Nonlinear model predictive control – Regulation

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Outline

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   - Inherent robustness of MPC
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System model[1]

- We consider systems of the form

$$x^+ = f(x, u)$$

  where the state x lies in $\mathbb{X} \subseteq \mathbb{R}^n$ and the control (input) u lies in $\mathbb{U} \subseteq \mathbb{R}^m$;
- In this formulation x and u denote, respectively, the current state and control, and x^+ the successor state.
- We assume in the sequel that the function $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is continuous, and the sets $\mathbb{X}$ and $\mathbb{U}$ are closed.
- Let

$$\phi(k; x, \boldsymbol{u})$$

  denote the solution of $x^+ = f(x, u)$ at time k if the initial state is $x(0) = x$ and the control sequence is $\boldsymbol{u} = (u(0), u(1), u(2), \ldots)$;
- The solution exists and is unique.

[1] Most of this preliminary material is taken from Rawlings, Mayne, and Diehl (2020, Appendix B). Downloadable from `engineering.ucsb.edu/~jbraw/mpc`.

Existence of solutions to model

- If a state-feedback control law $u = \kappa(x)$ has been chosen, the closed-loop system is described by $x^+ = f(x, \kappa(x))$.
- Let $\phi(k; x, \kappa(\cdot))$ denote the solution of this difference equation at time k if the initial state at time 0 is $x(0) = x$; the solution exists and is unique (even if $\kappa(\cdot)$ is discontinuous).
- If $\kappa(\cdot)$ is not continuous, as may be the case when $\kappa(\cdot)$ is a model predictive control (MPC) law, then $f((\cdot), \kappa(\cdot))$ may not be continuous.
- In this case we assume that $f((\cdot), \kappa(\cdot))$ is *locally bounded*.

Definition 1 (Locally bounded)

A function $f : X \to X$ is locally bounded if, for any $x \in X$, there exists a neighborhood $\mathcal{N}$ of x such that $f(\mathcal{N})$ is a bounded set, i.e., if there exists a $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathcal{N}$.

Stability and equilibrium point

We would like to be sure that the controlled system is "stable", i.e., that small perturbations of the initial state do not cause large variations in the subsequent behavior of the system, and that the state converges to a desired state or, if this is impossible due to disturbances, to a desired set of states.

If convergence to a specified state, x^* say, is sought, it is desirable for this state to be an *equilibrium* point:

Definition 2 (Equilibrium point)

A point x^* is an equilibrium point of $x^+ = f(x)$ if $x(0) = x^*$ implies $x(k) = \phi(k; x^*) = x^*$ for all $k \geq 0$. Hence x^* is an equilibrium point if it satisfies

$$x^* = f(x^*)$$

Positive invariant set

In other situations, for example when studying the stability properties of an oscillator, convergence to a specified closed set $\mathcal{A} \subset \mathbb{X}$ is sought.
If convergence to a set $\mathcal{A}$ is sought, it is desirable for the set $\mathcal{A}$ to be *positive invariant*:

Definition 3 (Positive invariant set)

A set $\mathcal{A}$ is positive invariant for the system $x^+ = f(x)$ if $x \in \mathcal{A}$ implies $f(x) \in \mathcal{A}$.

Clearly, any solution of $x^+ = f(x)$ with initial state in $\mathcal{A}$, remains in $\mathcal{A}$.
The (closed) set $\mathcal{A} = \{x^*\}$ consisting of a (single) equilibrium point is a special case; $x \in \mathcal{A}$ ($x = x^*$) implies $f(x) \in \mathcal{A}$ ($f(x) = x^*$).

Distance to a set; set addition

- Define distance from point x to set $\mathcal{A}$

$$|x|_{\mathcal{A}} := \inf_{z \in \mathcal{A}} |x - z|$$

  If $\mathcal{A} = \{x^*\}$, then $|x|_{\mathcal{A}} = |x - x^*|$ which reduces to $|x|$ when $x^* = 0$.
- Set addition: $A \oplus B := \{a + b \mid a \in A, b \in B\}$.

$\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions

Definition 4

- A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing;
- $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}_\infty$ if it is a class $\mathcal{K}$ and unbounded ($\sigma(s) \to \infty$ as $s \to \infty$).
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if it is continuous and if, for each $t \geq 0$, $\beta(\cdot, t)$ is a class $\mathcal{K}$ function and for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t\to\infty} \beta(s, t) = 0$.

Some useful properties of $\mathcal{K}$ functions

The following useful properties of these functions are established in Khalil (2002, Lemma 4.2):

- if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}$ functions ($\mathcal{K}_\infty$ functions), then $\alpha_1^{-1}(\cdot)$ and $(\alpha_1 \circ \alpha_2)(\cdot) := \alpha_1(\alpha_2(\cdot))$ are $\mathcal{K}$ functions ($\mathcal{K}_\infty$ functions).
- Moreover, if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}$ functions and $\beta(\cdot)$ is a $\mathcal{KL}$ function, then $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is a $\mathcal{KL}$ function.

Stability—Definitions

Definition 5 ((Classic) Asymptotic stability (constrained))

Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$, that $\mathcal{A} \subset X$ is closed and positive invariant for $x^+ = f(x)$. Then $\mathcal{A}$ is

1. locally stable in X if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $x \in X \cap (\mathcal{A} \oplus \delta\mathcal{B})$, implies $|\phi(i; x)|_{\mathcal{A}} < \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.[a]
2. locally attractive in X if there exists a $\eta > 0$ such that $x \in X \cap (\mathcal{A} \oplus \eta\mathcal{B})$ implies $|\phi(i; x)|_{\mathcal{A}} \to 0$ as $i \to \infty$.
3. attractive in X if $|\phi(i; x)|_{\mathcal{A}} \to 0$ as $i \to \infty$ for all $x \in X$.
4. asymptotically stable with a region of attraction X if it is locally stable in X and attractive in X.

[a] $\mathcal{B}$ denotes the unit ball in $\mathbb{R}^n$.

Asymptotic stability—stronger definition

Definition 6 (Asymptotic stability (constrained – KL versoin))

Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$, that the origin is an equilibrium of $x^+ = f(x)$, and that the origin is in X. The origin is *asymptotically stable in* X for $x^+ = f(x)$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that, for each $x \in X$

$$|\phi(i;x)| \leq \beta(|x|, i) \qquad \forall i \geq 0 \tag{1}$$

See Teel and Zaccarian (2006) and the "Notes on Recent MPC Literature" link on: `engineering.ucsb.edu/~jbraw/mpc` for further discussion of the differences in the two definitions.

If $f(\cdot)$ is *continuous*, the two definitions are equivalent.

Lyapunov function

Definition 7 (Lyapunov function (constrained))

Suppose that X is positive invariant and the origin is an equilibrium for $x^+ = f(x)$. A function $V : X \to \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function in X for the system $x^+ = f(x)$ if there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that for any $x \in X$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$V(f(x)) - V(x) \leq -\alpha_3(|x|)$$

Lyapunov stability theorem

Theorem 8 (Lyapunov stability theorem—constrained case)

Suppose that X is positive invariant and the origin is an equilibrium for $x^+ = f(x)$. If there exists a Lyapunov function in X for the system $x^+ = f(x)$ then the origin is asymptotically stable in X for $x^+ = f(x)$.

In other words, we don't have to analyze closed-loop stability of MPC on a case-by-case basis.

We instead establish that the optimal MPC cost function is a Lyapunov function for the closed-loop system!

Converse theorem for exponential stability

Exercise B.3: A converse theorem for exponential stability

1. Assume that the origin is globally exponentially stable (GES) for the system

$$x^+ = f(x)$$

in which f is Lipschitz continuous. Show that there exists a Lipschitz continuous Lyapunov function $V(\cdot)$ for the system satisfying for all $x \in \mathbb{R}^n$

$$a_1 |x|^\sigma \leq V(x) \leq a_2 |x|^\sigma$$
$$V(f(x)) - V(x) \leq -a_3 |x|^\sigma$$

in which $a_1, a_2, a_3, \sigma > 0$.
Hint: Consider summing the solution $|\phi(i; x)|$ on i as a candidate Lyapunov function $V(x)$.

