Tracking, disturbances and zero offset

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Outline

1. Setpoint Tracking
   - Deviation variables
2. Steady-state target problem
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Setpoint tracking

- In this section we show how to use the MPC regulator and MHE estimator to handle different kinds of control problems, including setpoint tracking and rejecting nonzero disturbances.
- It is a standard objective in applications to use a feedback controller to move the measured outputs of a system to a specified and constant setpoint. This problem is known as setpoint tracking.
- In nonlinear MPC theory we can consider the case in which the system is nonlinear and constrained, but here we consider linear model MPC in which y_{sp} is an arbitrary constant.

Deviation variables

- In the regulation problem we assumed that the goal was to take the state of the system to the origin. Such a regulator can be used to treat the setpoint tracking problem with a coordinate transformation.
- Denote the desired output setpoint as y_{sp}. Denote a steady state of the system model as (x_s, u_s). The steady state satisfies

$$\begin{bmatrix} I-A & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0$$

- For *unconstrained* systems, we also impose the requirement that the steady state satisfies $Cx_s = y_{\mathrm{sp}}$ for the tracking problem, giving the set of equations

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{\mathrm{sp}} \end{bmatrix} \tag{1}$$

Deviation variables

- If this set of equations has a solution, we can then define deviation variables

$$\tilde{x}(k) = x(k) - x_s$$
$$\tilde{u}(k) = u(k) - u_s$$

- They satisfy the dynamic model

$$\begin{aligned} \tilde{x}(k+1) &= x(k+1) - x_s \\ &= Ax(k) + Bu(k) - (Ax_s + Bu_s) \\ \tilde{x}(k+1) &= A\tilde{x}(k) + B\tilde{u}(k) \end{aligned}$$

- The deviation variables satisfy the same model equation as the original variables! This feature holds only for *linear* models.

Deviation variables

- The zero regulation problem applied to the system in deviation variables finds $\tilde{u}(k)$ that takes $\tilde{x}(k)$ to zero, or, equivalently, which takes $x(k)$ to x_s, so that at steady state, $Cx(k) = Cx_s = y_{\mathrm{sp}}$, which is the goal of the setpoint tracking problem.
- After solving the regulation problem in deviation variables, the input applied to the system is

$$u(k) = \tilde{u}(k) + u_s$$

- We next discuss when we can solve (1). We also note that for *constrained* systems, we must impose the constraints on the steady state (x_s, u_s).

More outputs than inputs: controlled variables

- The matrix in (1) is a $(n+p) \times (n+m)$ matrix. For (1) to have a solution for all y_{sp}, it is sufficient that the rows of the matrix are linearly independent.
- That requires $p \leq m$: we require at least as many inputs as outputs with setpoints. But it is not uncommon in applications to have many more measured outputs than manipulated inputs.
- To handle these more general situations, we choose a matrix H and denote a new variable $r = Hy$ as a selection of linear combinations of the measured outputs. The variable $r \in \mathbb{R}^{n_c}$ is known as the *controlled variable*.
- For cases in which $p > m$, we choose some set of outputs $n_c \leq m$, as controlled variables, and assign setpoints to r, denoted r_{sp}.

More inputs than outputs: input targets

- We also wish to treat systems with more inputs than outputs, $m > p$. For these cases, the solution to (1) may exist for some choice of H and r_{sp}, but cannot be unique.
- If we wish to obtain a unique steady state, then we also must provide desired values for the steady inputs, u_{sp}.
- To handle constrained systems, we simply impose the constraints on (x_s, u_s).

