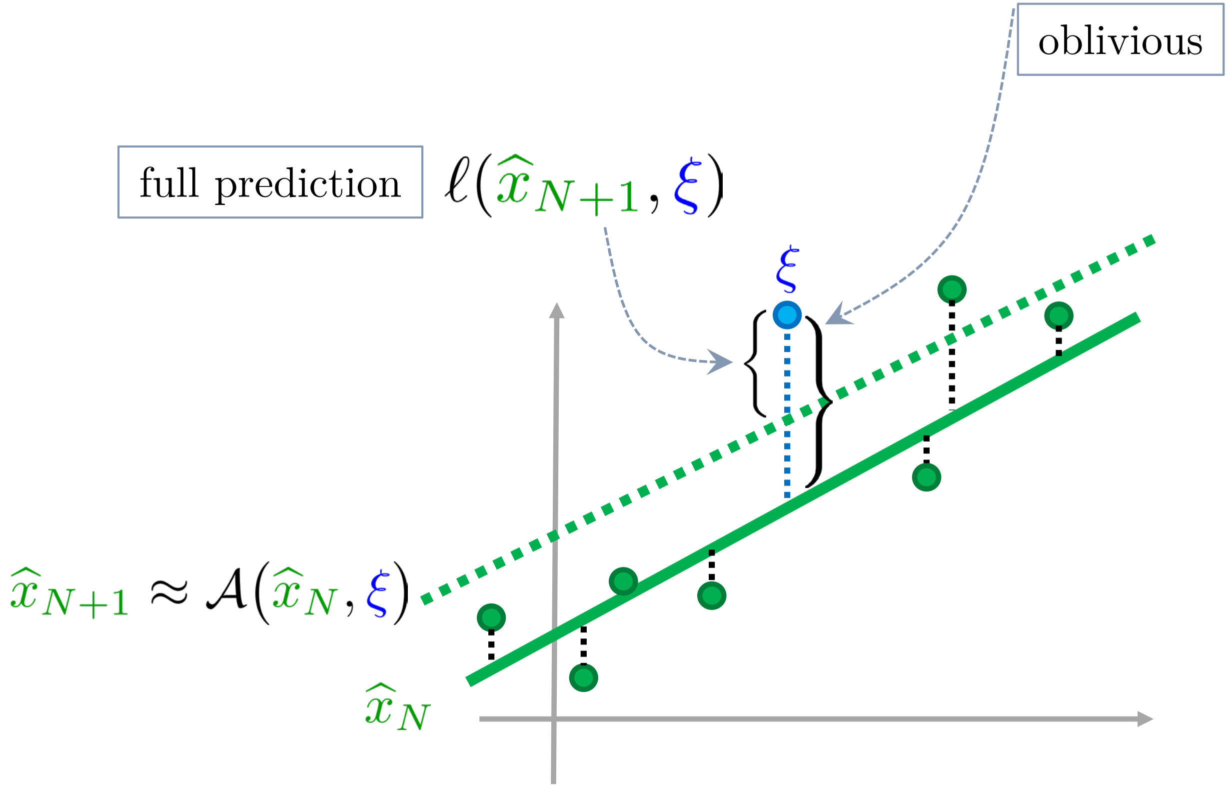

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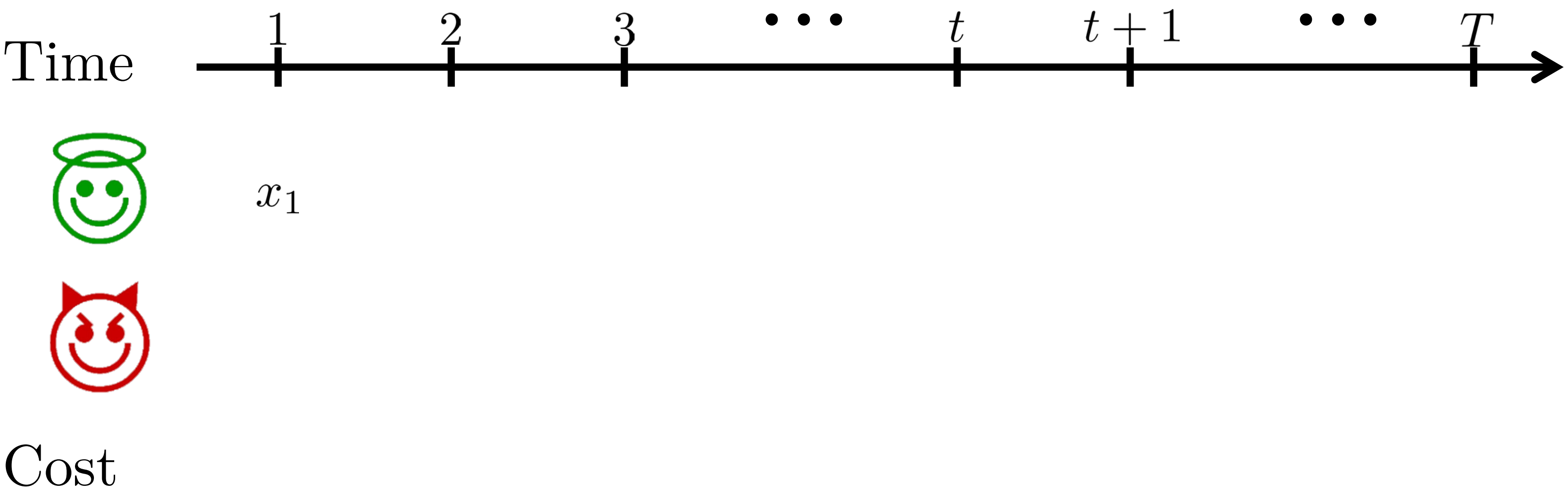
Pedro Zattoni Scroccaro

Online Optimization: Setting



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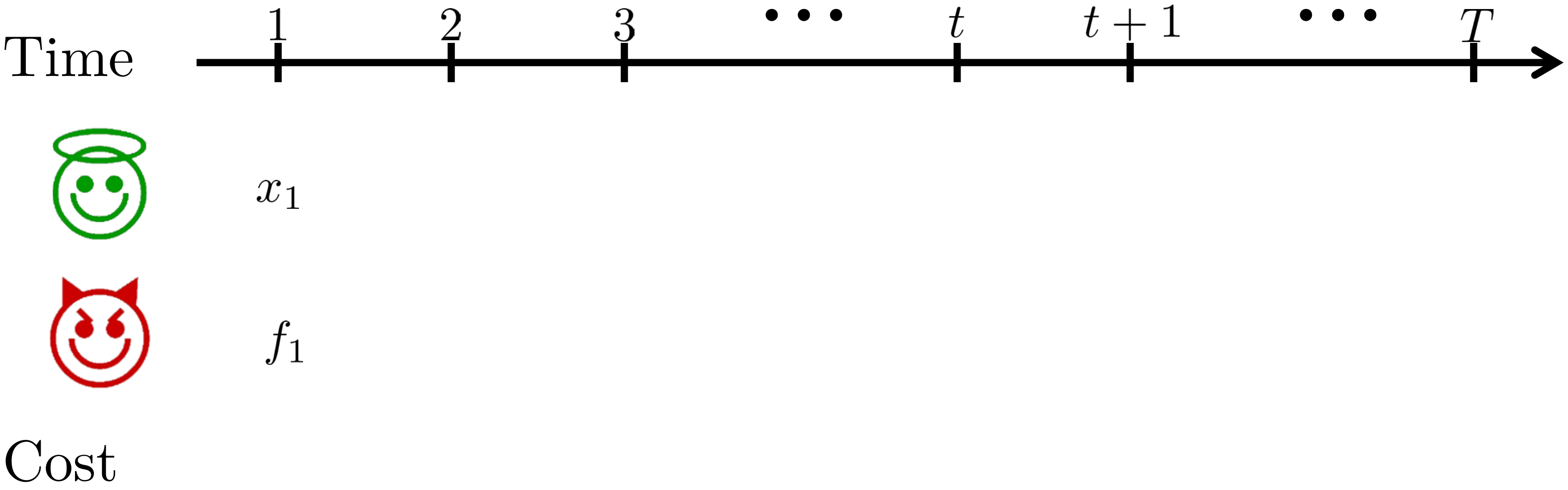

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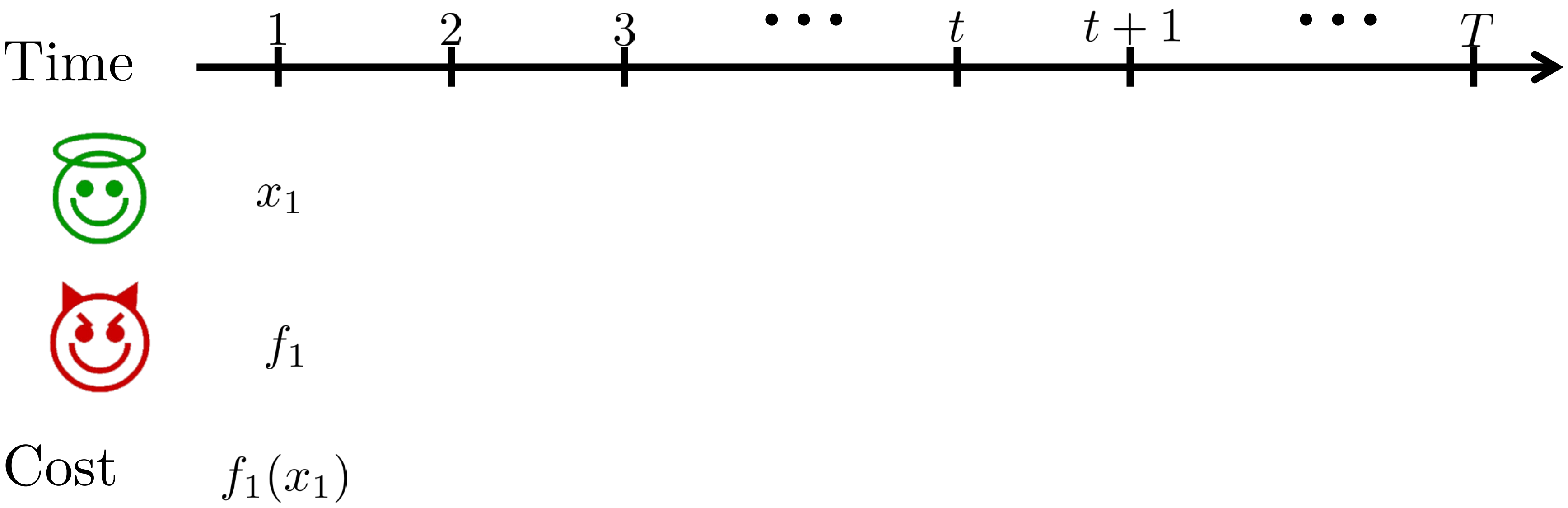

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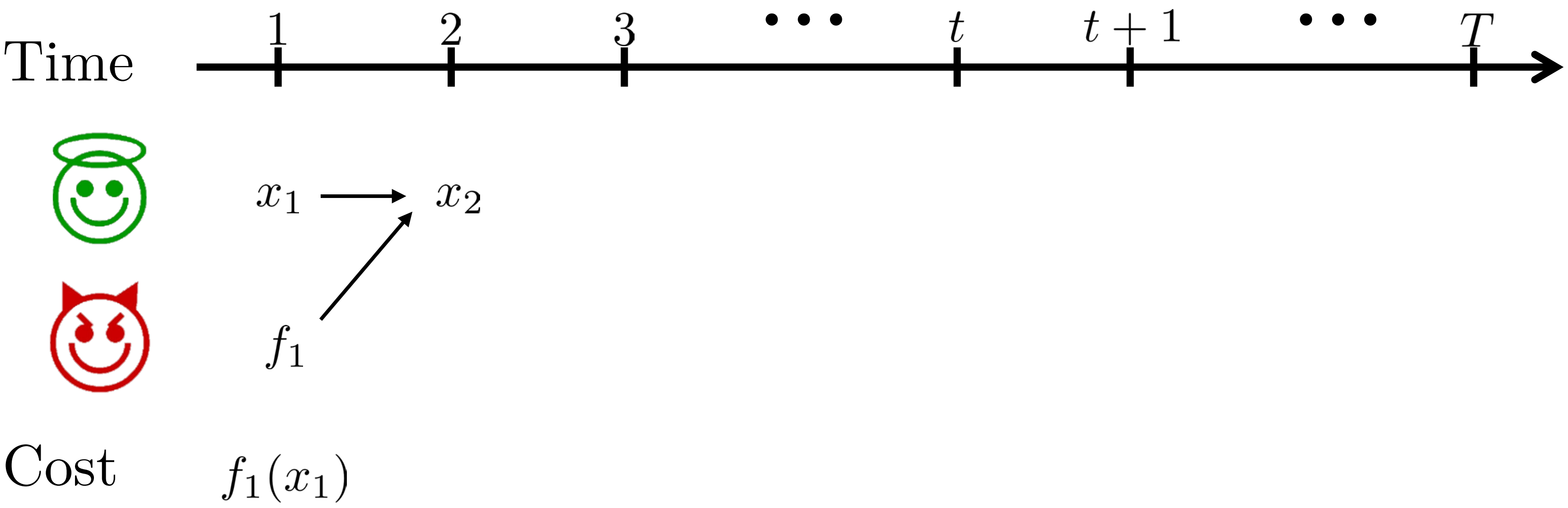

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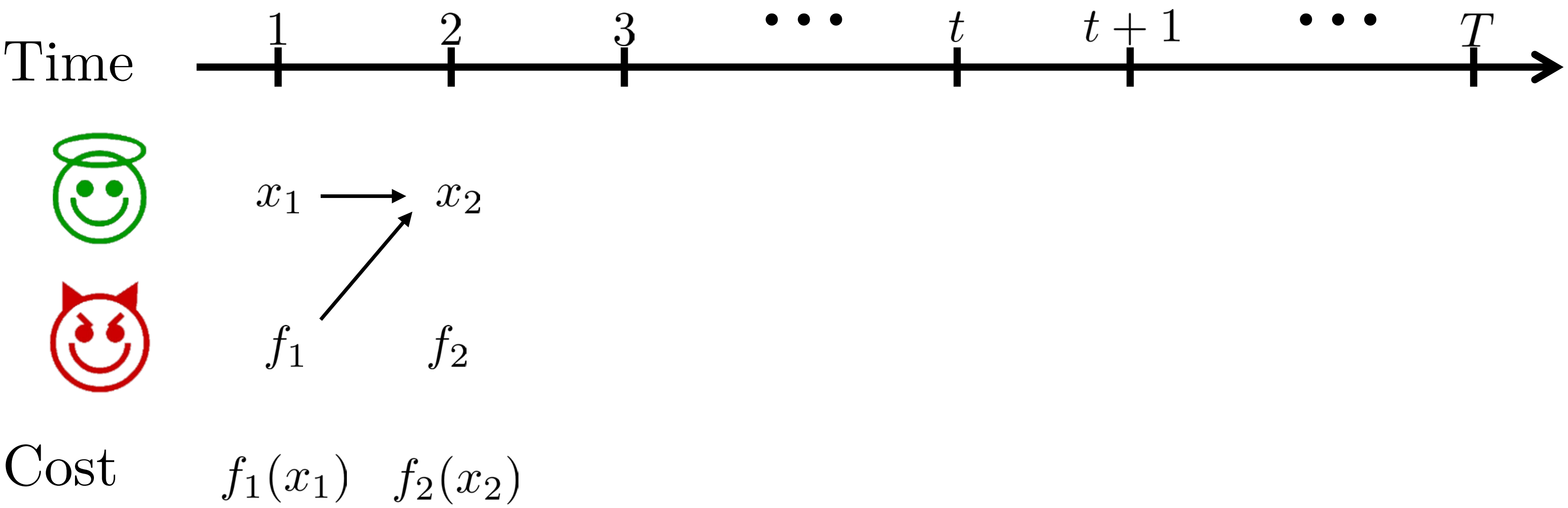