2. Establish also that in the Lyapunov function defined above, any $\sigma > 0$ is valid, and the constant a_3 can be chosen as large as one wishes.

The basic nonlinear, constrained MPC problem

- The system model is

$$x^+ = f(x, u) \tag{2}$$

- Both state and input are subject to constraints

$$x(k) \in \mathbb{X}\,, \qquad u(k) \in \mathbb{U} \qquad \text{for all } k \in \mathbb{I}_{\geq 0}$$

- Given an integer N (referred to as the finite horizon), and an input sequence $\boldsymbol{u}$ of length N, $\boldsymbol{u} = (u(0), u(1), \ldots, u(N-1))$, let $\phi(k; x, \boldsymbol{u})$ denote the solution of (2) at time k for a given initial state $x(0) = x$.
- Terminal constraint (and penalty)

$$\phi(N; x, \boldsymbol{u}) \in \mathbb{X}_f \subseteq \mathbb{X}$$

Feasible sets

- For an initial x, the corresponding set of feasible inputsequences is

$$\mathcal{U}_N(x) = \{\boldsymbol{u} \mid u(k) \in \mathbb{U},\ \phi(k; x, \boldsymbol{u}) \in \mathbb{X} \text{ for all } k \in \mathbb{I}_{0:N-1}, \\ \text{and } \phi(N; x, \boldsymbol{u}) \in \mathbb{X}_f\}$$

- The set of feasible initial states is

$$\mathcal{X}_N = \{x \in \mathbb{X} \mid \mathcal{U}_N(x) \neq \emptyset\} \tag{3}$$

Cost function and control problem

- For any state $x \in \mathbb{X}$ and input sequence $\boldsymbol{u} \in \mathbb{U}^N$, we define

$$V_N(x, \boldsymbol{u}) = \sum_{k=0}^{N-1} \ell(\phi(k; x, \boldsymbol{u}), u(k)) + V_f(\phi(N; x, \boldsymbol{u}))$$

- $\ell(x, u)$ is the stage cost; $V_f(x(N))$ is the terminal cost
- Consider the finite horizon optimal control problem

$$\mathbb{P}_N(x): \qquad \min_{\boldsymbol{u} \in \mathcal{U}_N} V_N(x, \boldsymbol{u})$$

Control law and closed-loop system

- The control law is

$$\kappa_N(x) = u^0(0; x)$$

The optimum may not be unique; then $\kappa_N(\cdot)$ is a point-to-set map

- Closed-loop system

$$x^+ = f(x, \kappa_N(x)) \qquad \text{difference equation}$$

$$x^+ \in f(x, \kappa_N(x)) \qquad \text{difference inclusion}$$

- Nominal closed-loop stability question; is the origin stable?
- If yes, what is the region of attraction? All of $\mathcal{X}_N$?

Basic MPC assumptions

Assumption 9 (Continuity of system and cost)

The functions $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$ and $V_f : \mathbb{X}_f \to \mathbb{R}_{\geq 0}$ are continuous, $f(0,0) = 0$, $\ell(0,0) = 0$, and $V_f(0) = 0$.

Assumption 10 (Properties of constraint sets)

The set $\mathbb{U}$ is compact and contains the origin. The sets $\mathbb{X}$ and $\mathbb{X}_f$ are closed and contain the origin in their interiors, $\mathbb{X}_f \subseteq \mathbb{X}$.

Basic MPC assumptions

Assumption 11 (Lower bound on stage cost)

The stage cost $\ell(\cdot)$ satisfies

$$\ell(x, u) \geq \alpha_1(|x|) \quad \forall x \in \mathcal{X}_N,\ \forall u \in \mathbb{U}$$

in which $\alpha_1(\cdot)$ is a $\mathcal{K}_\infty$ function.

Remark 12 (Upper bound on terminal cost)

Because $V_f(\cdot)$ is continuous and $V_f(0) = 0$, we also have that

$$V_f(x) \leq \alpha_2(|x|) \quad \forall x \in \mathbb{X}_f$$

in which $\alpha_2(\cdot)$ is a $\mathcal{K}_\infty$ function.

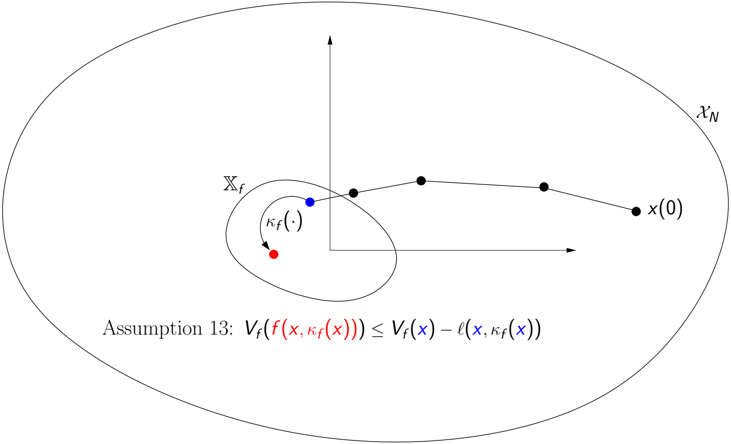
Basic MPC assumptions

Assumption 13 (Basic stability assumption)

For any $x \in \mathbb{X}_f$ there exists $u := \kappa_f(x) \in \mathbb{U}$ such that $f(x,u) \in \mathbb{X}_f$ and $V_f(f(x,u)) \leq V_f(x) - \ell(x,u)$.

Note: understanding this requirement created a big research challenge for the development of nonlinear MPC.

The MPC problem in pictures



Optimal MPC cost function as Lyapunov function

We show that the optimal cost $V_N^0(\cdot)$ is a Lyapunov function for the closed-loop system. We require three properties.

Lower bound.

$$V_N^0(x) \geq \alpha_1(|x|) \qquad \text{for all} \quad x \in \mathcal{X}_N$$

Given the definition of $V_N(x, \boldsymbol{u})$ as a sum of stage costs, we have using Assumption 11

$$V_N(x, \boldsymbol{u}) \geq \ell(x, u(0; x)) \geq \alpha_1(|x|) \quad \text{for all} \quad x \in \mathcal{X}_N, \boldsymbol{u} \in \mathbb{U}^N$$

so the first property is established.

MPC cost function as Lyapunov function – cost decrease

Next we require the **cost decrease**

$$V_N^0(f(x, \kappa_N(x))) \le V_N^0(x) - \alpha_3(|x|) \qquad \text{for all} \quad x \in \mathcal{X}_N$$

At state $x \in \mathcal{X}_N$, consider the optimal sequence $\boldsymbol{u}^0(x) = (u(0; x), u(1; x), \dots, u(N-1; x))$, and generate a *candidate sequence* for the successor state, $x^+ := f(x, \kappa_N(x))$

$$\tilde{\boldsymbol{u}} = (u(1; x), u(2; x), \dots, u(N-1; x), \kappa_f(x(N)))$$

with $x(N) := \phi(N; x, \boldsymbol{u})$. This candidate is *feasible* for x^+ because $\mathbb{X}_f$ is control invariant under control law $\kappa_f(\cdot)$ (Assumption 13).
The cost is

$$\begin{aligned} V_N(x^+, \tilde{\boldsymbol{u}}) = {} & V_N^0(x) - \ell(x, u(0; x)) \\ & - \underbrace{V_f(x(N)) + \ell(x(N), \kappa_f(x(N))) + V_f(f(x(N), \kappa_f(x(N))))} \end{aligned}$$

Cost decrease (cont.)