Steady-state target problem

Our candidate optimization problem is therefore

$$\min_{x_s, u_s} \frac{1}{2} \left(|u_s - u_{\mathrm{sp}}|^2_{R_s} + |Cx_s - y_{\mathrm{sp}}|^2_{Q_s} \right) \tag{2a}$$

subject to:

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r_{\mathrm{sp}} \end{bmatrix} \tag{2b}$$

$$Eu_s \leq e \tag{2c}$$

$$FCx_s \leq f \tag{2d}$$

Steady-state target problem

We make the following assumptions:

Assumption 1 (Target feasibility and uniqueness)

1. The target problem is feasible for the controlled variable setpoints of interest r_{sp}.
2. The steady-state input penalty R_s is positive definite.

- Assumption 1.1 ensures that the solution (x_s, u_s) exists
- Assumption 1.2 ensures that the solution is unique.
- If one chooses $n_c = 0$, then no controlled variables are required to be at setpoint, and the problem is feasible for any $(u_{\mathrm{sp}}, y_{\mathrm{sp}})$ because $(x_s, u_s) = (0, 0)$ is a feasible point.

Some exercises on the target problem

- Exercises 1.56 and 1.57 explore the connection between feasibility of the equality constraints and the number of controlled variables relative to the number of inputs and outputs.
- One restriction is that the number of controlled variables chosen to be offset free must be less than or equal to the number of manipulated variables and the number of measurements, $n_c \leq m$ and $n_c \leq p$.

Dynamic regulation problem

- Given the steady-state solution, we define the following multistage objective function

$$V(\tilde{x}(0), \tilde{\boldsymbol{u}}) = \frac{1}{2} \sum_{k=0}^{N-1} |\tilde{x}(k)|_Q^2 + |\tilde{u}(k)|_R^2 \qquad \text{s.t. } \tilde{x}^+ = A\tilde{x} + B\tilde{u}$$

- The initial state is

$$\tilde{x}(0) = \hat{x}(k) - x_s$$

i.e., the initial condition for the regulation problem comes from the state *estimate* shifted by the steady-state x_s.

Dynamic regulation problem

The regulator solves the following dynamic, zero-state regulation problem

$$\min_{\tilde{\boldsymbol{u}}} V(\tilde{x}(0), \tilde{\boldsymbol{u}})$$

subject to

$$E\tilde{u} \leq e - Eu_s$$
$$FC\tilde{x} \leq f - FCx_s$$

in which the constraints also are shifted by the steady state (x_s, u_s).

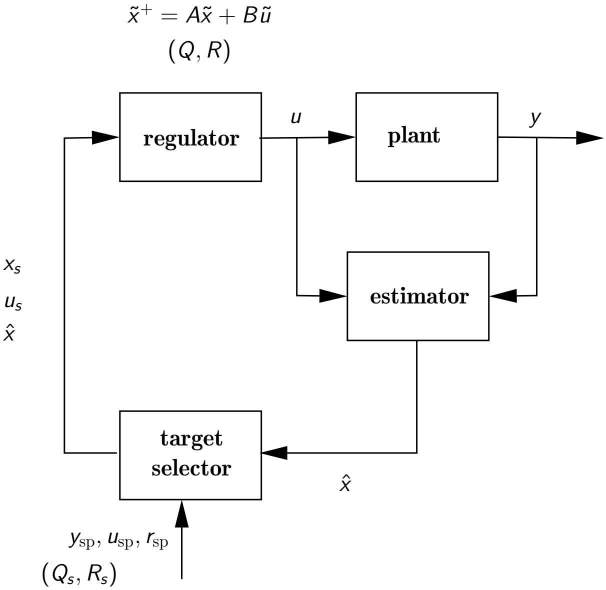
Dynamic regulation problem — control law

- The optimal cost and solution are $V^0(\tilde{x}(0))$ and $\tilde{\boldsymbol{u}}^0(\tilde{x}(0))$.
- The moving horizon control law uses the first move of this optimal sequence, $\tilde{u}^0(\tilde{x}(0)) = \tilde{\boldsymbol{u}}^0(0; \tilde{x}(0))$, so the controller output is

$$u(k) = \tilde{u}^0(\tilde{x}(0)) + u_s$$

- The control law is more complex than the PID control law, but the control is a function of the estimated state, and the estimated state depends on the measurements. That's the feedback in MPC!
- Designing the state estimator is crucial to good closed-loop control performance.