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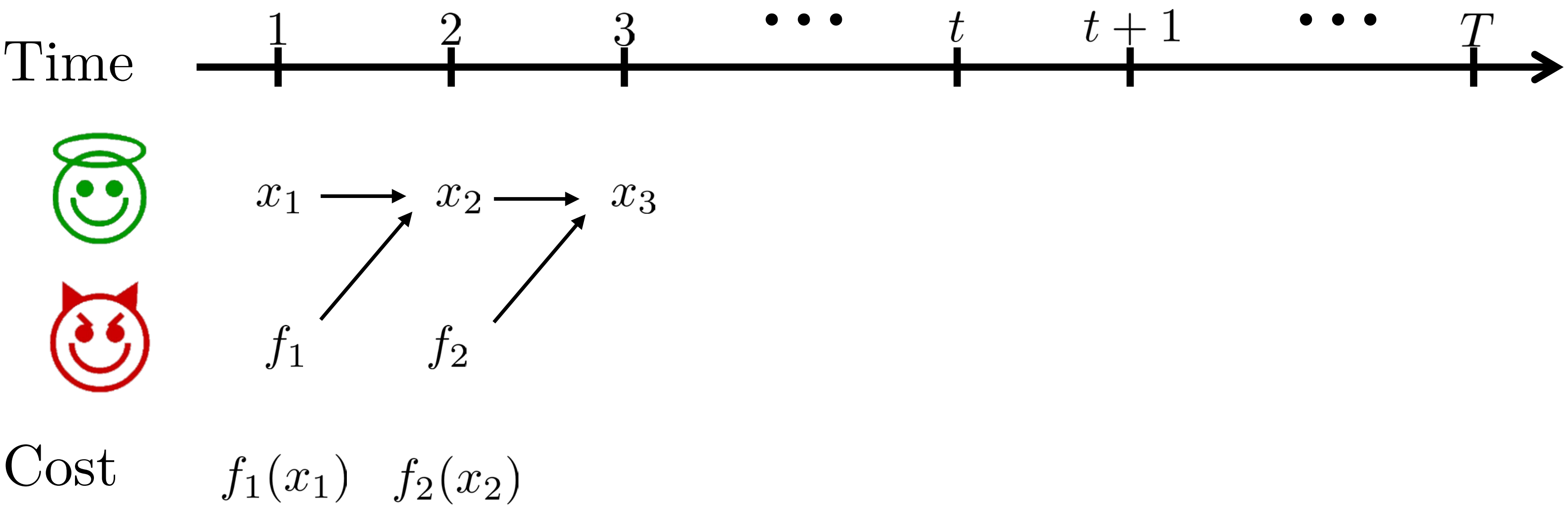

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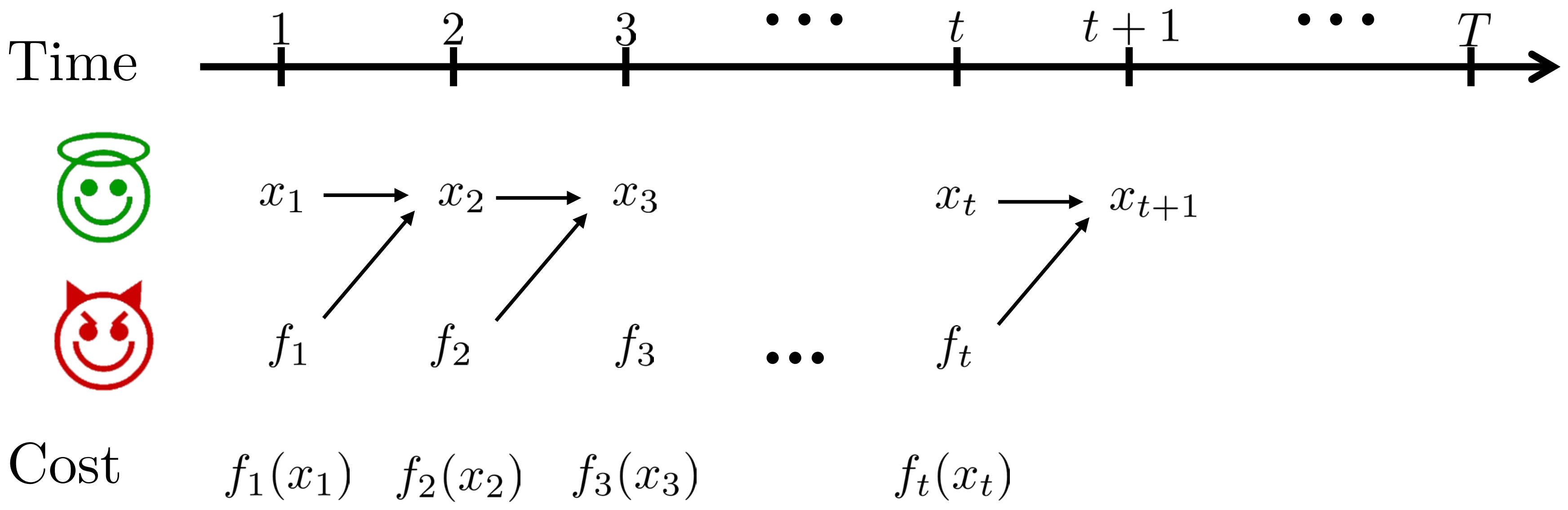

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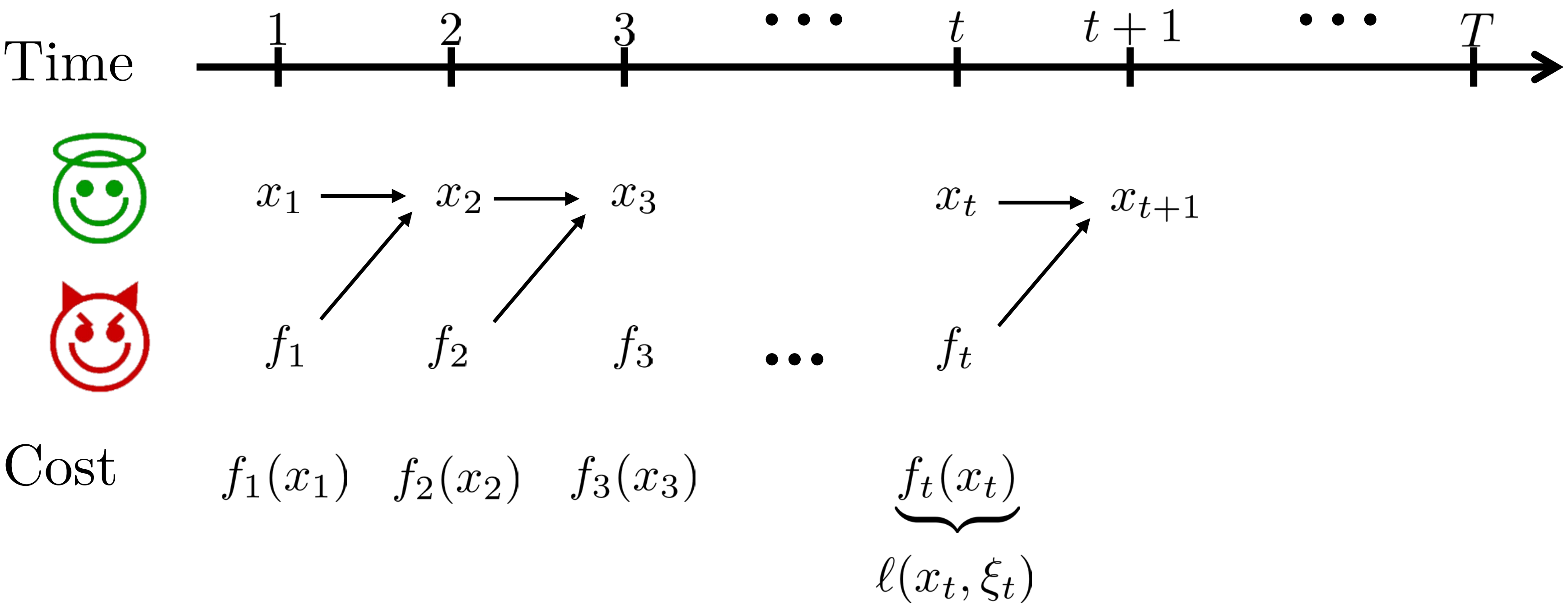

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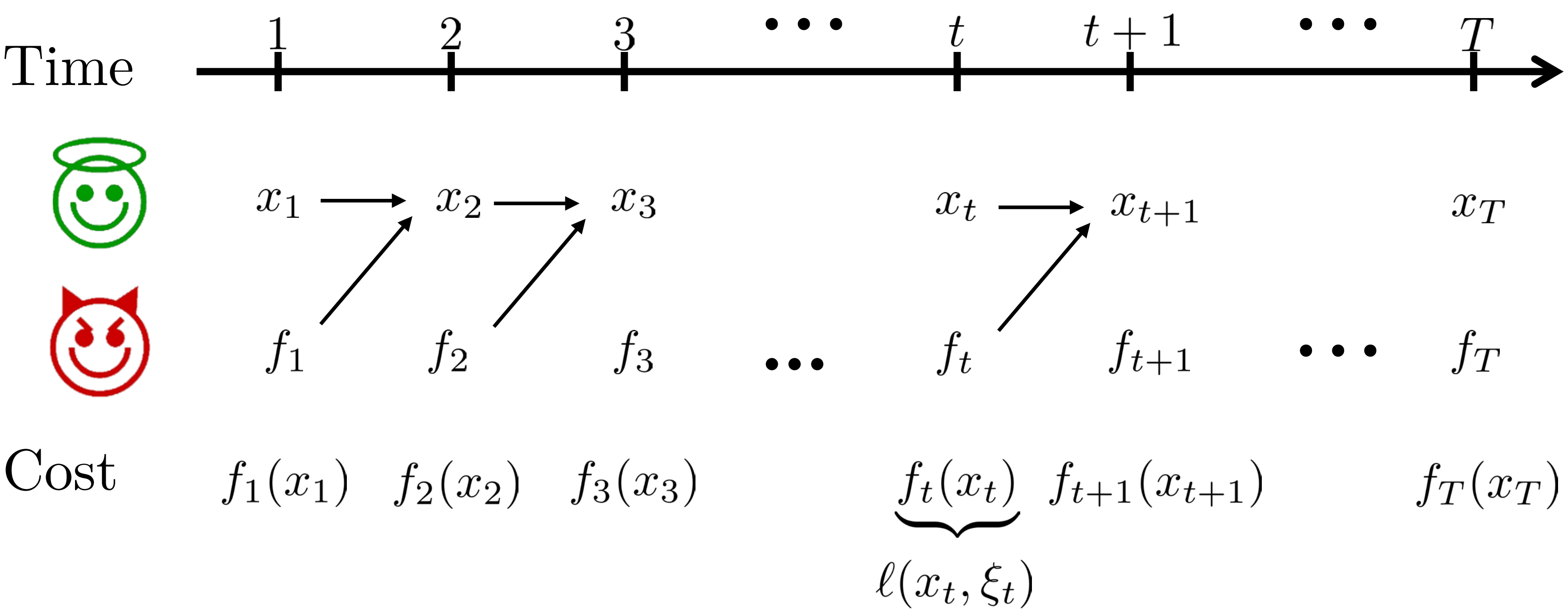

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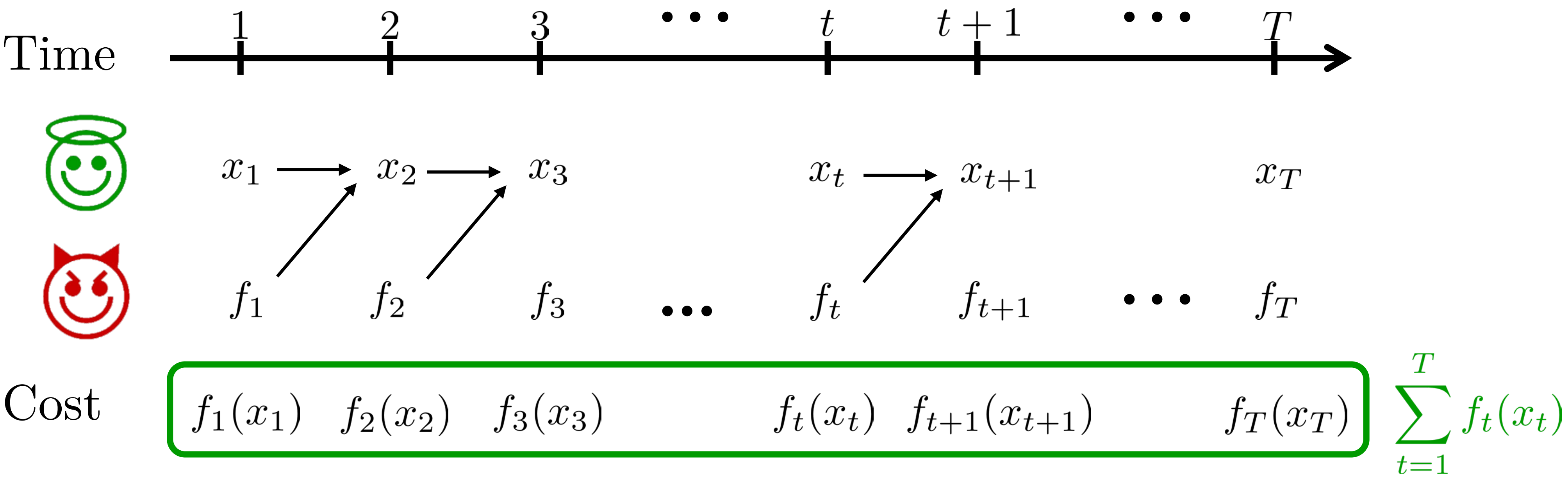

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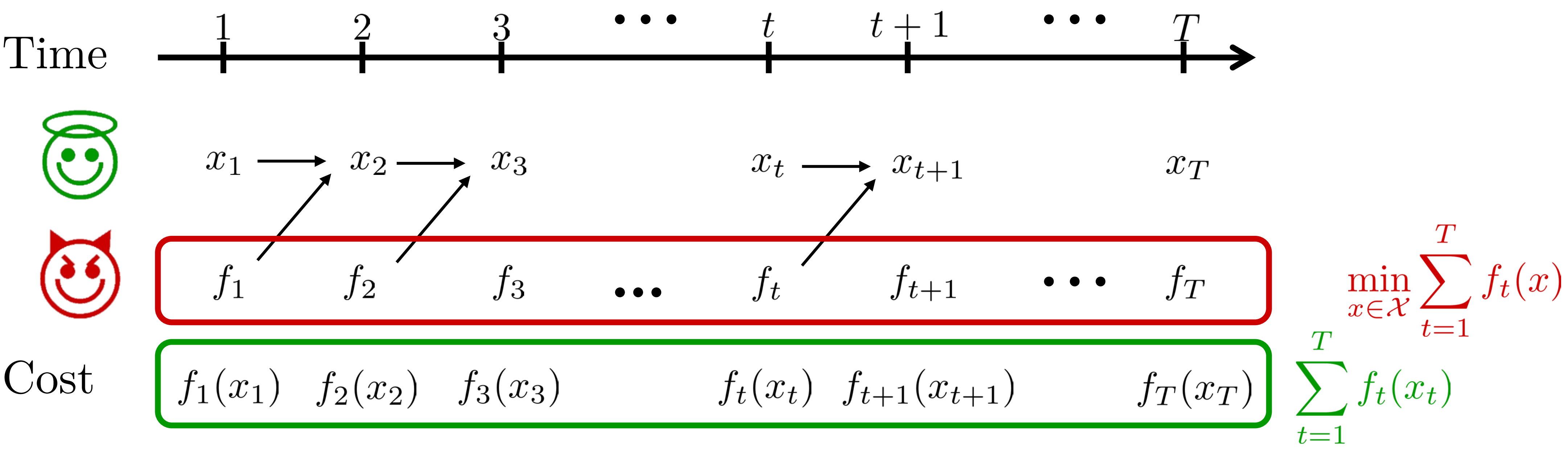