But by Assumption 13

$$V_f(f(x, \kappa_f(x))) - V_f(x) + \ell(x, \kappa_f(x)) \leq 0 \quad \text{for all} \quad x \in \mathbb{X}_f$$

so we have that

$$V_N(x^+, \tilde{\boldsymbol{u}}) \leq V_N^0(x) - \ell(x, u(0; x))$$

The optimal cost is certainly no worse, giving

$$\begin{aligned} V_N^0(x^+) &\leq V_N^0(x) - \ell(x, u(0; x)) \\ V_N^0(x^+) &\leq V_N^0(x) - \alpha_1(|x|) \qquad \text{for all} \quad x \in \mathcal{X}_N \end{aligned}$$

which is the desired cost decrease with the choice $\alpha_3(\cdot) = \alpha_1(\cdot)$.

Upper bound

Finally we require the **upper bound.**

$$V_N^0(x) \leq \alpha_2(|x|) \qquad \text{for all} \quad x \in \mathcal{X}_N$$

Surprisingly, this one turns out to be the most involved.
First, we have the bound from Assumption 11

$$V_f(x) \leq \alpha_2(|x|) \qquad \text{for all} \quad x \in \mathbb{X}_f$$

Next we show that $V_N^0(x) \leq V_f(x)$ for $x \in \mathbb{X}_f$, $N \geq 1$.
Consider $N = 1$,

$$\begin{aligned} V_1^0(x) &= \min_{u \in \mathbb{U}} \{\ell(x,u) + V_f(f(x,u)) \mid f(x,u) \in \mathbb{X}_f\} \\ &= \ell(x, \kappa_1(x)) + V_f(f(x, \kappa_1(x))) \quad x \in \mathcal{X}_1 \\ &\leq \ell(x, \kappa_f(x)) + V_f(f(x, \kappa_f(x))) \quad x \in \mathbb{X}_f \\ &\leq V_f(x) \quad x \in \mathbb{X}_f \qquad \text{(by Assumption 13)} \end{aligned}$$

Dynamic programming recursion

Next consider $N = 2$, and optimal control law $\kappa_2(\cdot)$

$$\begin{aligned}
V_2^0(x) &= \min_{u \in \mathbb{U}} \{\ell(x,u) + V_1^0(f(x,u)) \mid f(x,u) \in \mathcal{X}_1\} \quad x \in \mathcal{X}_2 \\
&= \ell(x, \kappa_2(x)) + V_1^0(f(x, \kappa_2(x))) \quad x \in \mathcal{X}_2 \\
&\leq \ell(x, \kappa_1(x)) + V_1^0(\underbrace{f(x, \kappa_1(x))}_{\in \mathbb{X}_f}) \quad x \in \mathcal{X}_1 \\
&\leq \ell(x, \kappa_1(x)) + V_f(f(x, \kappa_1(x))) \quad x \in \mathcal{X}_1 \\
&= V_1^0(x) \quad x \in \mathcal{X}_1
\end{aligned}$$

Therefore

$$V_2^0(x) \leq V_f(x) \quad x \in \mathbb{X}_f$$

Continuing this recursion gives for all $N \geq 1$

$$V_N^0(x) \leq V_f(x) \leq \alpha_2(|x|) \quad x \in \mathbb{X}_f$$

Extending the upper bound from $\mathbb{X}_f$ to $\boldsymbol{\mathcal{X}_N}$

- Question: When can we extend this bound from $\mathbb{X}_f$ to the (possibly unbounded!) set $\mathcal{X}_N$? Recall that $V_N^0(\cdot)$ is not necessarily continuous.
- Answer: The $\mathcal{K}_\infty$ upper bound of a function valid near the origin can be extended from $\mathbb{X}_f$ to the entire set $\mathcal{X}_N$ if and only if the function is locally bounded on $\mathcal{X}_N$.[2]
- We know from continuity of $f(\cdot)$ (Assumption 9) that $V_N(x, \boldsymbol{u})$ is a continuous function, hence locally bounded, and therefore so is $V_N^0(x)$.
  Therefore, there exists $\beta(\cdot) \in \mathcal{K}_\infty$ such that

$$V_N^0(x) \leq \beta(|x|) \qquad \text{for all} \quad x \in \mathcal{X}_N$$

- Be aware that the MPC literature has been confused about the requirements for this last result.

[2] See Proposition 11 of "Notes on Recent MPC Literature" link on: `engineering.ucsb.edu/~jbraw/mpc`. Thanks also to Andy Teel.

Asymptotic stability of constrained nonlinear MPC

Why you want a Lyapunov function

- We have established that the optimal cost $V_N^0(\cdot)$ is a Lyapunov function on $\mathcal{X}_N$ for the closed-loop system.
- Therefore, the origin is asymptotically stable (KL version) with region of attraction $\mathcal{X}_N$.
- We can also establish robust stability, but we'll do that later.
- If we strengthen the properties of $\ell(\cdot)$, we can strengthen the conclusion to exponential stability.
- Notice the essential role that $V_N^0(\cdot)$ plays in the stability analysis of MPC.
- In economic MPC we lose this Lyapunov function and have to do some work to bring it back.

A nice example (Example 2.6)

- System is linear (unstable, scalar)

$$x^+ = f(x, u) := x + u$$

- The stage cost and terminal cost are

$$\ell(x, u) := (1/2)(x^2 + u^2) \qquad V_f(x) := (1/2)x^2$$

- The control constraint is

$$u \in \mathbb{U} = [-1, 1]$$

- The horizon is $N = 2$. The feasible set is $\mathcal{U}_2 = \mathbb{U} \times \mathbb{U}$.

Nice example

- The cost function

$$V_N(x, \boldsymbol{u}) = (1/2)\big(x^2 + (x + u(0))^2 + (x + u(0) + u(1))^2 + u(0)^2 + u(1)^2\big)$$
$$= (3/2)x^2 + \begin{bmatrix} 2x & x \end{bmatrix} \boldsymbol{u} + (1/2)\boldsymbol{u}' H \boldsymbol{u}$$

in which

$$H = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

- The optimal control problem

$$\min_{\boldsymbol{u} \in \mathcal{U}_2} V_N(x, \boldsymbol{u})$$

- The optimal control problem is a quadratic program

The quadratic program as x varies

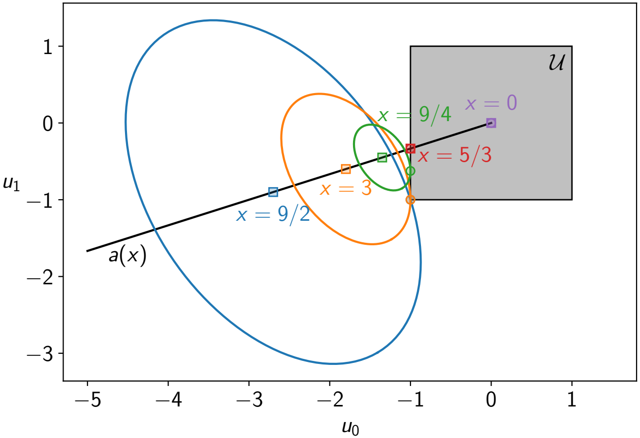


Figure 1: Feasible region $\mathcal{U}_2$, elliptical cost contours, and ellipse center, $a(x)$, and constrained minimizers for different values of x.

The simplest possible constrained control law

- The control law is piecewise affine ($u = Kx + b$) and continuous
- There are three regions: $-5/3 \leq x$, $-5/3 \leq x \leq 5/3$, $5/3 \leq x$

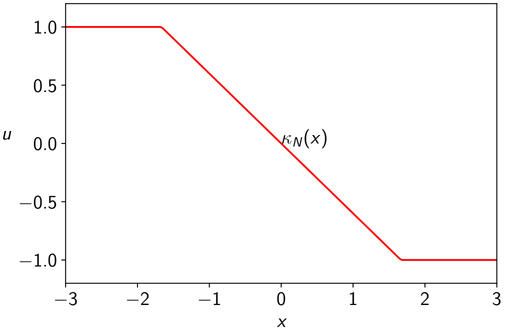


Figure 2: The optimal control law for $x^+ = x + u$, $N = 2$, $Q = R = 1$, $u \in [-1, 1]$.