The assembly so far



Disturbances and zero offset

- Another common objective in applications is to use a feedback controller to compensate for an unmeasured disturbance to the system with the input so the disturbance's effect on the controlled variable is mitigated.
- This problem is known as disturbance rejection. We may wish to design a feedback controller that compensates for nonzero disturbances such that the selected controlled variables asymptotically approach their setpoints without offset.
- This property is known as zero offset. In this section we show a simple method for constructing an MPC controller to achieve zero offset.

Disturbances and zero offset

- We will ensure that *if the system is stabilized in the presence of the disturbance*, then there is zero offset.
- This more limited objective is similar to what one achieves when using the integral mode in proportional-integral-derivative (PID) control of an unconstrained system: either there is zero steady offset, or the system trajectory is unbounded.
- In a constrained system, the statement is amended to: either there is zero steady offset, or the system trajectory is unbounded, or the system constraints are active at steady state.
- In both constrained and unconstrained systems, the zero-offset property *precludes* one undesirable possibility: the system settles at an unconstrained steady state, and the steady state displays offset in the controlled variables.

So why does the PI controller have zero offset?

- Here's the control law

$$u(t) = k_c\left(e(t) + \frac{1}{\tau_I}\int_0^t e(t')dt'\right), \qquad e = y_{\rm sp} - y$$

- If the tracking error goes to a (nonzero) constant, $e(t) \to e_s$, then $u(t) \to \infty$ as $t \to \infty$ because of the integral term.
- If we turn off the integrator

$$u_s = k_c e_s$$

  and we expect offset with a proportional controller.
- In PI, we obtain zero offset by integrating the tracking error. In MPC we will *not* integrate the tracking error. But we will integrate instead the *model error*.
- We can show that is also sufficient to remove offset. And we won't have windup when inputs saturate.

Disturbance model

- A simple method to compensate for an unmeasured disturbance is to
  1. model the disturbance
  2. use the measurements and model to estimate the disturbance
  3. find the inputs that minimize the effect of the disturbance on the controlled variables.
- The choice of disturbance model is motivated by the zero-offset goal. To achieve offset-free performance we augment the system state with an *integrating* disturbance d driven by the process noise w

$$d(k+1) = d(k) + w(k) \tag{3}$$

d integrates the driving noise w

$$d(k) = w(0) + w(1) + \cdots + w(k-1)$$

A little history on disturbance models

- This choice is motivated by the works of Davison and Smith (1971, 1974); Qiu and Davison (1993) and the Internal Model Principle of Francis and Wonham (1976).
- To remove offset, one designs a control system that can remove asymptotically constant, nonzero disturbances (Davison and Smith, 1971), (Kwakernaak and Sivan, 1972, p.278).
- To accomplish this end, the original system is augmented with a replicate of the constant, nonzero disturbance model, (3). Thus the states of the original system are moved to cancel the effect of the disturbance on the controlled variables.

System plus disturbance model

- The augmented system model used for the state estimator is given by

$$\begin{bmatrix} x \\ d \end{bmatrix}^+ = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + w \tag{4a}$$

$$y = \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + v \tag{4b}$$

- We are free to choose how the integrating disturbance affects the states and measured outputs through the choice of B_d and C_d.
- The only restriction is that the augmented system is detectable. That restriction can be easily checked using the following result.