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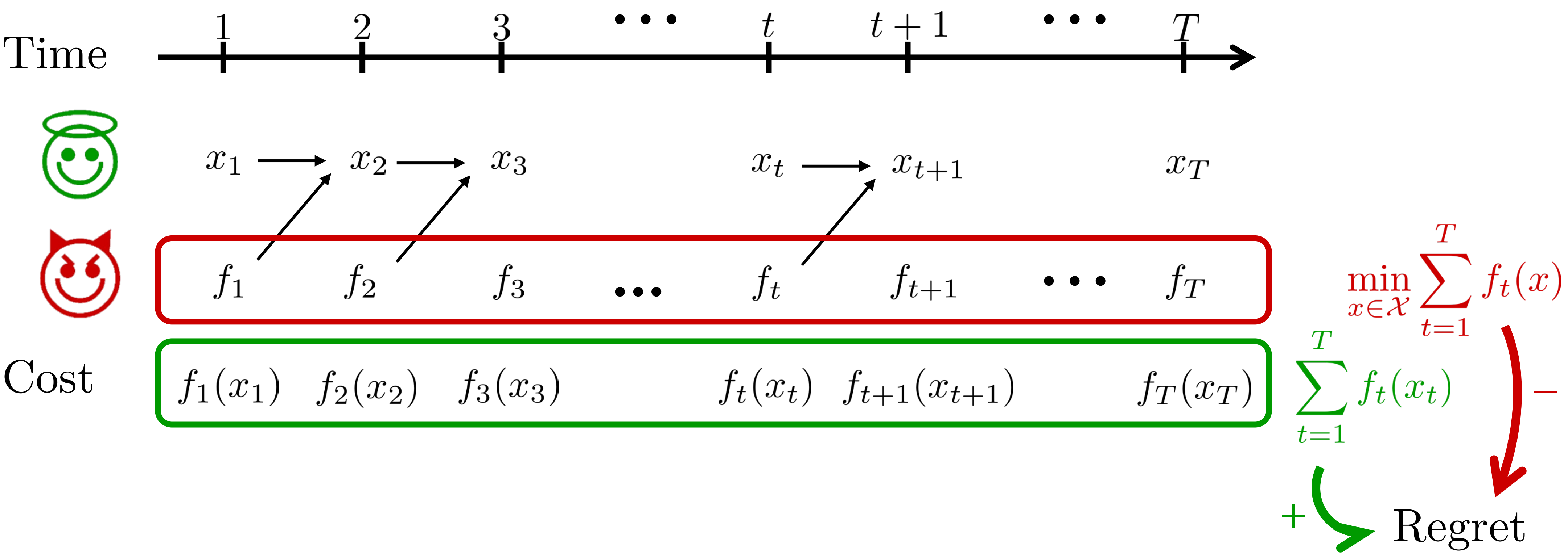

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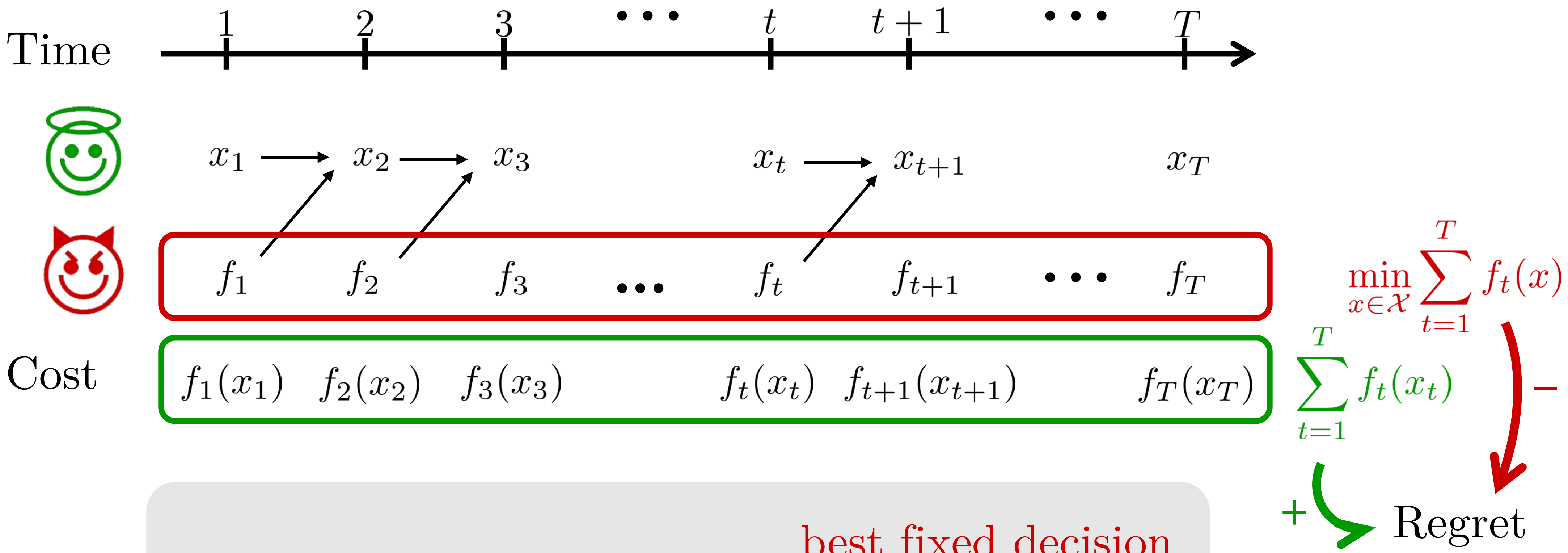
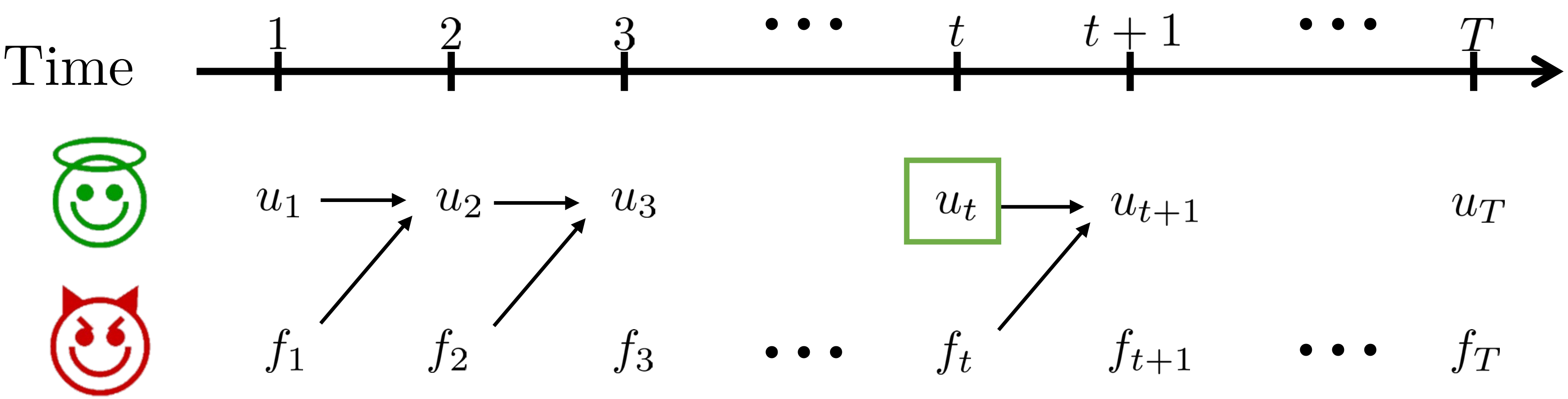


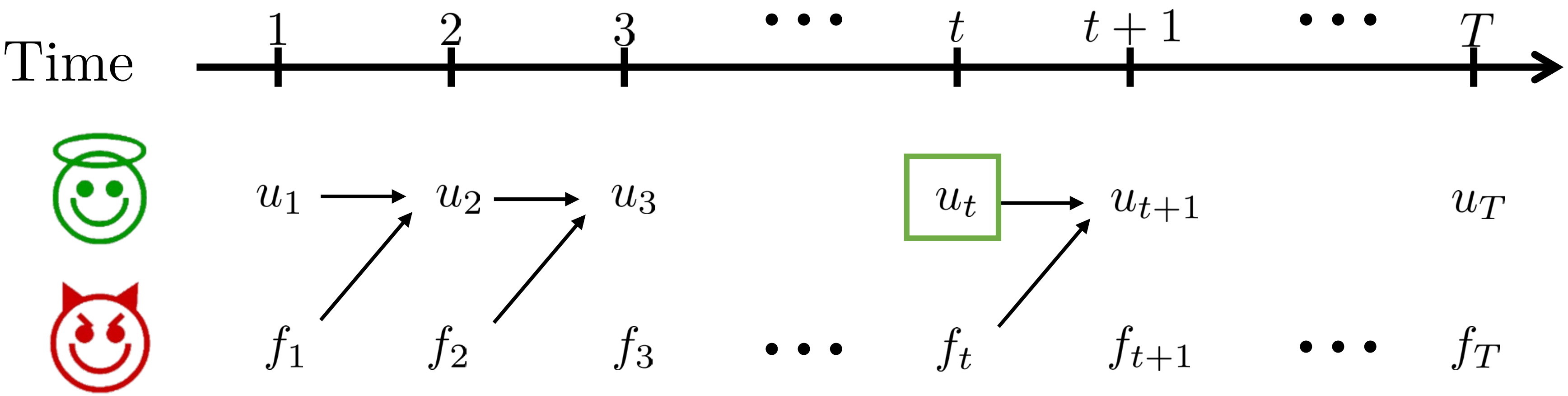
Regret $=$ online decisions $-$ best fixed decision in hindsight

$$\sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} f_t(x)$$

Online Optimization for Control



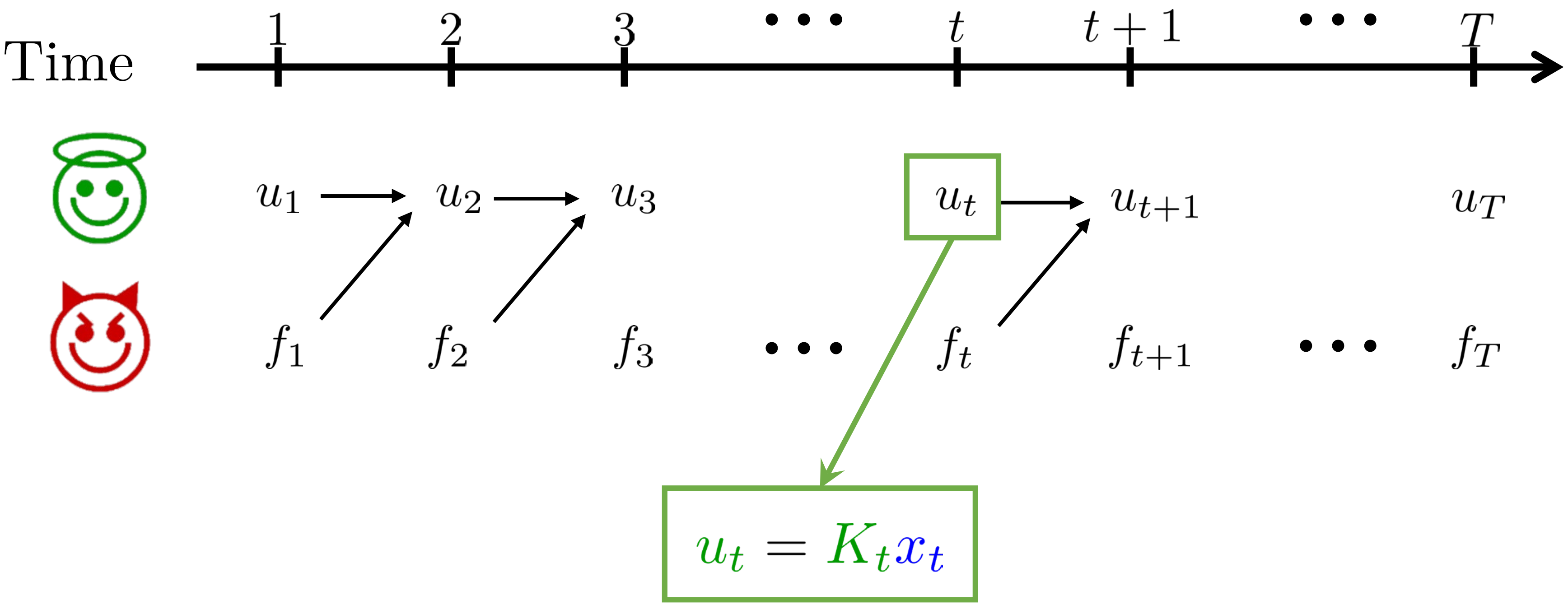
Online Optimization for Control



$$\text{Regret} = \sum_{t=1}^{T} f_t(u_t) - \min_{u \in \mathcal{U}} \sum_{t=1}^{T} f_t(u)$$

Optimal
open-loop!