The constrained control law can be complex

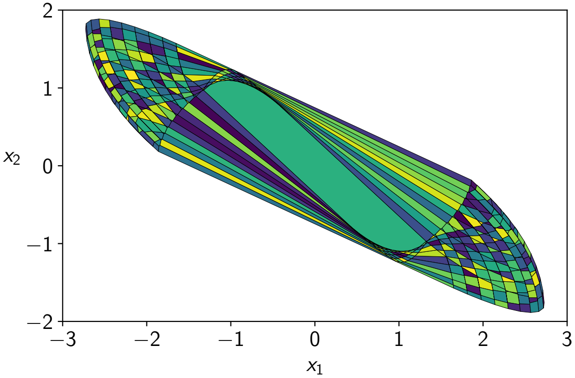


Figure 3: Regions with different linear (affine) control laws for a second-order example. (Rawlings et al., 2020, p.462)

- The number of regions increases exponentially with system order n, number of inputs, m, and horizon length N.
- Another example of Bellman's curse of dimensionality. It's difficult to store $\kappa_N(x)$, $x \in \mathbb{R}^n$, as n increases.

A troublesome example (Example 2.8)

$$x^+ = f(x, u)$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u \\ u^3 \end{bmatrix}$$

- Two state, single input example. The origin is the desired steady state: $u = 0$ at $x = 0$.
- Cannot be stabilized with continuous feedback $u = \kappa(x)$.
- Because (u, u^3) have the same sign, must use negative u to stabilize first quadrant.
- Must use positive u to stabilize third quadrant.
- But u cannot pass through zero or that point is a closed-loop steady state.
- Therefore discontinuous feedback.

And its troubled history

- Introduced by Meadows, Henson, Eaton, and Rawlings (1995) to show MPC control law and optimal cost can be discontinuous.
- Based on a CT example by Coron (1990).
- Grimm, Messina, Tuna, and Teel (2005) established robustness for MPC with horizon $N \geq 4$ with a terminal cost and no terminal region constraint.

MPC with terminal equality constraint

- Because we do not know even a local controller, we try a terminal constraint $x(N) = 0$ in the MPC controller.
- For what initial x is this constraint feasible?

$$(x_1(1), x_2(1)) = (x_1(0), x_2(0)) + (u_0, u_0^3)$$
$$(x_1(2), x_2(2)) = (x_1(1), x_2(1)) + (u_1, u_1^3)$$
$$(x_1(3), x_2(3)) = (x_1(2), x_2(2)) + (u_2, u_2^3)$$

- For $N = 1$, the feasible set $\mathcal{X}_1$ is only the line $x_2 = x_1^3$.
- For $N = 2$, to have real roots u_0, u_1, we require $-x_1^4 + 4x_1x_2 \geq 0$ which defines $\mathcal{X}_2$
- For $N = 3$, we have $\mathcal{X}_3$ is all of $\mathbb{R}^2$.
- So the shortest horizon that can globally stabilize the system is $N = 3$.

Feasibility sets $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_3$

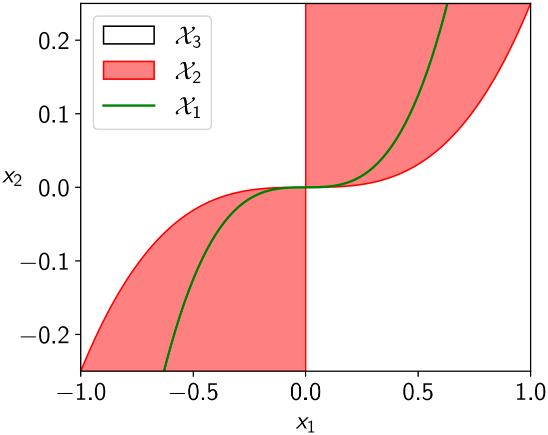
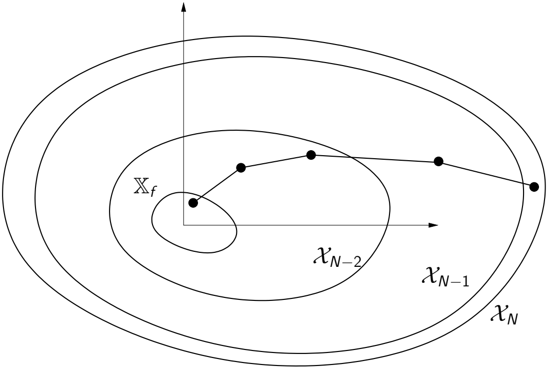


Figure 4: Feasibility sets $\mathcal{X}_N$ for $N = 1, 2, 3$.

Structure of Feasibility Sets



- The feasibility sets are nested: $\mathcal{X}_N \supseteq \mathcal{X}_{N-1} \supseteq \mathcal{X}_{N-2} \cdots \supseteq \mathbb{X}_f$
- The set $\mathcal{X}_N$ is forward invariant. Important for recursive feasibility of controller.
- The set $\mathcal{X}_{N-1}$ is also forward invariant!
- The sets $\mathcal{X}_{N-2}, \mathcal{X}_{N-3}, \ldots, \mathbb{X}_f$ are not necessarily forward invariant.

Optimal MPC with $N = 3$

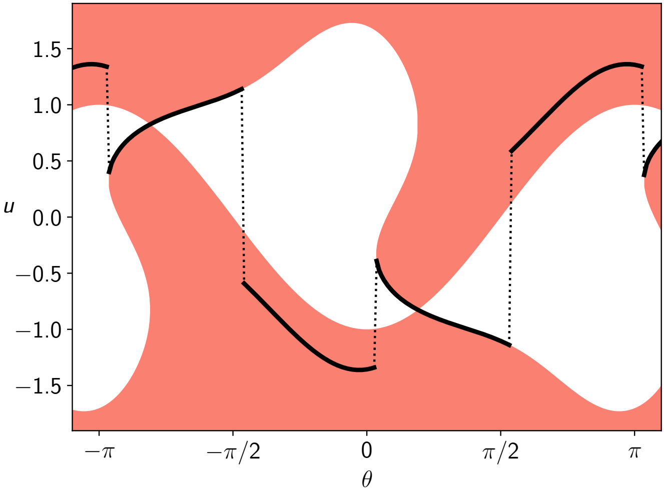


Figure 5: The control constraint set $\mathcal{U}_N(x)$ and optimal control $\kappa_N(x)$ for x on the unit circle (Rawlings et al., 2020, p. 106).

Optimal cost function with $N = 3$

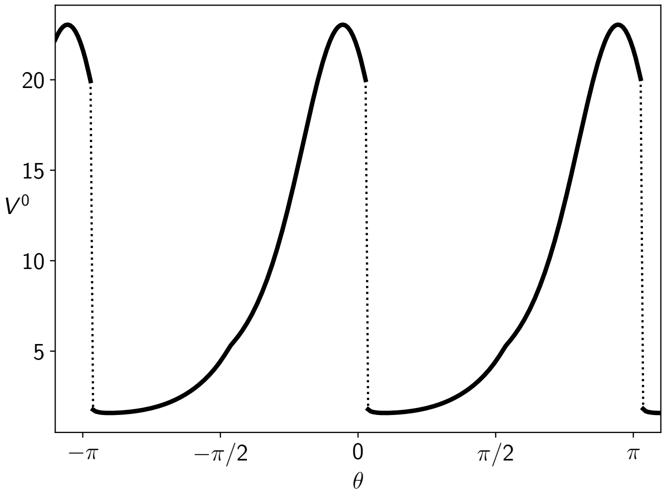


Figure 6: The discontinuity in the optimal cost for x on the unit circle

Discontinuities in optimal solution and value function

Optimal solution and parameter dependence

- Consider the general constrained optimization problem with parameter x

$$\min_{u \in \mathcal{U}(x)} V(u, x)$$

and optimal solution and value function

$$u^0(x) \qquad V^0(x)$$

- What does it take for $u^0(x)$ to be discontinuous?
- What does it take for $V^0(x)$ to be discontinuous?