Restrictions on the disturbance model

Lemma 2 (Detectability of the augmented system)

The augmented system (4) *is detectable if and only if the nonaugmented system* (A, C) *is detectable, and the following condition holds:*

$$\operatorname{rank} \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + n_d \tag{5}$$

Corollary 3 (Dimension of the disturbance)

The maximal dimension of the disturbance d *in* (4) *such that the augmented system is detectable is equal to the number of measurements, that is*

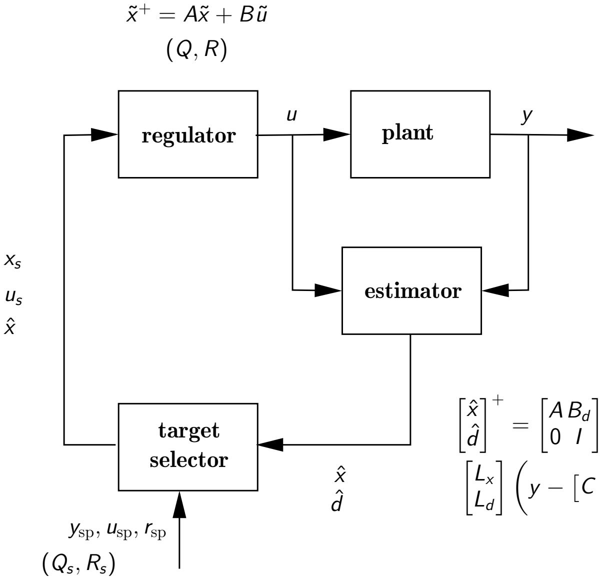
$$n_d \leq p$$

A pair of matrices (B_d, C_d) such that (5) is satisfied always exists.

Estimate state plus disturbance

- The state and the additional integrating disturbance are estimated from the plant measurement using a Kalman filter designed for the augmented system.
- The variances of the stochastic disturbances w and v may be treated as adjustable parameters or found from input-output measurements (Odelson, Rajamani, and Rawlings, 2006).

Overview of the final assembly



$$\begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}^+ = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L_x \\ L_d \end{bmatrix} \left(y - \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} \right)$$

Forecast of the disturbance

- The estimator provides $\hat{x}(k)$ and $\hat{d}(k)$ at each time k.
- The best forecast of the steady-state disturbance using (3) is simply

$$\hat{d}_s = \hat{d}(k)$$

The nonzero disturbance affects the steady-state target

The steady-state target problem is therefore modified to account for the nonzero disturbance $\hat{d}_s$

$$\min_{x_s, u_s} \frac{1}{2} \left(|u_s - u_{\mathrm{sp}}|^2_{R_s} + \left| Cx_s + C_d \hat{d}_s - y_{\mathrm{sp}} \right|^2_{Q_s} \right) \tag{6a}$$

subject to:

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d \hat{d}_s \\ r_{\mathrm{sp}} - HC_d \hat{d}_s \end{bmatrix} \tag{6b}$$

$$Eu_s \leq e \tag{6c}$$

$$FCx_s \leq f - FC_d \hat{d}_s \tag{6d}$$

Effect of the disturbance

Comparing (2) to (6), we see the disturbance model affects the steady-state target determination in four places.

1. The output target is modified in (6a) to account for the effect of the disturbance on the measured output ($y_{\mathrm{sp}} \to y_{\mathrm{sp}} - C_d \hat{d}_s$).
2. The output constraint in (6d) is similarly modified ($f \to f - FC_d \hat{d}_s$).
3. The system steady-state relation in (6b) is modified to account for the effect of the disturbance on the state evolution ($0 \to B_d \hat{d}_s$).
4. The controlled variable target in (6b) is modified to account for the effect of the disturbance on the controlled variable ($r_{\mathrm{sp}} \to r_{\mathrm{sp}} - HC_d \hat{d}_s$).

No change to the regulation problem, only the target!

- Given the steady-state target, the same dynamic regulation problem as presented in the tracking section is used for the regulator.
- In other words, the regulator is based on the deterministic system (A, B) in which the current state is $\hat{x}(k) - x_s$ and the goal is to take the system to the origin.