Online Optimization for Control



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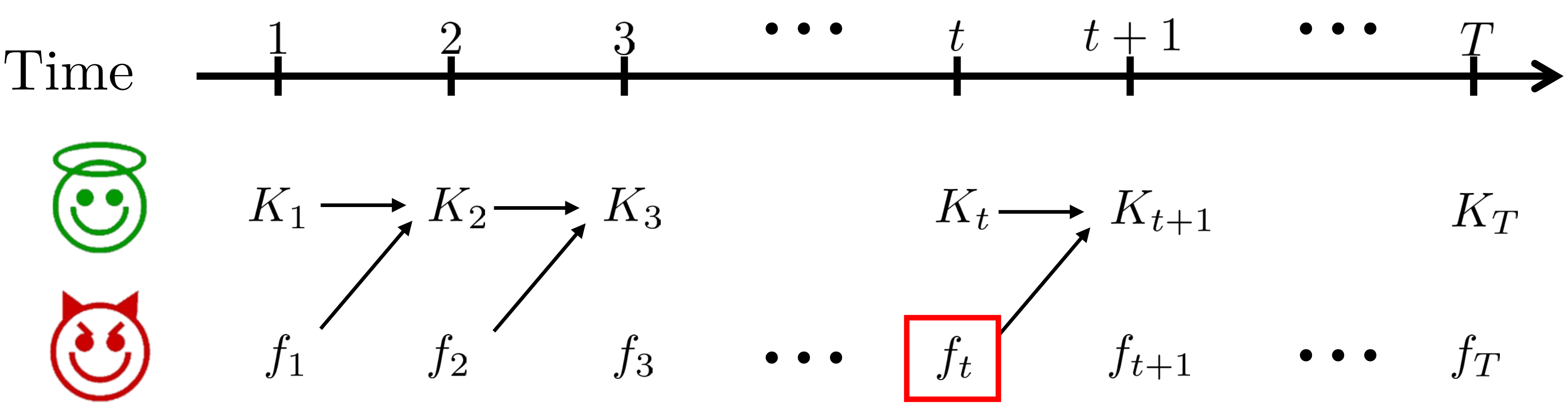
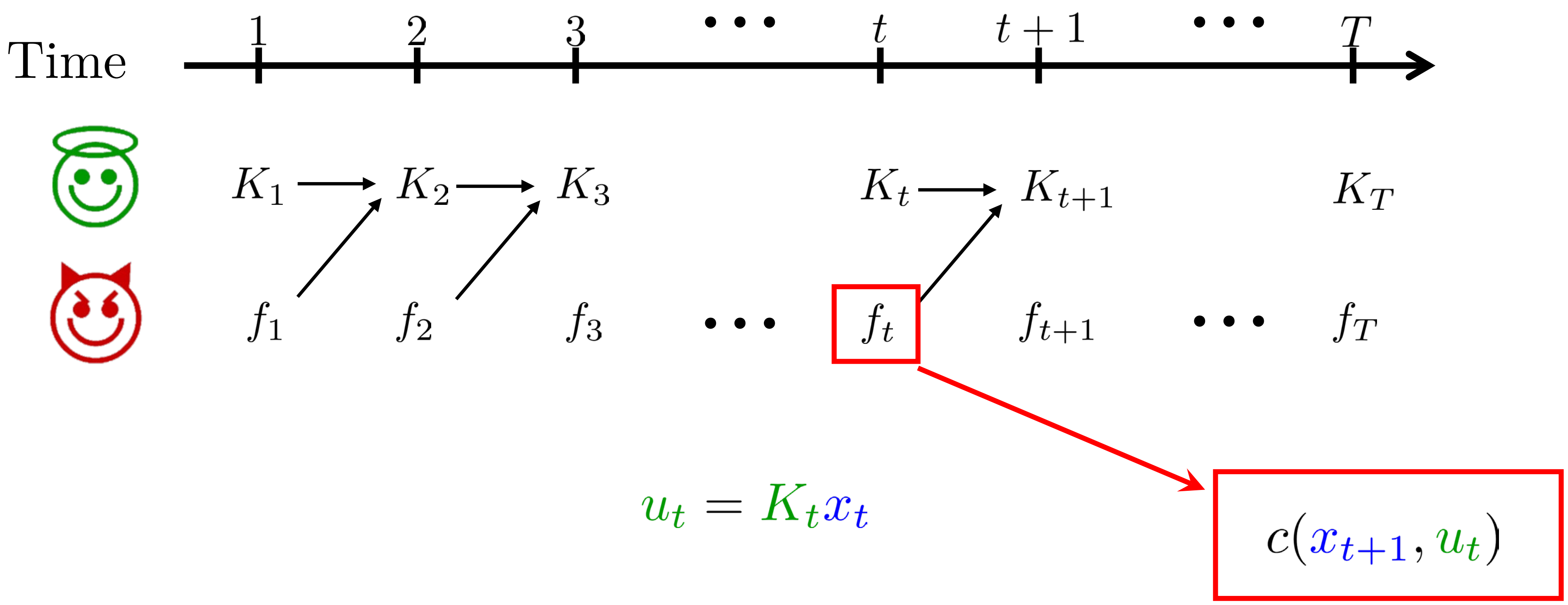


$$u_t = K_t x_t$$

Online Optimization for Control



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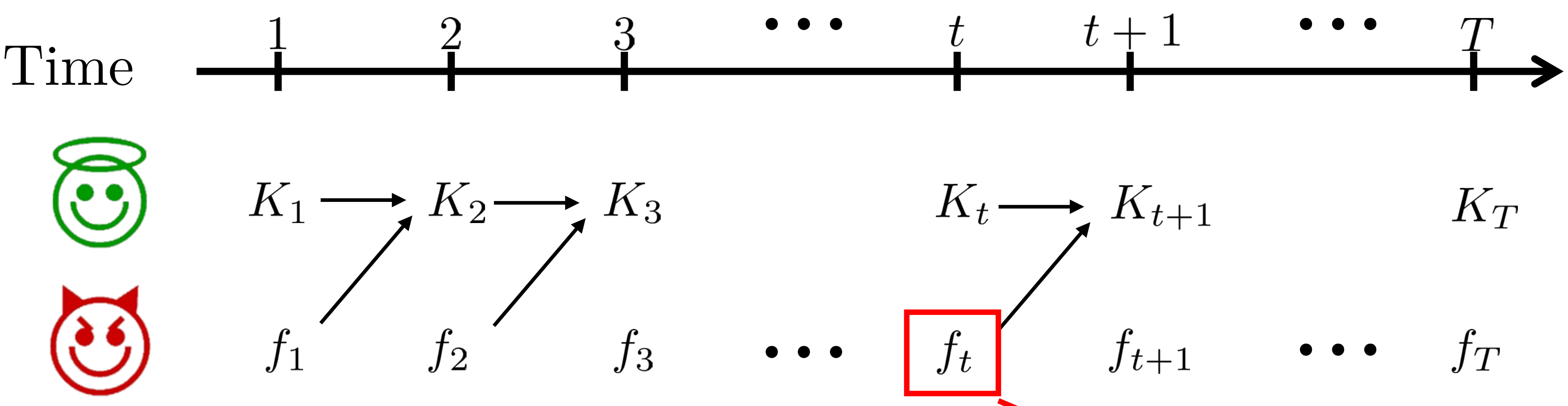
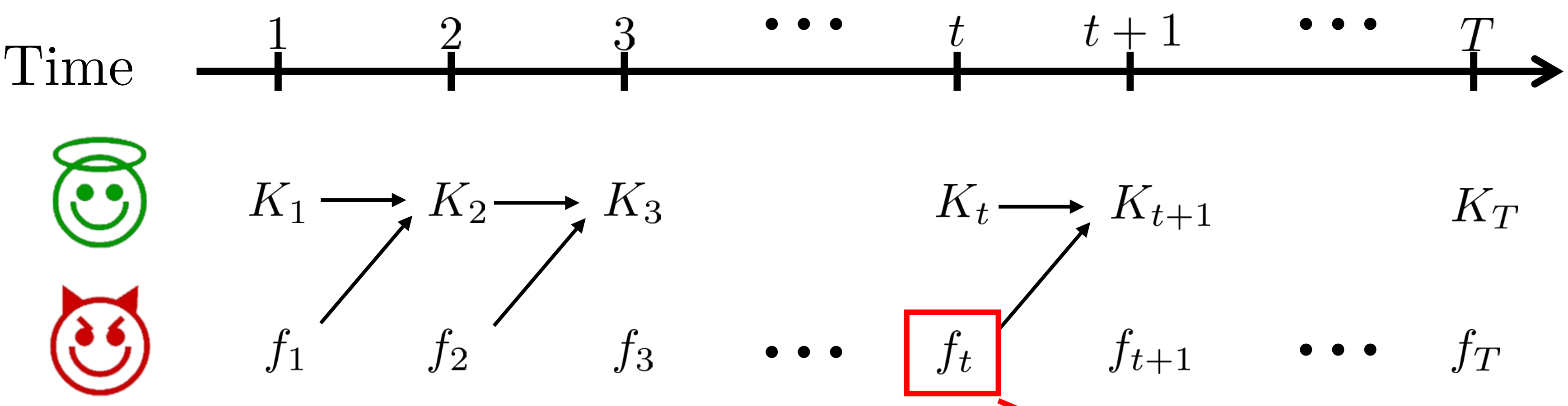


$$\boxed{c(x_{t+1}, u_t)}$$

$$
\begin{aligned}
u_t &= K_t x_t \\
x_{t+1} &= A x_t + B u_t + w_t \\
&= (A + B K_t) x_t + w_t
\end{aligned}
$$

Online Optimization for Control



$$\boxed{c(K_t \mid x_t, w_t)}$$

Adversarial disturbance!

$$
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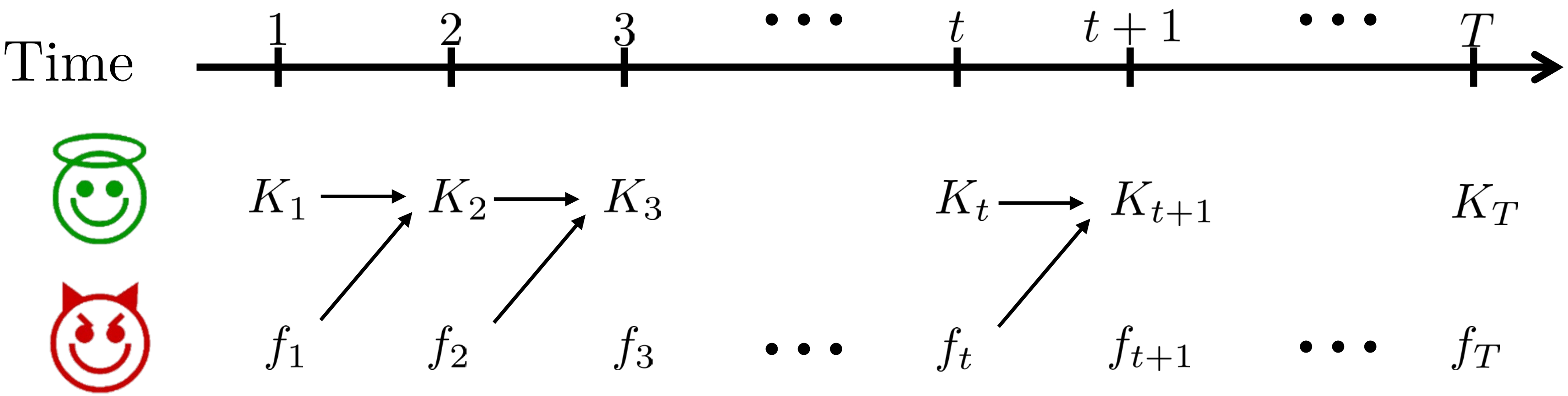
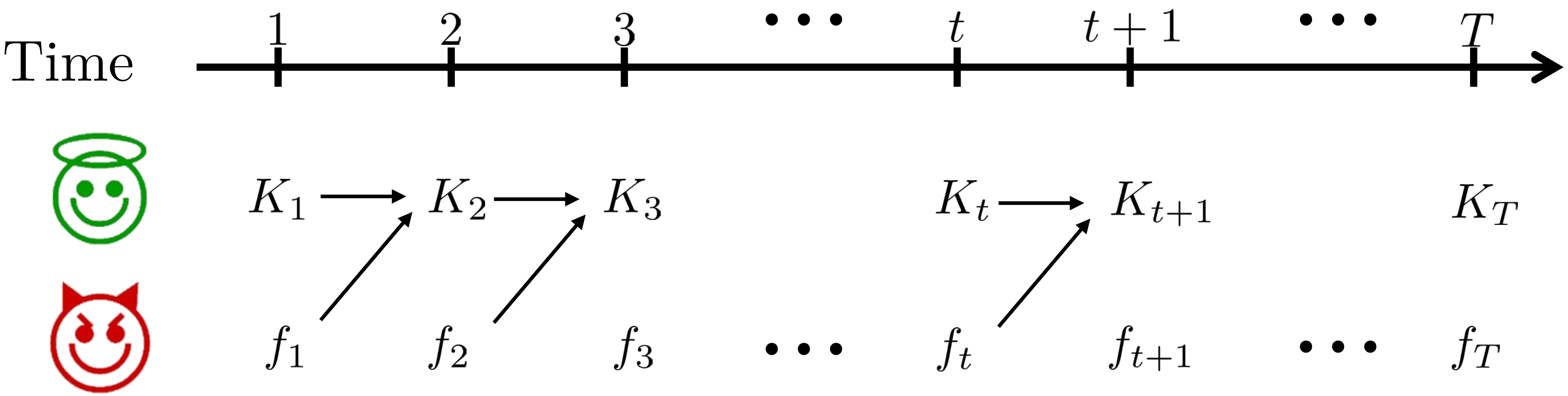
Online Optimization for Control



$$\text{Regret} = \sum_{t=1}^{T} c(K_t \mid x_t, w_t) - \min_{K} \sum_{t=1}^{T} c(K \mid x_t, w_t)$$

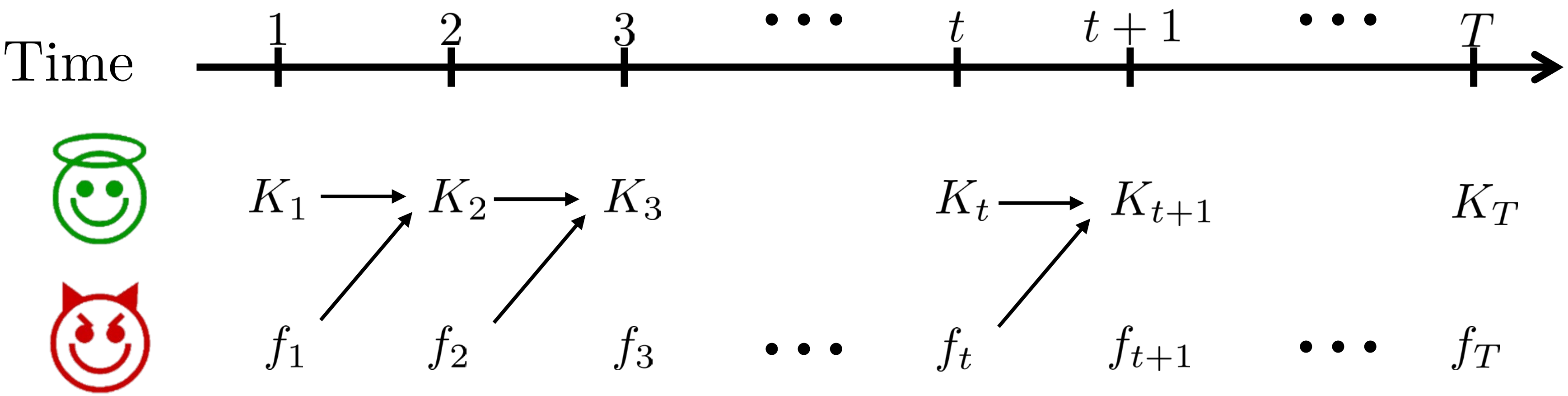
Online Optimization for Control



same states!

$$\text{Regret} = \sum_{t=1}^{T} c(K_t \mid x_t, w_t) - \min_{K} \sum_{t=1}^{T} c(K \mid x_t, w_t)$$