Discontinuous optimal solution $u^0(x)$

It is easy to generate a smooth $V(x, u)$ and continuous constraint set $\mathcal{U}(x)$ that has a discontinuous solution $u^0(x)$ (but continuous optimal value function $V^0(x)$). Consider the following nonconvex $V(x, u)$ with the constant constraint set $\mathcal{U}(x) = \mathbb{R}$.

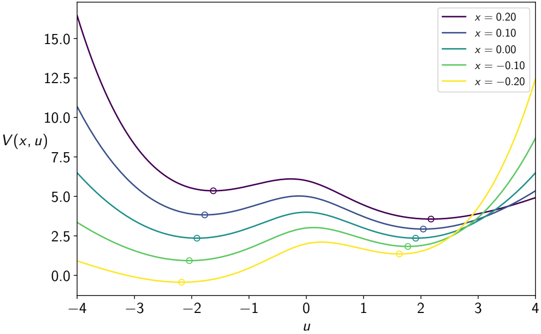


Figure 7: Smooth, nonconvex value function $V(x, u)$. There are two branches of local solutions and the optimal solution changes branches at $x = 0$.

Discontinuous optimal solution $u^0(x)$

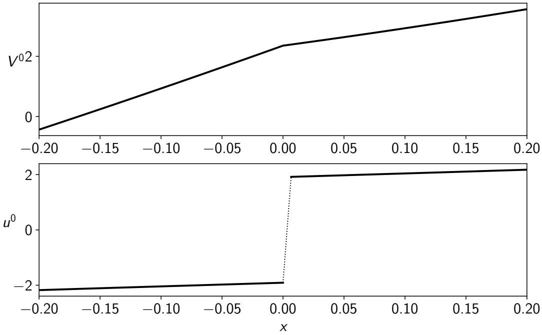


Figure 8: Smooth example with discontinuous solution and continuous value function. Note that the derivative of $V^0(x)$ is discontinuous.

Discontinuous optimal value function $V^0(x)$

To obtain a discontinuous optimal value function from a smooth $V(x, u)$, we have to make the constraint set $\mathcal{U}(x)$ discontinuous. The objective function $V(x, u)$ can be convex in this case. Consider

$$\mathcal{U}(x) = \{u \mid 1 \leq u \leq 3, \text{ or } \max(x, -1) \leq u \leq \min(-x, 1)\}$$

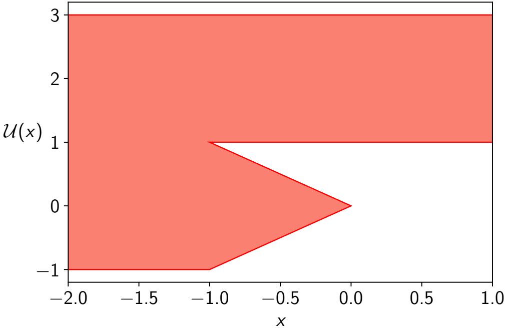


Figure 9: Discontinuous constraint set $\mathcal{U}(x)$. Note that $\mathcal{U}(x)$ at $x = 0^+$ contains no value near the point $0 \in \mathcal{U}(x)$ at $x = 0$.

Discontinuous optimal value function $V^0(x)$

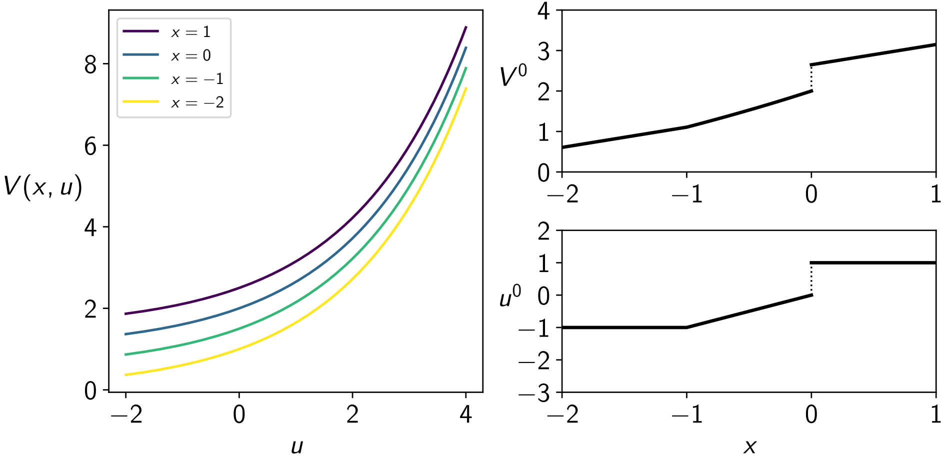


Figure 10: Smooth, convex value function $V(x,u)$ (left) and discontinuous optimal value function $V^0(x)$ and solution $u^0(x)$ (right).

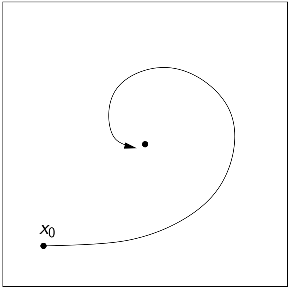
So far so good; now is the stability robust?

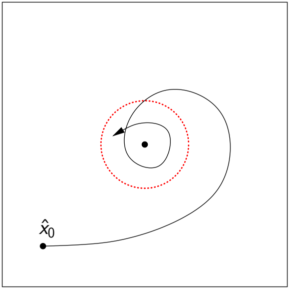
- Consider disturbances to the process (d) and state measurement (e)

$x^+ = f(x, \kappa_N(x))$	nominal system
$x^+ = f(x, \kappa_N(x + e)) + d$	nominal controller with disturbances

- How does the perturbed system behave?
- Study of *inherent* robustness motivated by Teel (2004) who showed examples for which arbitrarily small perturbations can destabilize the nominally stabilizing controller.
- If we cannot ensure desirable behavior with small disturbances, the control system will not be useful in practice.
- Every control system fails with large disturbances (think Fukushima nuclear reactor and a tsunami). But the inherent robustness of feedback control must ensure tolerance to small disturbances.

Desired behavior with and without disturbance





Nominal System

$$x^+ = f(x, u)$$
$$u = \kappa_N(x)$$

System with Disturbance

$$x^+ = f(x, u) + d$$
$$u = \kappa_N(x + e)$$

d is the process disturbance
e is the measurement disturbance

How do we define this desired behavior?

- Nominal controller with disturbances. Note $x_m = x + e$

$$\begin{aligned} x^+ &\in f(x, \kappa_N(x+e)) + d \\ x_m^+ &\in f(x_m - e, \kappa_N(x_m)) + d + e^+ \\ x^+ &\in F(x, w) \qquad w = (d, e) \text{ or } w = (d, e, e^+) \end{aligned}$$

- Inherent robustness: is the origin of the closed-loop system $x^+ \in F(x, w)$ input-to-state stable considering disturbance $w = (d, e)$ as the input?

Input-to-state stability (ISS)

Why ISS?