Offset-free control property of the control system

Lemma 4 (Offset-free control)

Consider a system controlled by the MPC algorithm as shown in the figure. The target problem (6) is assumed feasible. Augment the system model with a number of integrating disturbances equal to the number of measurements ($n_d = p$); choose any $B_d \in \mathbb{R}^{n \times p}$, $C_d \in \mathbb{R}^{p \times p}$ such that

$$\operatorname{rank} \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + p$$

If the plant output $y(k)$ goes to steady state y_s, the closed-loop system is stable, and constraints are not active at steady state, then there is zero offset in the controlled variables, that is

$$Hy_s = r_{\mathrm{sp}}$$

Remarks on offset

- The proof of this lemma is given in Pannocchia and Rawlings (2003).
- It may seem surprising that the number of integrating disturbances must be equal to the number of *measurements* used for feedback rather than the number of *controlled variables* to guarantee offset-free control.
- To gain insight into the reason, consider the disturbance part (bottom half) of the Kalman filter equations shown in the figure

$$\hat{d}^{+} = \hat{d} + L_d\left(y - \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}\right)$$

- Because of the integrator, the disturbance estimate cannot converge until

$$L_d\left(y - \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}\right) = 0$$

Number of disturbances equals number of measurements

Let the output prediction error be

$$L_d e = 0 \qquad e = y - \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} \qquad \begin{array}{cc} & p \\ n_d & \begin{bmatrix} L_d \end{bmatrix} \end{array}$$

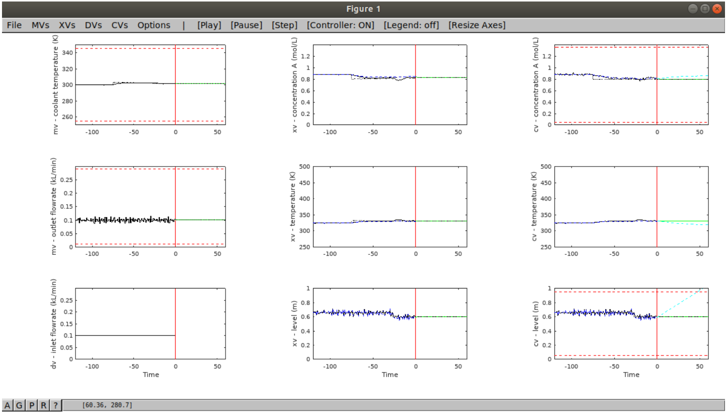
- If we choose $n_d = n_c < p$, then the number of columns of L_d is greater than the number of rows and $L_d e = 0$ does *not* force $e = 0$.
- In general, we require the output prediction error to be *zero* to achieve zero offset independently of the regulator tuning.
- For $L_d e = 0$ to force $e = 0$, we require $n_d \geq p$.
- Since we also know $n_d \leq p$ from Corollary 3, we conclude $n_d = p$.

Also removes offset due to model error!

- Notice also that Lemma 4 does not require that the plant output be generated by the model. The theorem applies regardless of what generates the plant output. *If the plant is identical to the system plus disturbance model assumed in the estimator*, then the conclusion can be strengthened.
- In the nominal case without measurement or process noise ($w = 0$, $v = 0$), *for a set of plant initial states*, the closed-loop system *converges to a steady state* and the feasible steady-state target is achieved leading to zero offset in the controlled variables.
- Characterizing the set of initial states in the region of convergence, and stabilizing the system when the plant and the model differ, are treated in Chapters 3 and 5 of (Rawlings, Mayne, and Diehl, 2020).
- We conclude this section with a nonlinear example that demonstrates the use of Lemma 4.