Online Optimization for Control



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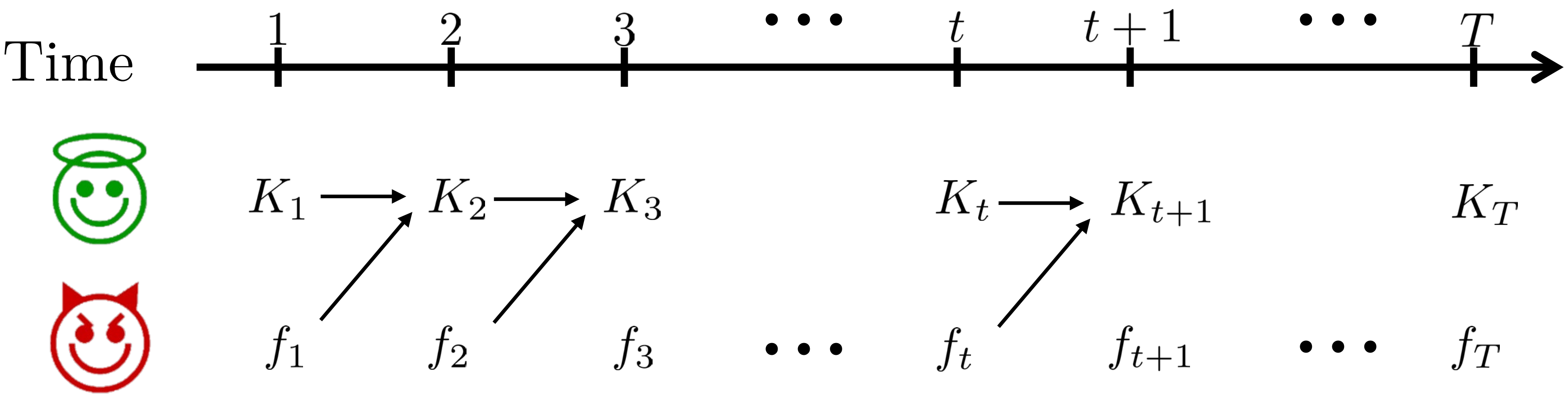
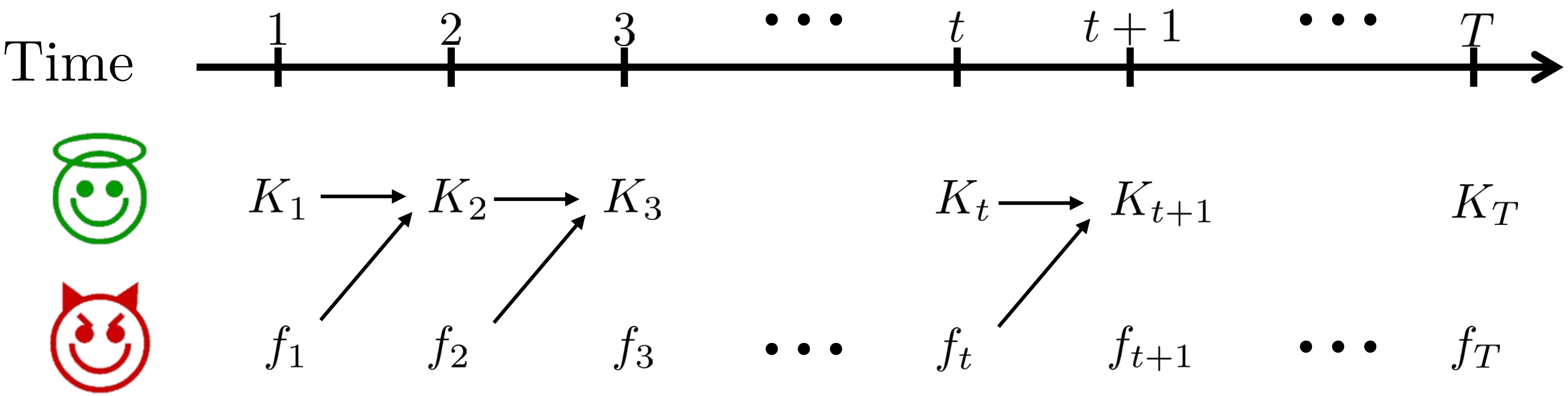
meaningless!

Online Optimization for Control



$$\text{Regret} = \sum_{t=1}^{T} c(K_t \mid x_t(K_{1:t-1}), w_t) - \min_{K} \sum_{t=1}^{T} c(K \mid x_t(K), w_t)$$

Online Optimization for Control



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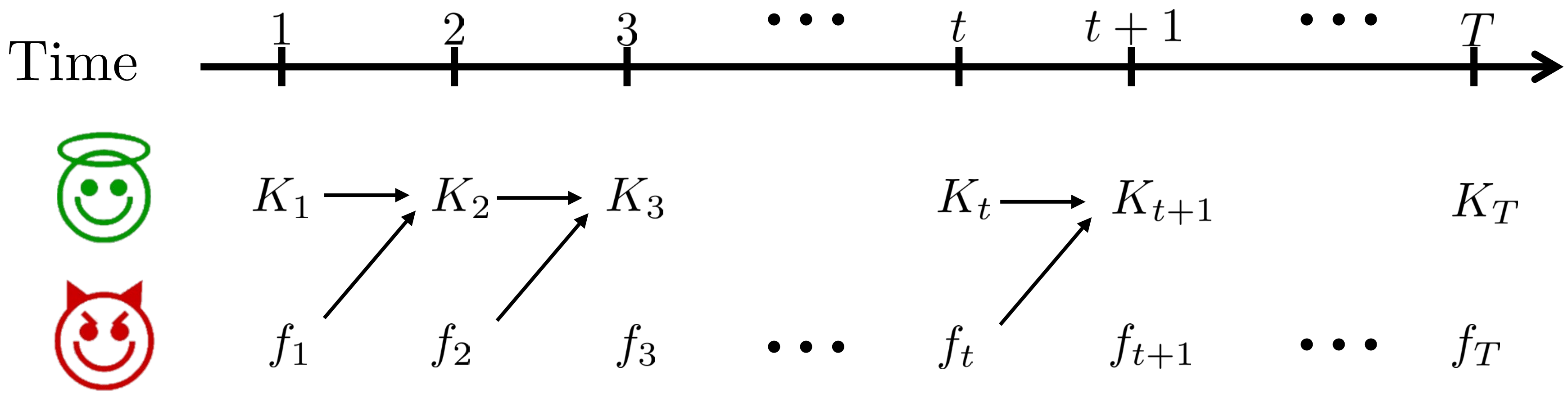
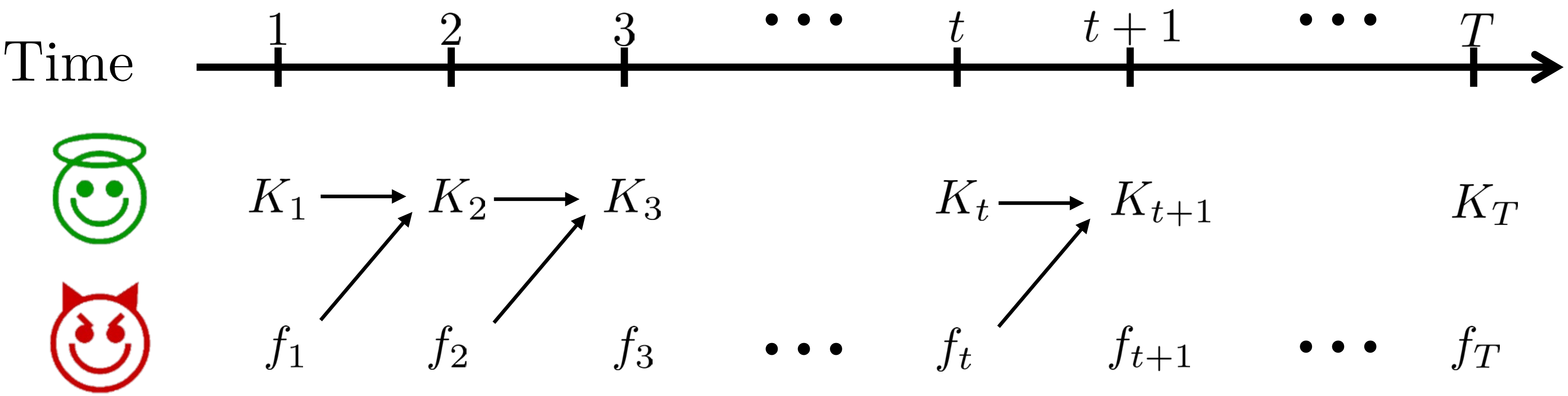
state-feedback
non-convex cost

Online Optimization for Control



$u_t = K_t x_t$

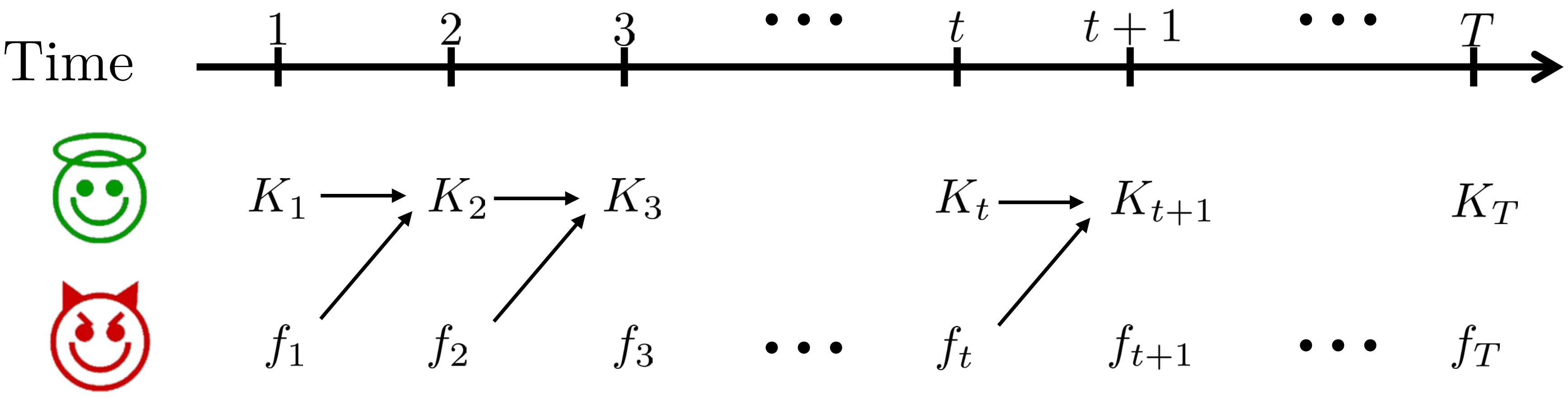
Online Optimization for Control



$$u_t = K_t x_t(K_{1:t-1})$$

state feedback

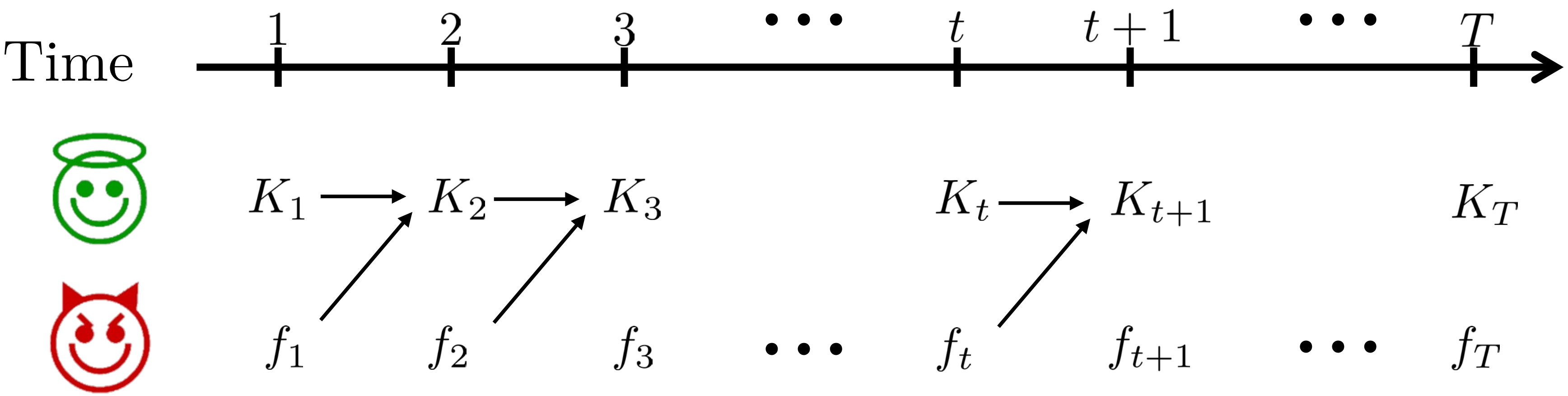
Online Optimization for Control



$$u_t = K_t x_t(K_{1:t-1}) = \sum_{i=1}^{t-1} \mathrm{poly}_i(K_1, \ldots, K_t) w_i$$

state feedback

Online Optimization for Control

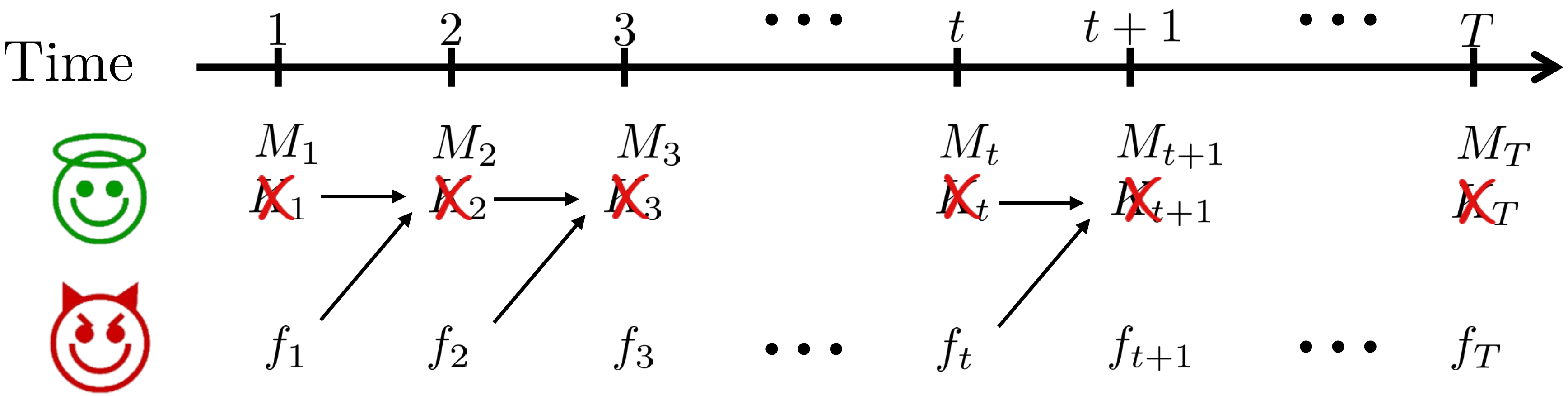


$$\color{green}{u_t} = \color{green}{K_t}\color{blue}{x_t(}\color{green}{K_{1:t-1}}\color{blue}{)} = \sum_{i=1}^{t-1} \mathrm{poly}_i(\color{green}{K_1}, \ldots, \color{green}{K_t})\color{red}{w_i} \xrightarrow{①} \color{green}{u_t} = \sum_{i=1}^{t-1} \color{green}{M_t^{[i]}}\color{red}{w_i}$$