- Consider a system $x^+ = f(x, w)$ with input w

Definition 14 (Input-to-state stable)

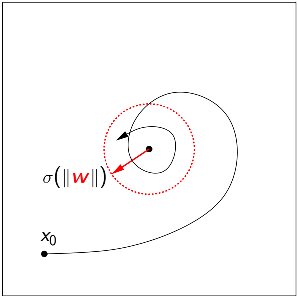
The system $x^+ = f(x, w)$ is (globally) input-to-state stable (ISS) if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ and a $\mathcal{K}$ function $\sigma(\cdot)$ such that, for each $x_0 \in \mathbb{R}^n$, and each bounded disturbance sequence $\boldsymbol{w} = (w(0), w(1), \ldots)$

$$|x(k; x_0, \boldsymbol{w})| \leq \beta(|x_0|, k) + \sigma(\|\boldsymbol{w}\|_{0:k-1})$$

for all $k \in \mathbb{I}_{\geq 0}$, $\|\boldsymbol{w}\|_{a:b} := \max_{j \in \mathbb{I}_{[a:b]}} |w(j)|$

- The main ingredient of robust stability is that the closed-loop system is ISS considering the disturbance as the input

Desired behavior with disturbance



ISS in pictures

$$x^+ \in f(x, w)$$
$$|x(k; x_0, \mathbf{w})| \leq \beta(|x_0|, k) + \sigma(\|\mathbf{w}\|_{0:k-1})$$

Note also that ISS implies the desirable behavior that if $w(k) \to 0$ as $k \to \infty$, then $x(k; x_0, \mathbf{w}) \to 0$ also.

Robust positive invariance

We also require that the system not leave an invariant set due to the disturbance.

Definition 15 (Robust Positive Invariance)

A set $\mathcal{X} \subseteq \mathbb{R}^n$ is robustly positive invariant with respect to a difference inclusion $x^+ \in f(x, w)$ if there exists some $\delta > 0$ such that $f(x, w) \subseteq \mathcal{X}$ for all $x \in \mathcal{X}$ and all disturbance sequences $\boldsymbol{w}$ satisfying $\|\boldsymbol{w}\| \leq \delta$.

Robust asymptotic stability

So, we define robust asymptotic stability as input-to-state stability on a robust positive invariant set.

Definition 16 (Robust Asymptotic Stability)

The origin of a perturbed difference inclusion $x^+ \in f(x, w)$ is robustly asymptotically stable in $\mathcal{X}$ if there exists functions $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ and $\delta > 0$ such that for all $x \in \mathcal{X}$ and $\|\boldsymbol{w}\| \leq \delta$, $\mathcal{X}$ is robustly positive invariant and all solutions $\phi(k; x, \boldsymbol{w})$ satisfy

$$|\phi(k; x, \boldsymbol{w})| \leq \beta(|x|, k) + \gamma(\|\boldsymbol{w}\|) \tag{4}$$

for all $k \in \mathbb{I}_{\geq 0}$.

Input-to-state stability Lyapunov function

In order to establish ISS, we define an ISS Lyapunov function for a difference inclusion, similar to ISS Lyapunov function defined in Jiang and Wang (2001) and Lazar, Heemels, and Teel (2013).

Definition 17 (ISS Lyapunov Function)

$V(\cdot)$ is an ISS Lyapunov function in the robust positive invariant set $\mathcal{X}$ for the difference inclusion $x^+ \in f(x, w)$ if there exists some $\delta > 0$, functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$, and function $\sigma(\cdot) \in \mathcal{K}$ such that for all $x \in \mathcal{X}$ and $\|\boldsymbol{w}\| \leq \delta$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{5}$$

$$\sup_{x^+ \in f(x,w)} V(x^+) \leq V(x) - \alpha_3(|x|) + \sigma(\|\boldsymbol{w}\|) \tag{6}$$

ISS Lyapunov function implies ISS

Proposition 18 (ISS Lyapunov stability theorem)

If a difference inclusion $x^+ \in f(x, w)$ admits an ISS Lyapunov function in a robust positive invariant set $\mathcal{X}$ for all $\|\boldsymbol{w}\| \leq \delta$ for some $\delta > 0$, then the origin is robustly asymptotically stable in $\mathcal{X}$ for all $\|\boldsymbol{w}\| \leq \delta$.

- This is a valuable result to know when trying to establish robustness of stability.
- Let's skip this proof (hooray!), but it's not difficult (Jiang and Wang, 2001; Allan, Bates, Risbeck, and Rawlings, 2017).

Inherent robustness of nominal MPC

- Our strategy now is to establish that $V_N^0(x)$ is an ISS Lyapunov function for the perturbed closed-loop system.
- We have already established the upper and lower bounding inequalities

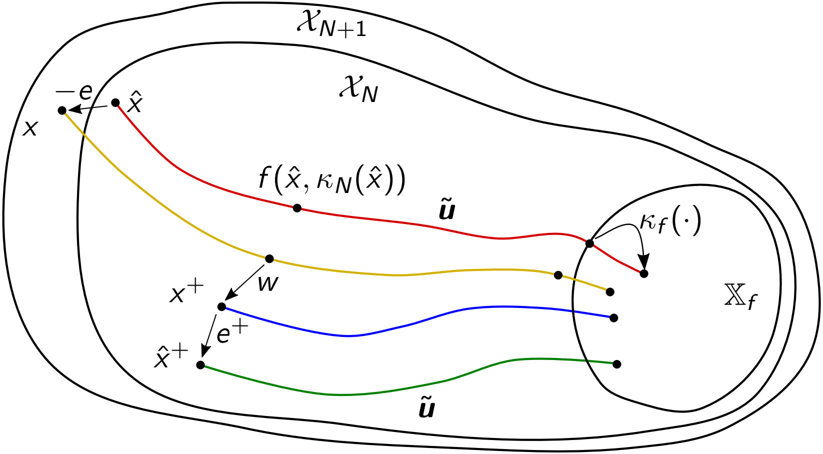
$$\alpha_1(|x|) \leq V_N^0(x) \leq \alpha_2(|x|)$$

- So we require only

$$\sup_{x^+ \in f(x,w)} V_N^0(x^+) \leq V_N^0(x) - \alpha_3(|x|) + \sigma(\|\mathbf{w}\|)$$

- That plus robust positive invariance, and we've established RAS of the controlled system.

Picture of the argument we are going to make



We have that $\hat{x}^+ = f(\hat{x} - e, \kappa_N(\hat{x})) + w + e^+$

We next compute difference in cost of red and green using $\tilde{\boldsymbol{u}}$

Note that $\tilde{\boldsymbol{u}}$ is feasible also for green, i.e., terminates in $\mathbb{X}_f := \text{lev } V_f$.

A useful tool for invoking continuity

Continuity in the language of K-functions

The usual ϵ-δ definition of continuity is equivalent to the following K-function definition (Rawlings and Risbeck, 2015).

Definition 19 (Continuity: K-function)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x if there exists a K-function $\gamma(\cdot)$ (note that the function $\gamma(\cdot)$ may depend on x) such that

$$|f(x+p) - f(x)| \leq \gamma(|p|) \quad \text{for all } |p| \in \mathrm{Dom}(\gamma) \tag{7}$$

OK, let's jump in (Allan et al., 2017)

Since $V_N(x, \boldsymbol{u})$ is a continuous function

$$\left|V_N(\hat{x}^+, \tilde{\boldsymbol{u}}) - V_N(f(\hat{x}, \kappa_N(\hat{x})), \tilde{\boldsymbol{u}})\right| \leq \sigma_V(\left|\hat{x}^+ - f(\hat{x}, \kappa_N(\hat{x}))\right|)$$

with $\sigma_V(\cdot) \in \mathcal{K}$ (note we are *not* using the possibly discontinuous $V_N^0(x)$ here). Since $f(x, u)$ is also continuous

$$\begin{aligned}
\left|\hat{x}^+ - f(\hat{x}, \kappa_N(\hat{x}))\right| &= \left|f(\hat{x} + e, \kappa_N(\hat{x})) + w + e^+ - f(\hat{x}, \kappa_N(\hat{x}))\right| \\
&\leq \left|f(\hat{x} + e, \kappa_N(\hat{x})) - f(\hat{x}, \kappa_N(\hat{x}))\right| + |w| + \left|e^+\right| \\
&\leq \sigma_f(|e|) + |w| + \left|e^+\right| \\
&\leq \sigma_f(|d|) + 2\,|d| \leq \tilde{\sigma}_f(|d|)
\end{aligned}$$

with $d := (e, w, e^+)$ and $\tilde{\sigma}_f(\cdot) := \sigma_f(\cdot) + 2(\cdot) \in \mathcal{K}$. Therefore

$$\begin{aligned}
\left|V_N(\hat{x}^+, \tilde{\boldsymbol{u}}) - V_N(f(\hat{x}, \kappa_N(\hat{x})), \tilde{\boldsymbol{u}})\right| &\leq \sigma_V(\tilde{\sigma}_f(|d|)) := \sigma(|d|) \\
V_N(\hat{x}^+, \tilde{\boldsymbol{u}}) &\leq V_N(f(\hat{x}, \kappa_N(\hat{x})), \tilde{\boldsymbol{u}}) + \sigma(|d|)
\end{aligned}$$

with $\sigma(\cdot) \in \mathcal{K}$.