The mpcsim software

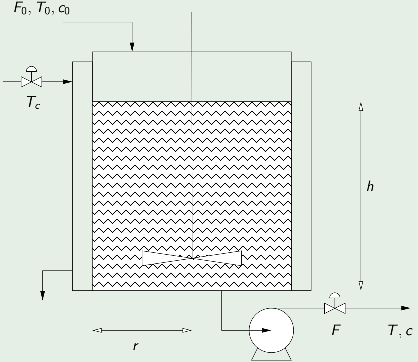
- Provides a GUI for MPCTools/CasADi
- Allows interactive simulation of closed-loop systems
- Trending capability is similar to modern MPC implementations
- Written in Python/Tkinter, with calls to MPCTools/CasADI



Simulation exercise: LQG control of an exothermic CSTR

Example 5

We consider a well-stirred chemical reactor as in Pannocchia and Rawlings (2003). An irreversible, first-order reaction A$\longrightarrow$B occurs in the liquid phase and the reactor temperature is regulated with external cooling.



Mass and energy balances

Mass and energy balances lead to the following nonlinear state space model:

$$\begin{aligned}
\frac{dc}{dt} &= \frac{F_0(c_0 - c)}{\pi r^2 h} - k_0 \exp\left(-\frac{E}{RT}\right) c \\
\frac{dT}{dt} &= \frac{F_0(T_0 - T)}{\pi r^2 h} + \frac{-\Delta H}{\rho C_p} k_0 \exp\left(-\frac{E}{RT}\right) c + \frac{2U}{r\rho C_p}(T_c - T) \\
\frac{dh}{dt} &= \frac{F_0 - F}{\pi r^2}
\end{aligned}$$

Steady-state operating point

- The controlled variables are h, the level of the tank, and c, the molar concentration of species A. The additional state variable is T, the reactor temperature
- The manipulated variables are T_c, the coolant liquid temperature, and F, the outlet flowrate.
- Moreover, it is assumed that the inlet flowrate acts as an unmeasured disturbance.
- The open-loop stable steady-state operating conditions are the following:

$$c^s = 0.878\,\mathrm{kmol/m^3} \qquad T^s = 324.5\,\mathrm{K} \qquad h^s = 0.659\,\mathrm{m}$$
$$T_c^s = 300\,\mathrm{K} \qquad F^s = 0.1\,\mathrm{m^3/min}$$

- The model parameters in nominal conditions are reported in the following table.

Plant parameter values

Parameter	Nominal value	Units
F_0	0.1	$\mathrm{m}^3/\mathrm{min}$
T_0	350	K
c_0	1	$\mathrm{kmol}/\mathrm{m}^3$
r	0.219	m
k_0	7.2×10^{10}	min^{-1}
E/R	8750	K
U	54.94	$\mathrm{kJ}/\mathrm{min}{\cdot}\mathrm{m}^2{\cdot}\mathrm{K}$
ρ	1000	kg/m^3
C_p	0.239	$\mathrm{kJ}/\mathrm{kg}{\cdot}\mathrm{K}$
ΔH	-5×10^4	kJ/kmol

Using a sampling time of 1 min, a linearized discrete state space model is obtained and, assuming that all the states are measured, the state space variables are:

$$x = \begin{bmatrix} c - c^s \\ T - T^s \\ h - h^s \end{bmatrix} \qquad u = \begin{bmatrix} T_c - T_c^s \\ F - F^s \end{bmatrix} \qquad y = \begin{bmatrix} c - c^s \\ T - T^s \\ h - h^s \end{bmatrix} \qquad p = F_0 - F_0^s$$

The corresponding linear model is:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_p p \\ y(k) &= Cx(k) \end{aligned}$$

in which

$$A = \begin{bmatrix} 0.2681 & -0.00338 & -0.00728 \\ 9.703 & 0.3279 & -25.44 \\ 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.00537 & 0.1655 \\ 1.297 & 97.91 \\ 0 & -6.637 \end{bmatrix} \qquad B_p = \begin{bmatrix} -0.1175 \\ 69.74 \\ 6.637 \end{bmatrix}$$

CSTR disturbance model options

We will test a LQG control with three different disturbance models:

1. output disturbances on c and h

$$C_d = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad B_d = 0$$

2. output disturbances on c, h, T

$$C_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_d = 0$$

3. output disturbances on c and h, input disturbance on F

$$C_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_d = \begin{bmatrix} 0 & 0 & 0.1655 \\ 0 & 0 & 97.91 \\ 0 & 0 & -6.637 \end{bmatrix}$$

CSTR disturbance model exercise - LQG control

1. Navigate to /mpcsim, start up Octave and run mpcsim_setup.m
2. Run cstr_mpcsim.m
3. Toggle the Controller switch to ON and press the Play button
4. Using the CVs menu change the c setpoint to 0.882 kmol/m3. Note that the control decreases T_c to accomplish the c setpoint change.
5. Using the CVs menu change the h setpoint to 0.70 m. Note that the control drops F and then brings it back to its original value to accomplish the h setpoint change. Note also that the control must increase T_c in order to keep c at its setpoint.
6. Change the c setpoint back to 0.878 kmol/m3 and change the h setpoint back to 0.66 m. Let the process line out.
7. On the options menu set the Disturbance Model Number to 1 to select the first disturbance model.

CSTR disturbance model exercise, continued

1. On the DVs menu select F_0 and enter a value of 0.102 m3/min. Observe how the closed-loop system responds to this unmeasured disturbance. With this disturbance model is offset-free control achieved? Note that the disturbance model matrices B_d and C_d and the result of the rank test and disturbance number test are written to standard output.
2. Return the F_0 value to 0.100 m3/min.
3. Repeat the disturbance test for the remaining disturbance models 2 and 3.
4. Experiment with other setpoint changes and tuning parameter settings in the remaining time.

a. Since we have two inputs, T_c and F, we try to remove offset in two controlled variables, c and h. Model the disturbance with *two* integrating output disturbances on the two controlled variables. Assume that the covariances of the state noises are zero except for the two integrating states. Assume that the covariances of the three measurements' noises are also zero.
Notice that although there are only two controlled variables, this choice of *two* integrating disturbances does not follow the prescription of Lemma 4 for zero offset.
Simulate the response of the controlled system after a 10% increase in the inlet flowrate F_0 at time $t = 10\,\mathrm{min}$. Use the nonlinear differential equations for the plant model. Do you have steady offset in any of the outputs? Which ones?

- (b) Follow the prescription of Lemma 4 and choose a disturbance model with *three* integrating modes. Can you choose three integrating output disturbances for this plant? If so, prove it. If not, state why not.
- (c) Again choose a disturbance model with three integrating modes; choose two integrating output disturbances on the two controlled variables. Choose one integrating input disturbance on the outlet flowrate F. Is the augmented system detectable?
  Simulate again the response of the controlled system after a 10% increase in the inlet flowrate F_0 at time $t = 10\,\mathrm{min}$. Again use the nonlinear differential equations for the plant model. Do you have steady offset in any of the outputs? Which ones?
  Compare and contrast the closed-loop performance for the design with two integrating disturbances and the design with three integrating disturbances. Which control system do you recommend and why?

Solution—two integrating disturbances

Solution

a

- Integrating disturbances are added to the two controlled variables (first and third outputs) by choosing

$$C_d = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad B_d = 0$$

- The results with two integrating disturbances are shown in the next figures.
- Notice that despite adding integrating disturbances to the two controlled variables, c and h, both of these controlled variables as well as the third output, T, all display nonzero offset at steady state.