state feedback

disturbance feedback

Online Optimization for Control

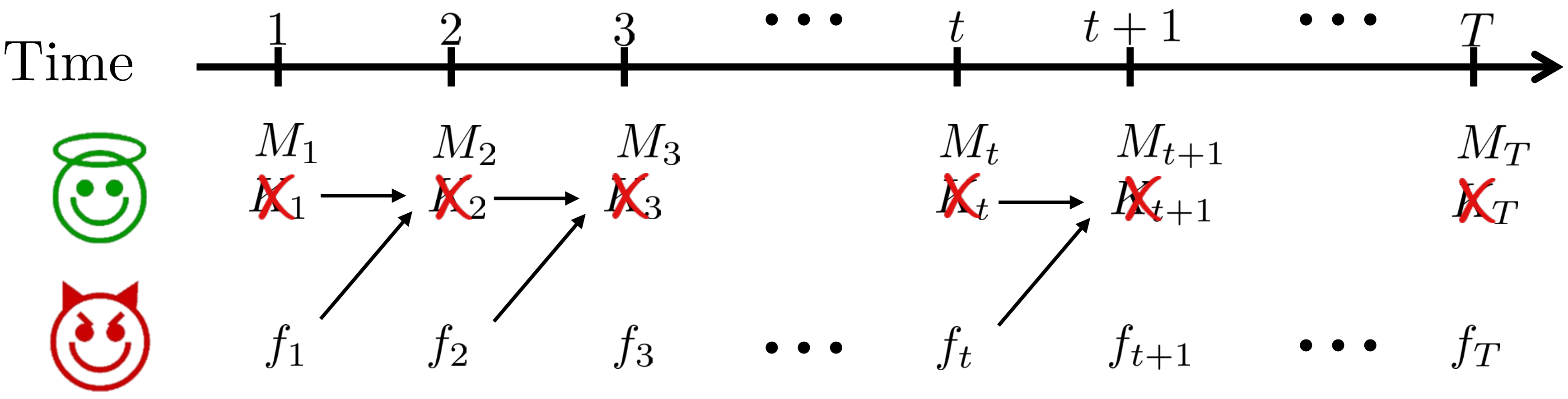


$$u_t = K_t x_t(K_{1:t-1}) = \sum_{i=1}^{t-1} \mathrm{poly}_i(K_1, \ldots, K_t) w_i \xrightarrow{1} u_t = \sum_{i=1}^{t-1} M_t^{[i]} w_i \qquad M_t := \left(M_t^{[0]}, \ldots, M_t^{[t-1]}\right)$$

state feedback

disturbance feedback

Online Optimization for Control



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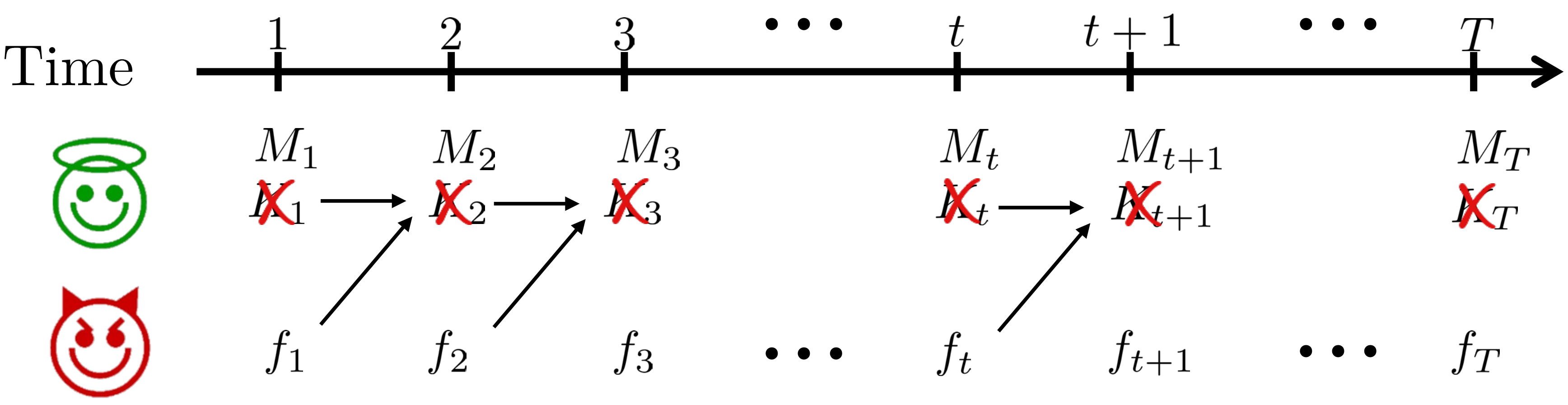
$$c(x_{t+1}, u_t) = c(K_{1:t} \mid w_1, \ldots, w_t)$$

Online Optimization for Control

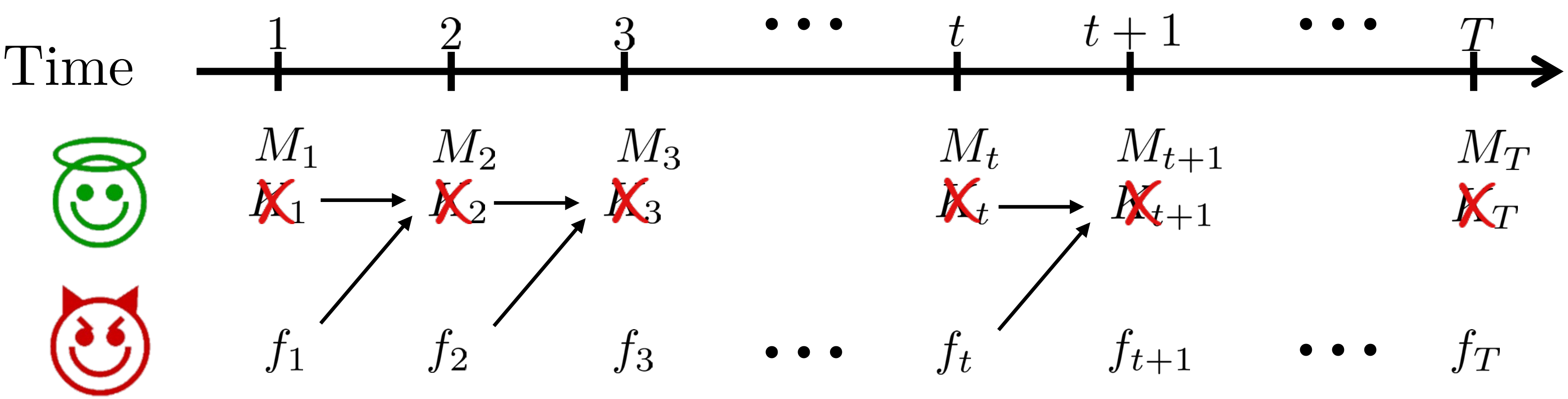


$$u_t = K_t x_t(K_{1:t-1}) = \sum_{i=1}^{t-1} \mathrm{poly}_i(K_1, \dots, K_t) w_i \xrightarrow{1} u_t = \sum_{i=1}^{t-1} M_t^{[i]} w_i \qquad M_t := \left(M_t^{[0]}, \dots, M_t^{[t-1]}\right)$$

$$c(x_{t+1}, u_t) = c(K_{1:t} \mid w_1, \dots, w_t) \xrightarrow{1} c(M_{1:t} \mid w_1, \dots, w_t)$$

convexification by lifting!

Online Optimization for Control



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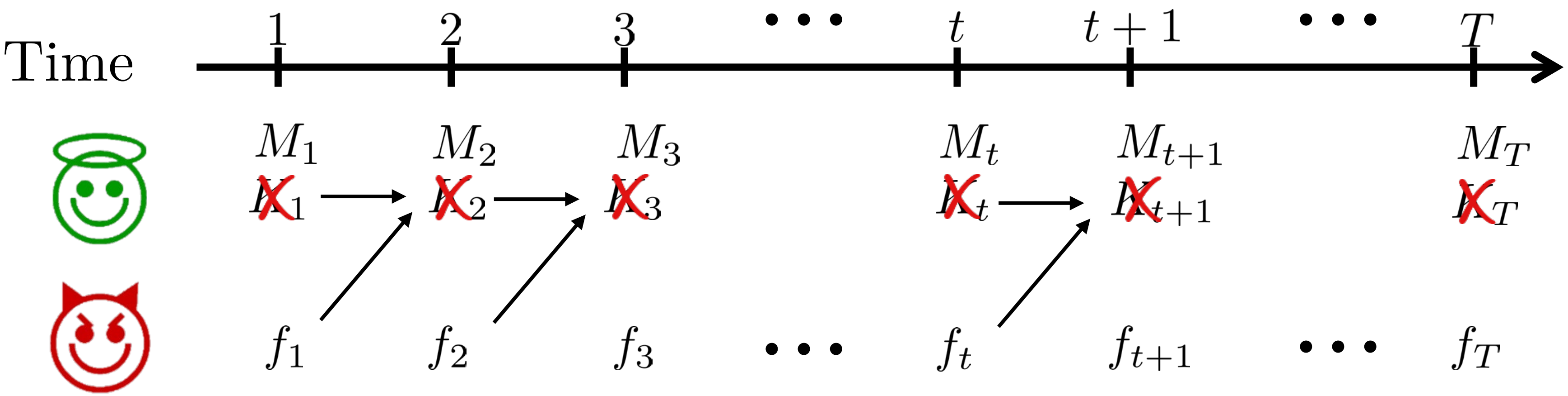
$$c(x_{t+1}, u_t) = c(K_{1:t} \mid w_1,\dots,w_t) \xrightarrow{1} c(M_{\cancel{1}:t} \mid w_{\cancel{1}},\dots,w_t)$$ convexification by lifting!

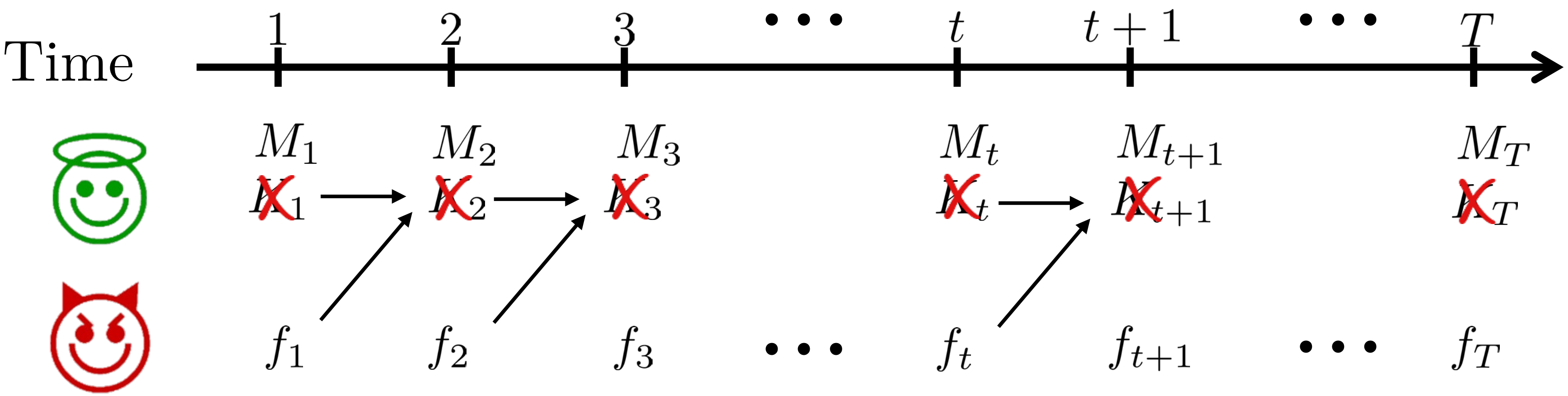
$$\xrightarrow{2} c(M_{t-h:t} \mid w_{t-h},\dots,w_t)$$ stability assumption

Online Optimization for Control



$$u_t \stackrel{1}{=} \sum_{\underset{2}{i=t-h}}^{t-1} M_t^{[i]} w_i \qquad M_t \stackrel{2}{:=} \left(M_t^{[t-h]}, \dots, M_t^{[t-1]}\right)$$

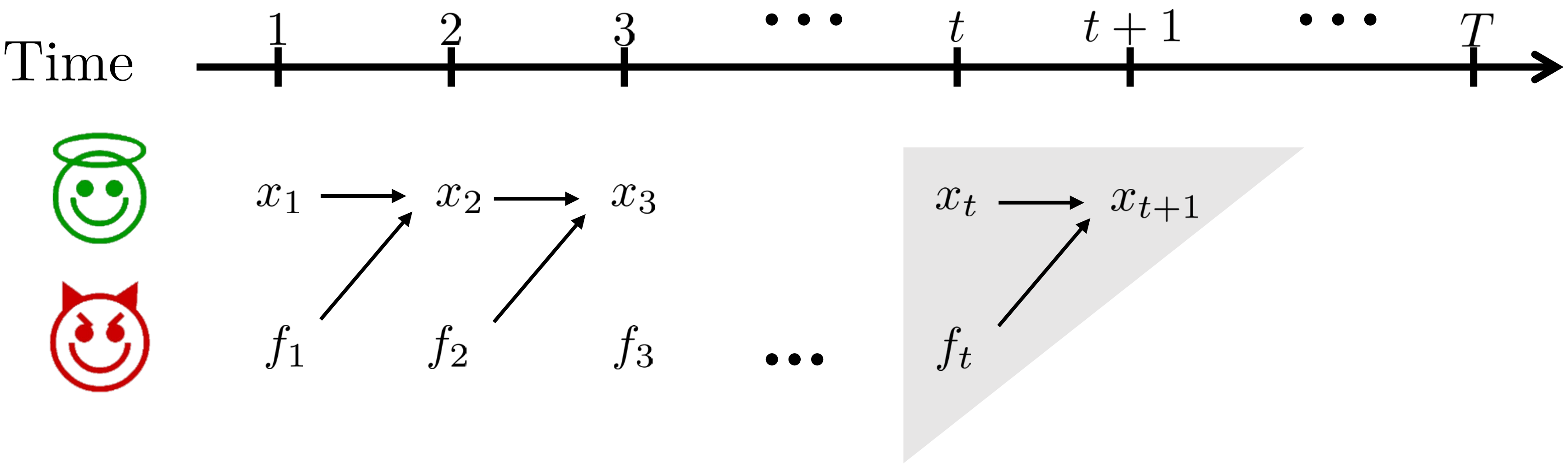
Online Optimization for Control



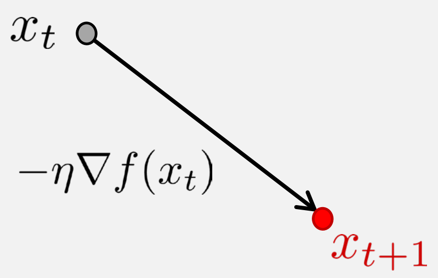
$$u_t = \sum_{i=t-h}^{t-1} M_t^{[i]} w_i \qquad M_t := \left(M_t^{[t-h]}, \ldots, M_t^{[t-1]}\right)$$

$$\text{Regret} = \sum_{t=1}^{T} c(M_{t-h}, \ldots, M_t \mid w_{1:t}) - \min_{M} \sum_{t=1}^{T} c(M, \ldots, M \mid w_{1:t})$$

Online Optimization: Algorithms



Standard Optimization Algorithms

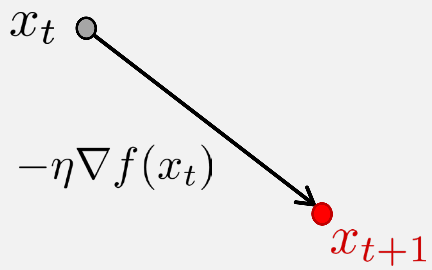


Explicit Gradient Descent

$$x_{t+1} = x_t - \eta\nabla f(x_t)$$

Standard Optimization Algorithms

Converges if
$\eta \leq \frac{1}{\beta}$



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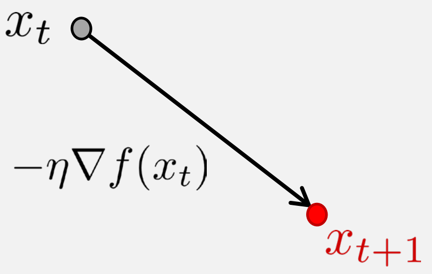