Last steps

Note that for the candidate sequence $V_N(f(\hat{x}, \kappa_N(\hat{x})), \tilde{\boldsymbol{u}}) \leq V_N^0(\hat{x}) - \ell(\hat{x}, \kappa_N(\hat{x}))$ so we have that

$$V_N(f(\hat{x}, \kappa_N(\hat{x})), \tilde{\boldsymbol{u}}) \leq V_N^0(\hat{x}) - \alpha_1(|\hat{x}|)$$

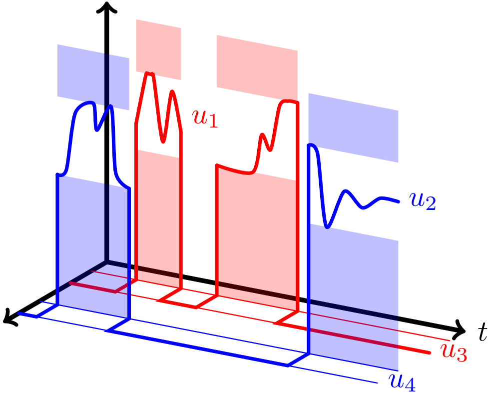
since $\alpha_1(|x|) \leq \ell(x, \kappa_N(x))$ for all x. Therefore, we finally have

$$\begin{aligned} V_N(\hat{x}^+, \tilde{\boldsymbol{u}}) &\leq V_N^0(\hat{x}) - \alpha_1(|\hat{x}|) + \sigma(|d|) \\ V_N^0(\hat{x}^+) &\leq V_N^0(\hat{x}) - \alpha_1(|\hat{x}|) + \sigma(\|\boldsymbol{d}\|) \end{aligned}$$

and we have established that $V_N^0(\cdot)$ is an ISS-Lyapunov function!
That plus robust invariance gives robust asymptotic stability of $\hat{x}$. Since $x = \hat{x} + e$, that gives also RAS of x.
Notice that neither $V_N^0(\cdot)$ nor $\kappa_N(\cdot)$ need be continuous for MPC to be inherently robust.

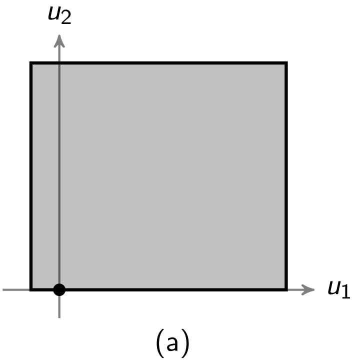
Discrete actuators

In addition to continuous actuators, many process systems also have discrete actuators that are constrained to be *integers*.



- Processes with banks of furnaces, heaters, chillers, etc.
- Scheduling models with discrete decisions.
- Switched systems with input-dependent dynamics.
- Semi-continuous variables (e.g. $u \in \{0\} \cup [1, 2]$).

Continuous and mixed continuous-discrete actuators

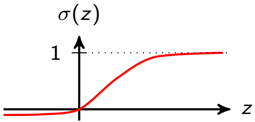


Typical input constraint sets $\mathbb{U}$ for (a) continuous actuators and (b) mixed continuous-discrete actuators; the origin ($\bullet$) is the equilibrium of interest.

Example: Driving a manual transmission

- State: vehicle velocity v
- Inputs: engine RPM $\omega \in [0, \omega_{\max}]$
  gear $\gamma \in \{1, 2, 3, 4, 5\}$

$$\frac{dv}{dt} = a_{\max}(\gamma)\, \sigma\left(R(\gamma)\omega - v\right)$$



- Maximum acceleration $a_{\max}(\gamma)$ decreases for higher gears
- Final velocity $v = R(\gamma)\omega$ increases for higher gears

Choose setpoint v_{sp} and use tracking stage cost

$$\ell(v, \omega, \gamma) = \underbrace{20\left(\frac{v}{v_{sp}} - 1\right)^2}_{\text{Track } v_{sp}} + \underbrace{8 \max\left(0, \frac{\omega - \omega_{ss}}{\omega_{\max}}\right)}_{\text{Minimize excessive } \omega} + \underbrace{(\Delta\gamma)^2}_{\text{Restrict switching}}$$

Example Simulation

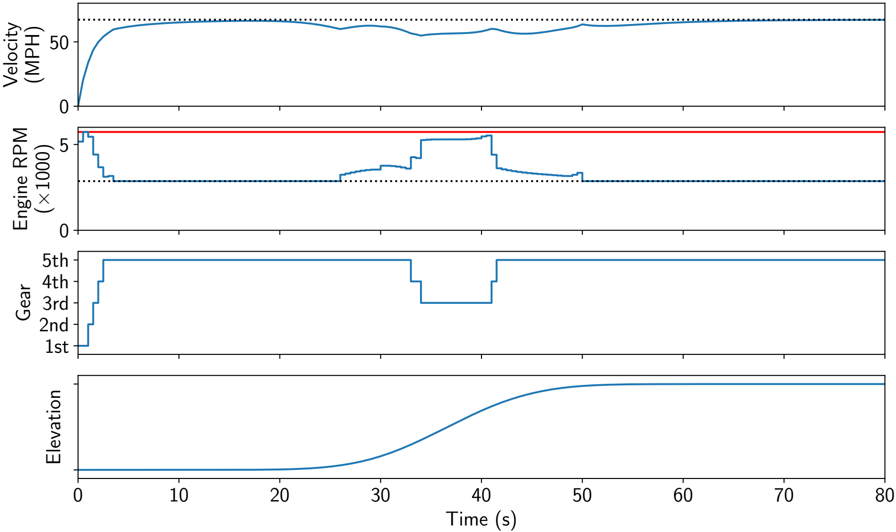


Figure 11: Closed-loop evolution of car system. Optimization performed using Bonmin.

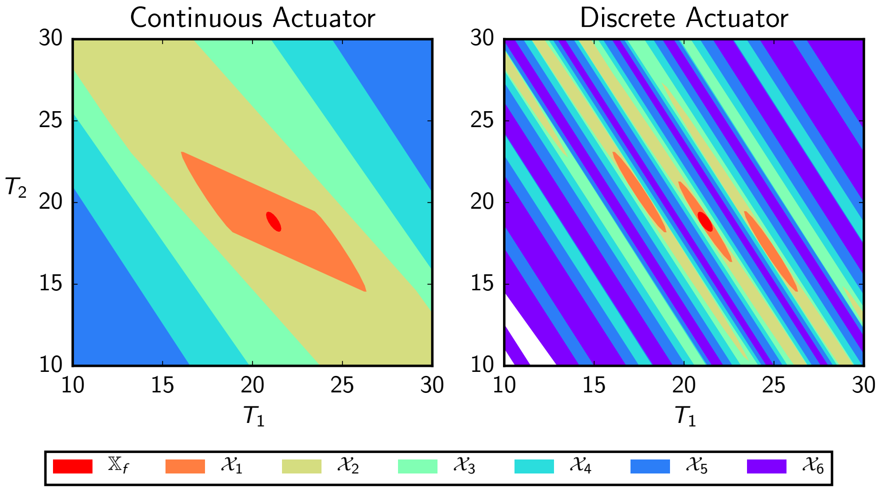
Inherent Robustness—Extension to discrete actuators