System trajectory

Solution

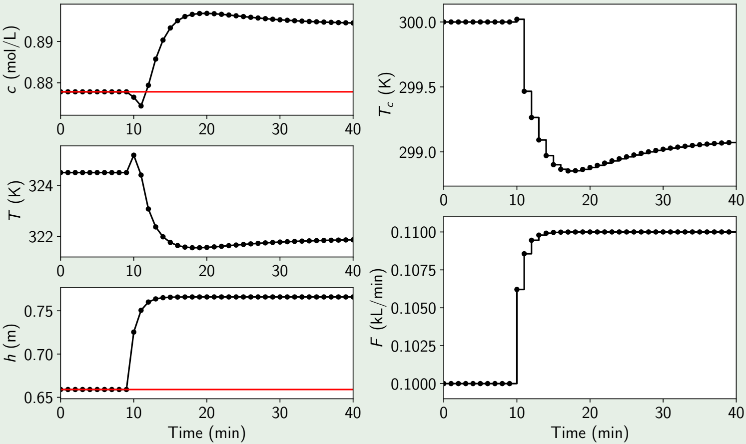


Figure 1: Outputs and inputs versus time after a step change ininlet flowrate at 10 minutes; $n_d = 2$. Notice the steady-state offset in all three output variables.

Solution—third integrator at an output

Solution

b

- A third integrating disturbance is added to the second output giving

$$C_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_d = 0$$

- The augmented system is not detectable with this disturbance model! The rank of $\begin{bmatrix} I-A & -B_d \\ C & C_d \end{bmatrix}$ is only 5 instead of 6.
- The problem here is that the system level is itself an integrator, and we cannot distinguish h from the integrating disturbance added to h.

Solution—third integrator at an input

Solution

c

- Next we try three integrating disturbances: two added to the two controlled variables, and one added to the second manipulated variable

$$C_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_d = \begin{bmatrix} 0 & 0 & 0.1655 \\ 0 & 0 & 97.91 \\ 0 & 0 & -6.637 \end{bmatrix}$$

- The augmented system is detectable for this disturbance model.
- The results for this choice of three integrating disturbances are shown in the next figure. Notice that we have zero offset in the two controlled variables, c and h, and have successfully forced the steady-state effect of the inlet flowrate disturbance entirely into the second output, T.

System trajectory

Solution

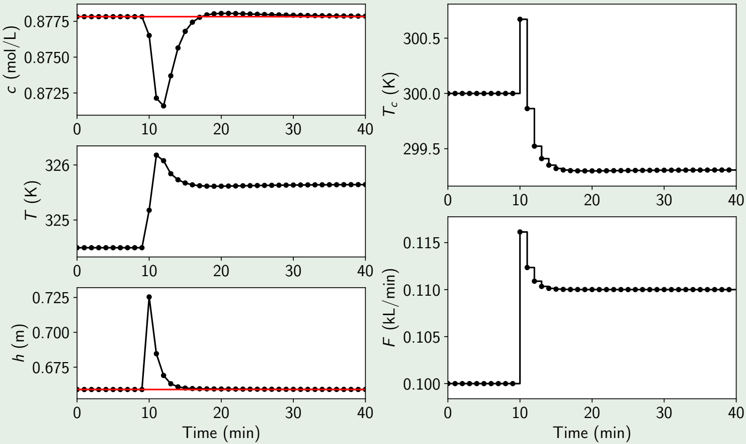


Figure 2: Output and inputs versus time after a step change ininlet flowrate at 10 minutes with a good disturbance model; $n_d = 3$.

Solution

- Notice also that the dynamic behavior of all three outputs is superior to that achieved with the model using two integrating disturbances.
- The true disturbance, which is a step at the inlet flowrate, is better represented by including the integrator in the outlet flowrate. With a more accurate disturbance model, better overall control is achieved.
- The controller uses smaller manipulated variable action and also achieves better output variable behavior.
- An added bonus is that steady offset is removed in the maximum possible number of outputs. ■

Further reading I

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Further reading II

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Tracking, disturbances, and zero offset

Review

Lab Exercises

- Exercise 1.56
- Exercise 1.57
- Exercise 1.58
- Exercise 1.60
- Reproduce Figure 4 of Rajamani, Rawlings, and Qin (2009)
- Reproduce Figure 6 of Rajamani et al. (2009)