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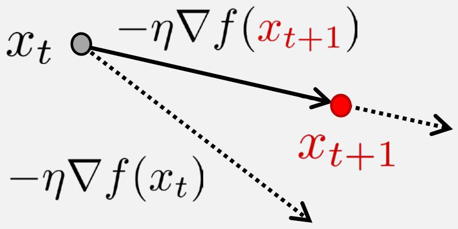

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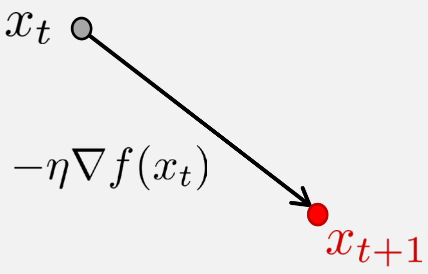
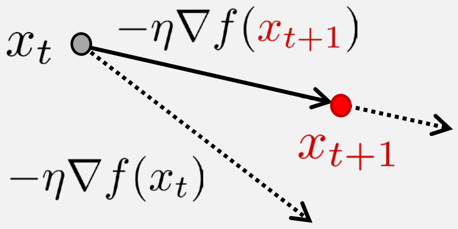


Explicit Gradient Descent

$$x_{t+1} = x_t - \eta \nabla f(x_t)$$

Converges
$\forall \eta > 0$



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Standard Optimization Algorithms

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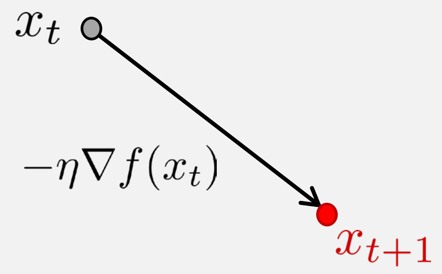

Explicit Gradient Descent

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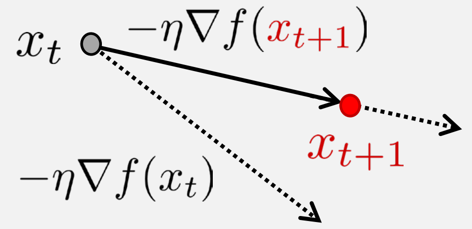

Implicit Gradient Descent

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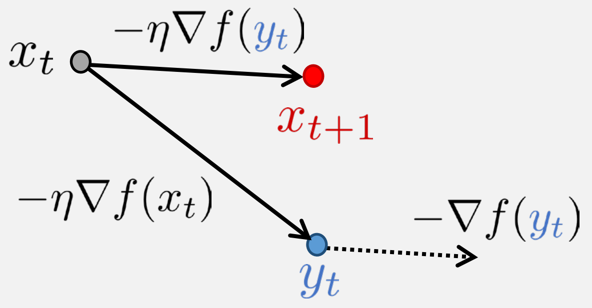
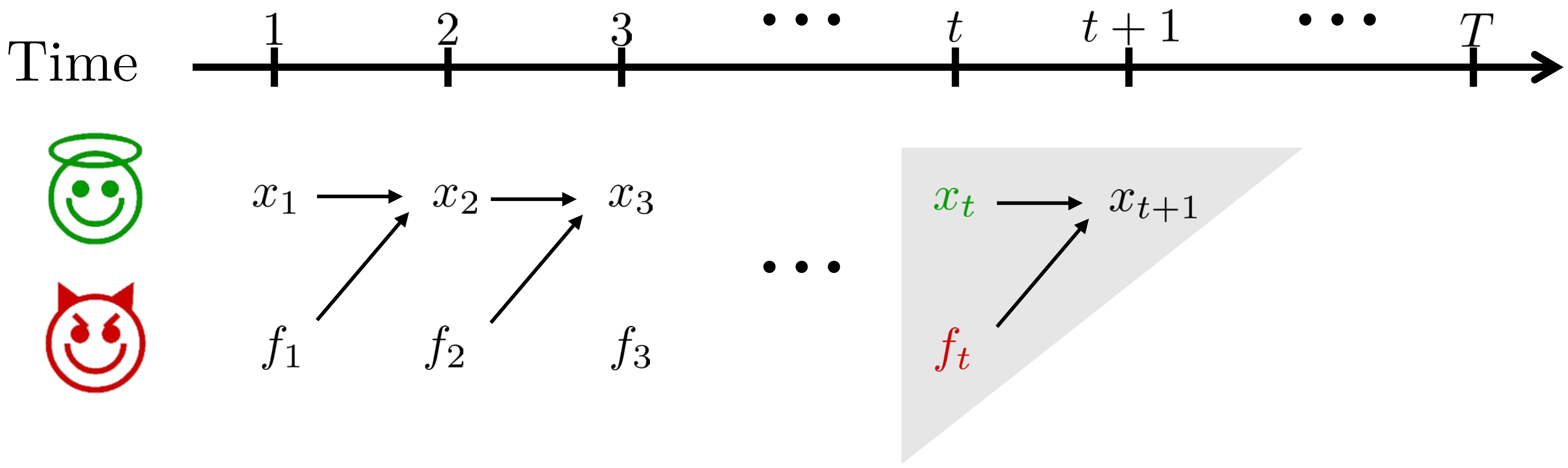


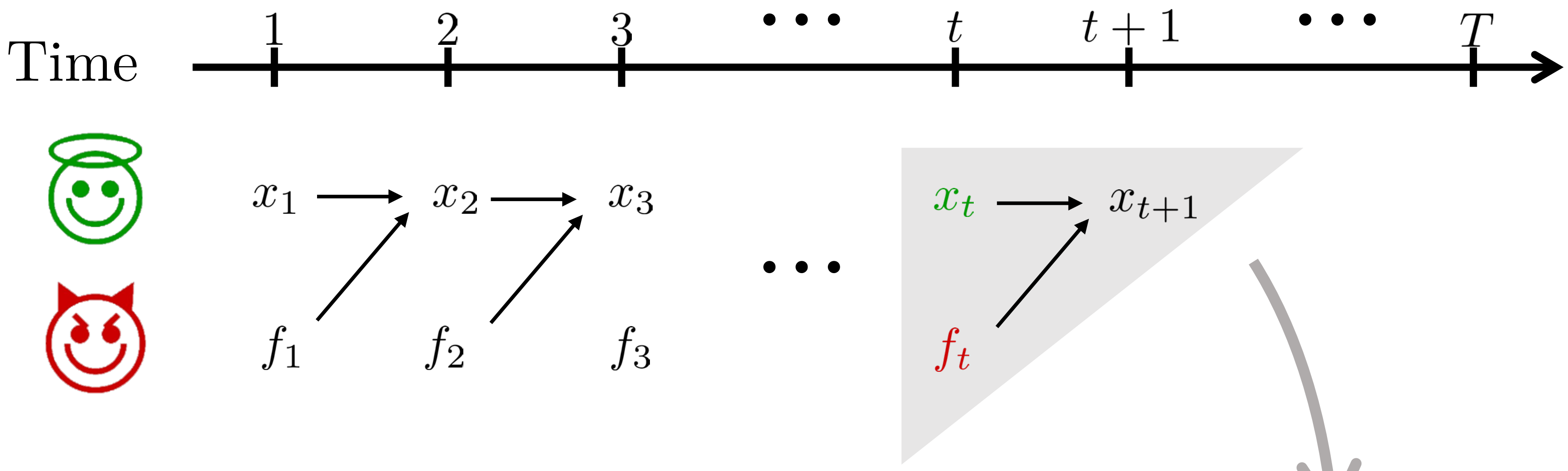
Extra Gradient Descent

$$y_t = x_t - \eta \nabla f(x_t)$$
$$x_{t+1} = x_t - \eta \nabla f(y_t)$$

Online Optimization: Algorithms



Online Optimization: Algorithms

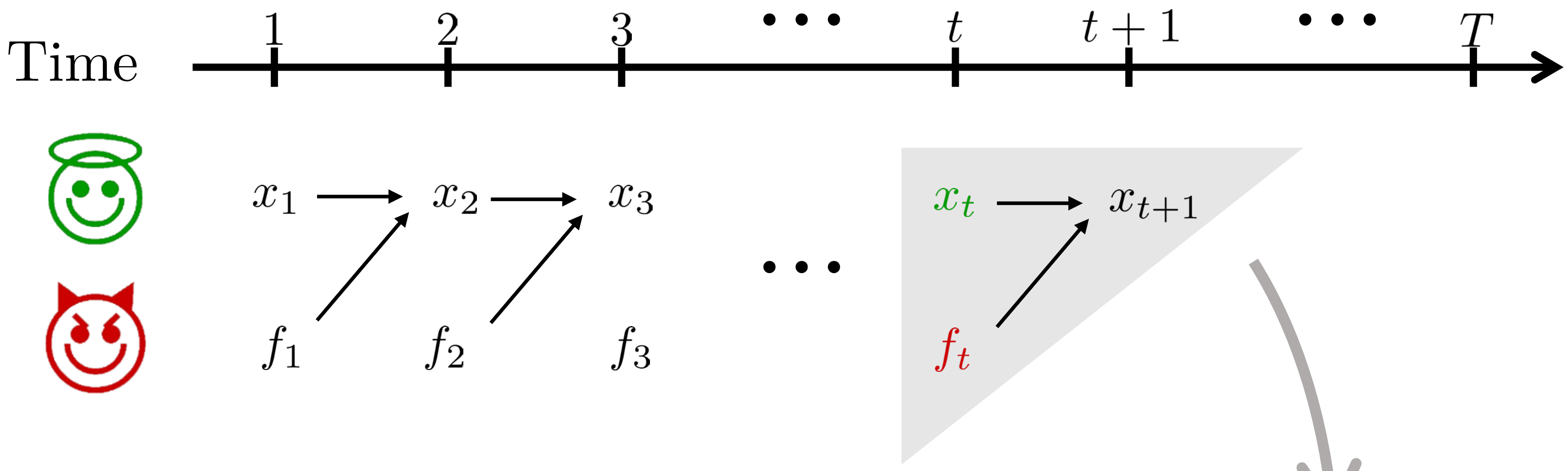


Online Gradient Descent [1]

$$x_{t+1} = \underbrace{\boxed{\Pi_{\mathcal{X}}}}_{\text{Euclidean projection onto } \mathcal{X}}\big(x_t - \eta_t \nabla f_t(x_t)\big)$$

[1] Zinkevich (2003)

Online Optimization: Algorithms

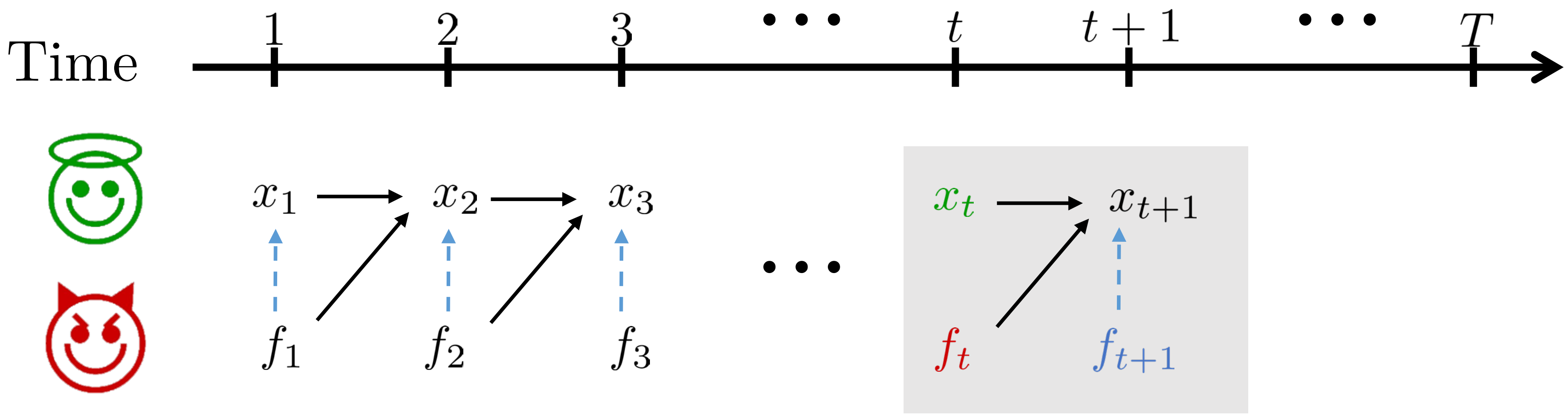


Online Gradient Descent [1]

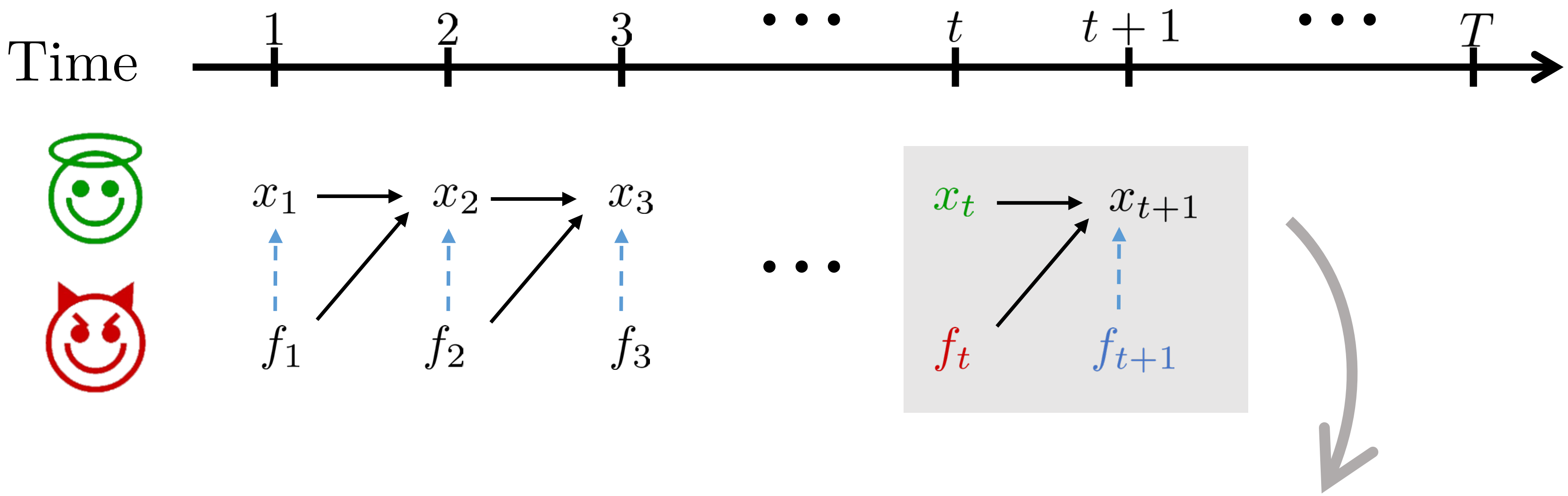
$$x_{t+1} = \underbrace{\boxed{\Pi_{\mathcal{X}}}}_{\text{Euclidean projection onto } \mathcal{X}}\big(x_t - \eta_t \nabla f_t(x_t)\big) \xrightarrow[\eta_t \,\propto\, 1/\sqrt{t}]{f_t \text{ convex}} \text{Regret} = O(\sqrt{T})$$

[1] Zinkevich (2003)

Online Optimization with Prediction



Online Optimization with Prediction



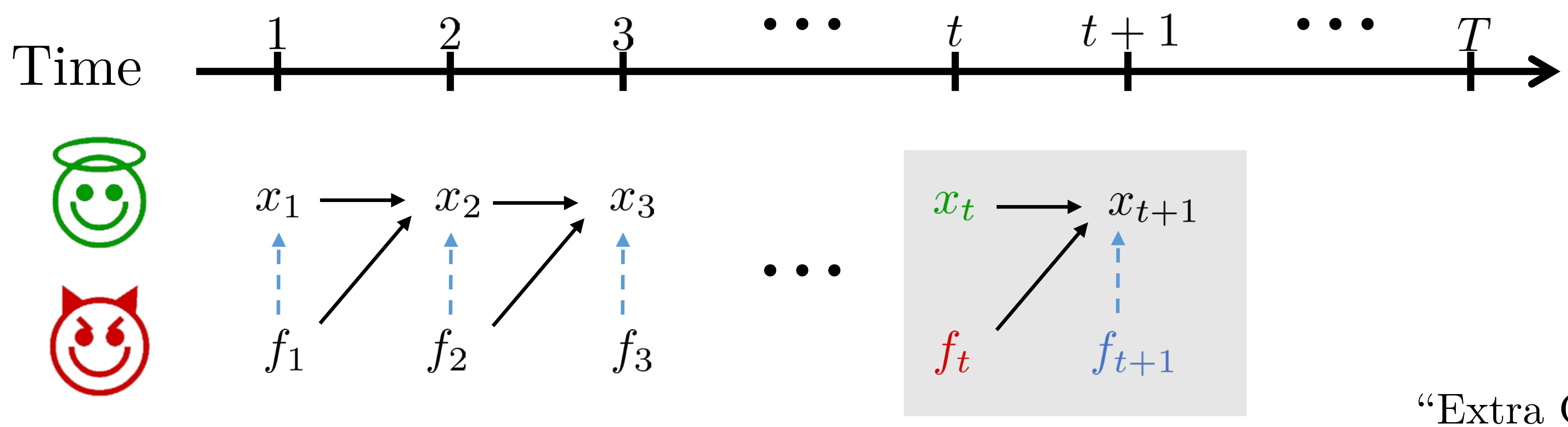
[1]

$$y_t = \Pi_{\mathcal{X}}\big(y_{t-1} - \eta_t \nabla f_t(x_t)\big)$$

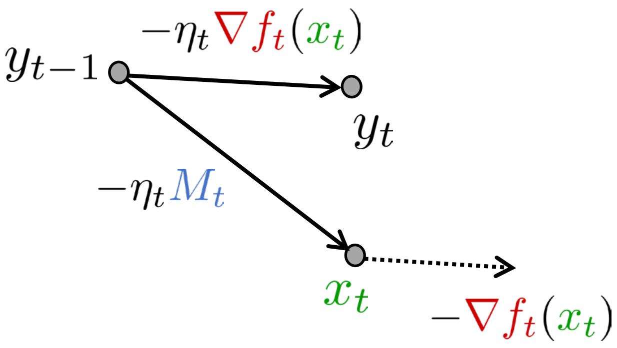
$$x_{t+1} = \Pi_{\mathcal{X}}\big(y_t - \eta_{t+1} M_{t+1}\big)$$

[1] Rakhlin and Sridharan (2013)

Online Optimization with Prediction



"Extra Gradient Descent"



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[1] Rakhlin and Sridharan (2013)

Online Optimization with Prediction

[1]

$$
\begin{aligned}
y_t &= \Pi_{\mathcal{X}}\left(y_{t-1} - \eta_t \nabla f_t(x_t)\right) \\
x_{t+1} &= \Pi_{\mathcal{X}}\left(y_t - \eta_{t+1} M_{t+1}\right)
\end{aligned}
$$

[1] Rakhlin and Sridharan (2013)

Online Optimization with Prediction

[1]

$$y_t = \Pi_{\mathcal{X}}\left(y_{t-1} - \eta_t \nabla f_t(x_t)\right)$$
$$x_{t+1} = \Pi_{\mathcal{X}}\left(y_t - \eta_{t+1} M_{t+1}\right)$$

- $\eta_t = O\left(1/\sqrt{\sum_{i=1}^{t-1} \|\nabla f_i(x_i) - M_i\|^2}\right)$

- Regret $= O\left(1 + \sqrt{\sum_{t=1}^{T} \|\nabla f_t(x_t) - M_t\|^2}\right)$

[1] Rakhlin and Sridharan (2013)

Online Optimization with Prediction

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Constant regret
$\Rightarrow M_t = \nabla f_t(x_t)$

[1] Rakhlin and Sridharan (2013)

Online Optimization with Prediction

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- $\eta_t = O\left(1/\sqrt{\sum_{i=1}^{t-1} \|\nabla f_i(x_i) - M_i\|^2}\right)$

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Constant regret
$\Rightarrow M_t = \nabla f_t(x_t)$

Implicit Gradient Descent

$$\boxed{x_{t+1} = \Pi_{\mathcal{X}}\left(y_t - \eta_{t+1} \nabla f_{t+1}(x_{t+1})\right)}$$

[1] Rakhlin and Sridharan (2013)

Online Optimization with Prediction

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$$y_t = \Pi_{\mathcal{X}}\left(y_{t-1} - \eta_t \nabla f_t(x_t)\right)$$
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- $\eta_t = O\left(1/\sqrt{\sum_{i=1}^{t-1} \|\nabla \cancel{f_i(x_i)} - M_i\|^2 + 4\beta^2}\right)$

  $\nabla f_i(y_{i-1})$

- Regret $= O\left(1 + \sqrt{\sum_{t=1}^{T} \|\nabla \cancel{f(x_t)} - M_t\|^2}\right)$

  $\nabla f_t(y_{t-1})$

Constant regret
$\Rightarrow M_t = \nabla \cancel{f(x_t)}$

$\nabla f_t(y_{t-1})$

Explicit Gradient Descent

$$\boxed{x_{t+1} = \Pi_{\mathcal{X}}\left(y_t - \eta_{t+1} \nabla \cancel{f_{t+1}(x_{t+1})}\right)}$$

$\nabla f_{t+1}(y_t)$

[1] Rakhlin and Sridharan (2013)
[2] Zattoni Scroccaro, Kolarijani, and PME (2022)

Online Optimization: Convex Costs

[3]
$$y_t = \Pi_{\mathcal{X}}\left(y_{t-1} - \eta_t \nabla f_t(x_t)\right)$$
$$x_{t+1} = \Pi_{\mathcal{X}}\left(y_t - \eta_{t+1} \widetilde{\nabla f}_{t+1}(y_t)\right)$$

Gradient Prediction Error

$$D_t := \sum_{i=1}^{t} \|\nabla f_i(y_{i-1}) - \widetilde{\nabla f}_i(y_{i-1})\|^2$$

[1] Zinkevich (2003)
[2] Ho-Nguyen and Kılınç-Karzan (2019)
[3] Zattoni Scroccaro, Kolarijani, and PME (2022)

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$$D_t := \sum_{i=1}^{t} \|\nabla f_i(y_{i-1}) - \widetilde{\nabla f}_i(y_{i-1})\|^2$$

	Worst-case [1]	**General prediction [3]**	**Perfect prediction [2]**
Prediction		$\widetilde{\nabla f}_{t+1}$	
η_t		$1/\sqrt{D_{t-1} + 4\beta^2}$	
Regret		$O(1 + \sqrt{D_T})$	

[1] Zinkevich (2003)
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Gradient Prediction Error
$$D_t := \sum_{i=1}^{t} \|\nabla f_i(y_{i-1}) - \widetilde{\nabla f}_i(y_{i-1})\|^2$$

	Worst-case [1]	General prediction [3]	Perfect prediction [2]
Prediction	$\widetilde{\nabla f}_{t+1} = 0$	$\widetilde{\nabla f}_{t+1}$	
η_t	$O(1/\sqrt{t})$	$1/\sqrt{D_{t-1} + 4\beta^2}$	
Regret	$O(\sqrt{T})$	$O(1 + \sqrt{D_T})$	

[1] Zinkevich (2003)
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Gradient Prediction Error $\quad D_t := \sum_{i=1}^{t} \|\nabla f_i(y_{i-1}) - \widetilde{\nabla f}_i(y_{i-1})\|^2$

	Worst-case [1]	**General prediction [3]**	**Perfect prediction [2]**
Prediction	$\widetilde{\nabla f}_{t+1} = 0$	$\widetilde{\nabla f}_{t+1}$	$\widetilde{\nabla f}_{t+1} = \nabla f_{t+1}$
η_t	$O(1/\sqrt{t})$	$1/\sqrt{D_{t-1} + 4\beta^2}$	$1/2\beta$
Regret	$O(\sqrt{T})$	$O(1 + \sqrt{D_T})$	$O(1)$

[1] Zinkevich (2003)
[2] Ho-Nguyen and Kılınç-Karzan (2019)
[3] Zattoni Scroccaro, Kolarijani, and PME (2022)

Online Optimization: Strongly Convex Costs

[3]
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$$x_{t+1} = \Pi_{\mathcal{X}}\left(y_t - \eta_{t+1} \widetilde{\nabla f}_{t+1}(y_t)\right)$$

Gradient Prediction Error $\quad D_t := \sum_{i=1}^{t} \|\nabla f_i(y_{i-1}) - \widetilde{\nabla f}_i(y_{i-1})\|^2$

	Worst-case [1]	**General prediction [3]**	**Perfect prediction [2]**
Prediction	$\widetilde{\nabla f}_{t+1} = 0$	$\widetilde{\nabla f}_{t+1}$	$\widetilde{\nabla f}_{t+1} = \nabla f_{t+1}$
η_t	$O(1/t)$	$O(1/(D_{t-1} + 2\beta))$	$1/2\beta$
Regret	$O(\log(T))$	$O(1 + \log(1 + D_T))$	$O(1)$

Strongly convex costs

[1] Zinkevich (2003)
[2] Ho-Nguyen and Kılınç-Karzan (2019)
[3] Zattoni Scroccaro, Kolarijani, and PME (2022)

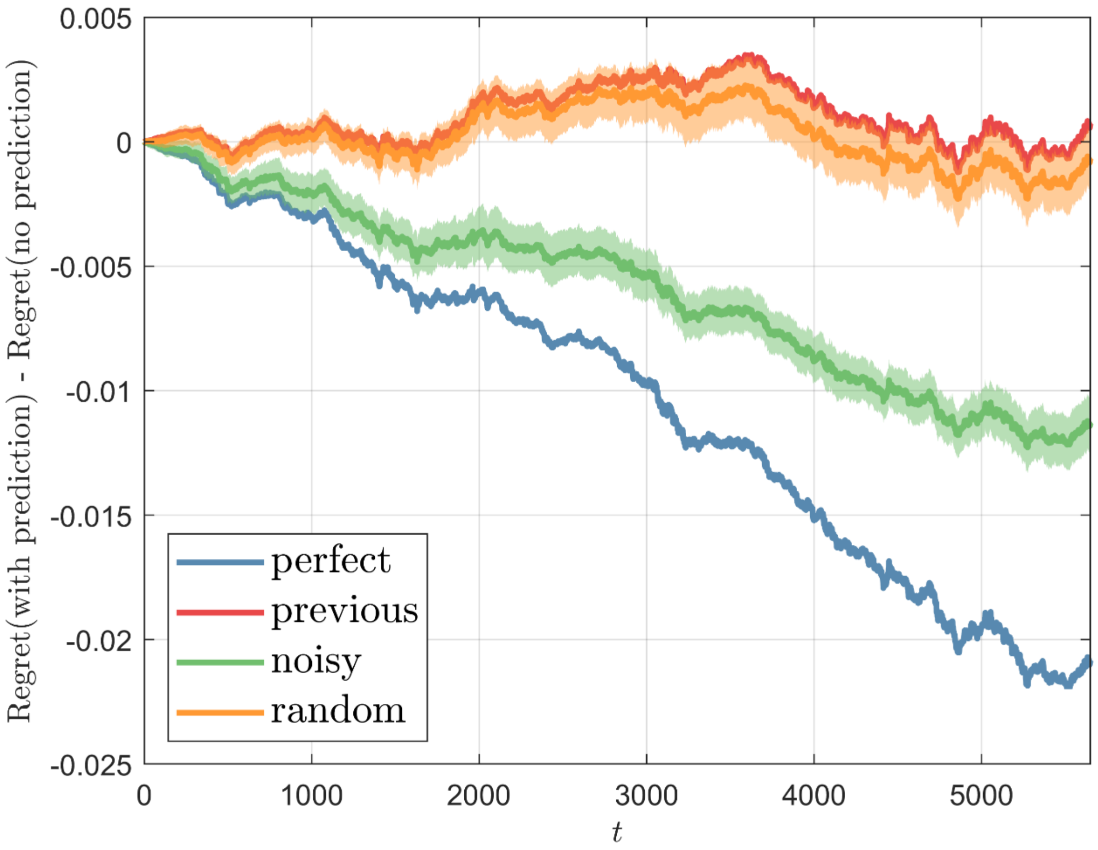
Numerical Example: Portfolio Selection

- Player is an investor. Nature is the stock market.
- The Player chooses $x_t \in \Delta_n$, a distributions of his/her wealth over n assets.
- Nature chooses the returns vector $r_t \in \mathbb{R}^n_+$
- At time t, the Player suffers the loss $f_t = -\log(\langle r_t, x_t \rangle)$
- $\widetilde{\nabla f}_t(y_{t-1}) = -\tilde{r}_t / \langle \tilde{r}_t, y_{t-1} \rangle$, where $\tilde{r}_t$ is the prediction of r_t

Numerical Example: Portfolio Selection

PREDICTION MODELS

- perfect: $\hat{r}_t = r_t$
- previous: $\hat{r}_t = r_{t-1}$
- noisy: $\hat{r}_t = r_t + w_t, \; w_t \sim N(0, I)$
- random: $\hat{r}_t \sim U(0, 2)$



NYSE dataset.
Zattoni Scroccaro, Kolarijani, and PME (2022)

Other Extensions

- Non-smooth cost functions

$$f_t(x) = s_t(x) + \underbrace{r_t(x)}_{\text{Non-smooth}}$$

$\Rightarrow$ Extra Gradient
Composite Mirror Descent

- Dynamic regrets

$$\sum_{t=1}^{T} f_t(x_t) - \underbrace{\min_{x \in \mathcal{X}}}_{\text{fixed decision}} \sum_{t=1}^{T} f_t(x) \quad \Longrightarrow \quad \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(\underbrace{u_t)}_{\text{reference trajectory}}$$

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[1] Zattoni Scroccaro, Kolarijani, and PME (2022)

References

- Zattoni Scroccaro, Kolarijani, and Mohajerin Esfahani, "Adaptive Composite Online Optimization: Predictions in Static and Dynamic Environments", *IEEE Transaction on Automatic Control (TAC)*, 2023.
- Mohajerin Esfahani, Sutter, and Lygeros "Performance Bounds for the Scenario Approach and an Extension to a Class of Non-convex Programs", *IEEE Transaction on Automatic Control (TAC)*, 2015
- Zinkevich, "Online convex programming and generalized infinitesimal gradient ascent", *International Conference on Machine Learning (ICML)*, 2003.
- Hazan, Agarwal, and Kale, "Logarithmic regret algorithms for online convex optimization", Machine Learning, 2007.
- Rakhlin and Sridharan, "Optimization, Learning, and Games with Predictable Sequences", *Conference on Neural Information Processing Systems (NeurIPS)*, 2013.
- Ho-Nguyen and Kılınç-Karzan, "Exploiting problem structure in optimization under uncertainty via online convex optimization", *Mathematical Programming (MP)*, 2019.