- The extension to discrete actuators is immediate
- The set $\mathbb{U}$ need not be convex, connected, etc.—it need only contain the origin

However, design choices become more striking with discrete actuators:

- Theory forbids "large" control action near the setpoint
  - System must be locally stabilizable using only unsaturated actuators
  - Discrete actuators are always saturated
- Single setpoint stabilization may no longer be an appropriate goal

- MPC is stabilizing on $\mathcal{X}_N$ but $\mathcal{X}_N$ may not be what you expect



Conclusion

- We have extended standard MPC theory to handle discrete actuators for robust stabilization of an equilibrium point
- This theory extends to periodic trajectories and economic MPC
- Based on these results we offer the following conjecture:

Theorem 20 (Folk theorem)

Any result that holds for standard MPC holds also for MPC with discrete actuators. (Rawlings and Risbeck, 2017)

- Applications include a rich class of commercial building energy optimization problems
- A current challenge is to develop better software tools for efficient, reliable *online* solution of the mixed-integer optimal control problems. See `casadi.org`

Further reading I

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Further reading II

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J. B. Rawlings, D. Q. Mayne, and M. M. Diehl. *Model Predictive Control: Theory, Design, and Computation*. Nob Hill Publishing, Santa Barbara, CA, 2nd, paperback edition, 2020. 770 pages, ISBN 978-0-9759377-5-4.

A. R. Teel. Discrete time receding horizon control: is the stability robust. In Marcia S. de Queiroz, Michael Malisoff, and Peter Wolenski, editors, *Optimal control, stabilization and nonsmooth analysis*, volume 301 of *Lecture notes in control and information sciences*, pages 3–28. Springer, 2004.

A. R. Teel and L. Zaccarian. On "uniformity" in definitions of global asymptotic stability for time-varying nonlinear systems. *Automatica*, 42:2219–2222, 2006.

Review

Recommended exercises

- Stability definitions. Exercise B.8.[3]
- Lyapunov functions. Exercise B.2–B.3.[3]
- Dynamic programming. Exercise C.1–C.2.[3]
- MPC stability results. Exercises 2.12, 2.13[3]

[3] Rawlings et al. (2020, Chapter 2, Appendices B and C). Downloadable from `engineering.ucsb.edu/~jbraw/mpc`.

Computational Exercise

Consider the following system:

$$\frac{d}{dt}x = f(x) + g(x)u$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_2 & 0 \\ x_1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix} \leq \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- For fixed u_1, system is linear.
- Far from the origin, system is difficult to stabilize along the x_2-axis.

Exercise

Design a nonlinear MPC controller to regulate the system to the origin.

- Cost functions: $\ell(x, u) = 100x'x + u'u$, $P_f(x) = 1000x'x$
- State is measured.
- No disturbances.

Compare results to linear MPC.

- Why might linear MPC be a bad idea for this system?
- Can linear MPC stabilize the system? Where?

Hints

Start with the linearized problem.

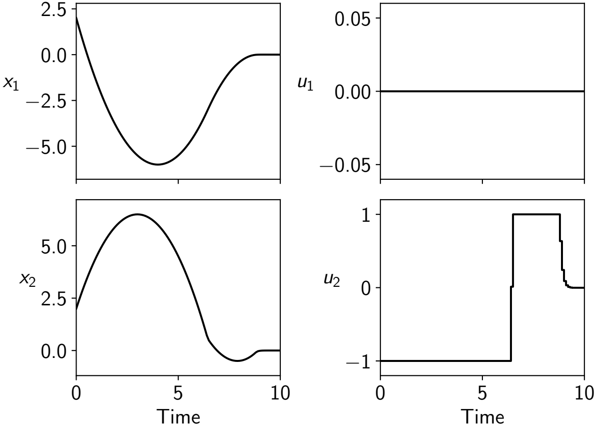


Figure 12: Trajectory using linearized system and linear MPC.

Hints

Adding nonlinearities, you should get something like this:

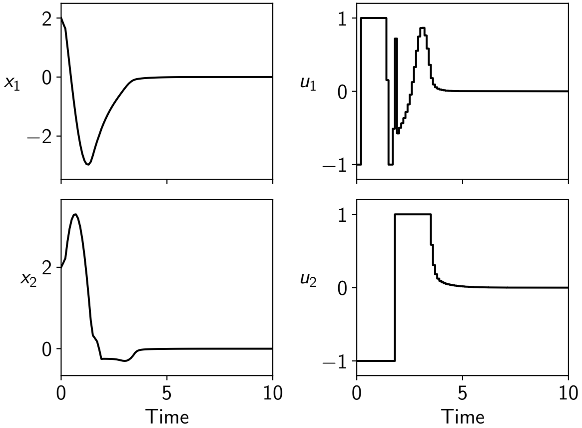


Figure 13: Trajectory using nonlinear MPC.

Finally, you can compare both on the same axes:

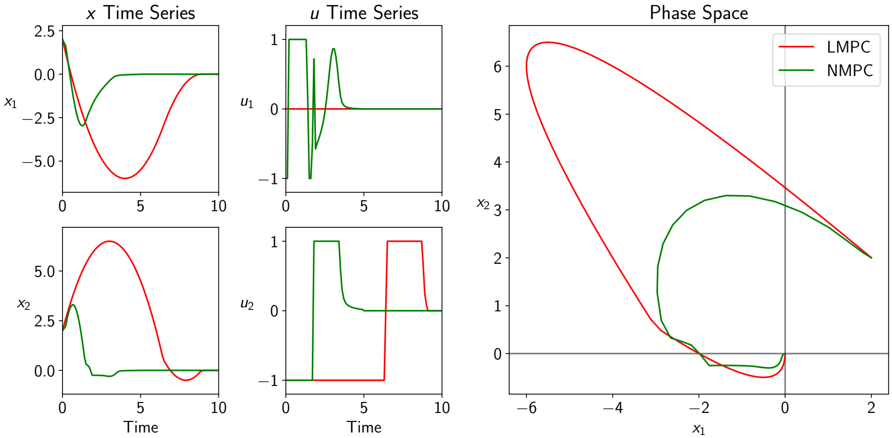


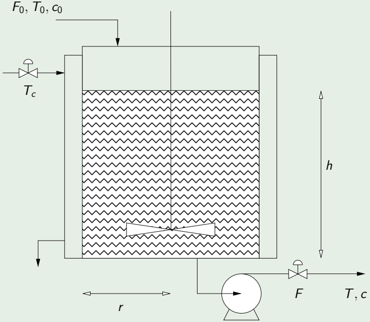
Figure 14: Comparison of linear and nonlinear MPC trajectories.

Computational Exercise 2

Consider the CSTR Example from earlier

Nonlinear CSTR

An irreversible, first-order reaction A→B occurs in the liquid phase and the reactor temperature is regulated with external cooling.



Computational Exercise 2

Simulation Parameters

1. Initial Condition and Sample Time

$$x_0 = \begin{bmatrix} 0.05c^s \\ 0.75\,T^s \\ 0.5h^s \end{bmatrix} \qquad \Delta = 0.25\,\mathrm{min}$$

2. Input Constraints

$$\begin{bmatrix} 0.975\,T_c^s \\ 0.75F^s \end{bmatrix} \leq u \leq \begin{bmatrix} 1.025\,T_c^s \\ 1.25F^s \end{bmatrix}$$

Computational Exercise 2

Reactor Startup

Using the model and parameters provided previously,

1. Simulate the performance of an uncontrolled startup by injecting the steady-state input into the system. Does the system reach the desired operating point?
2. Use linear MPC to simulate the same startup. Does the system reach the desired operating point with a linear controller?
3. Repeat the startup, but with nonlinear MPC. Does the system reach the desired operating point with a nonlinear controller? Comment on the performance differences between the various approaches.

The uncontrolled startup does not drive the reactor to the desired steady state, however both the linear and nonlinear MPC controllers do.

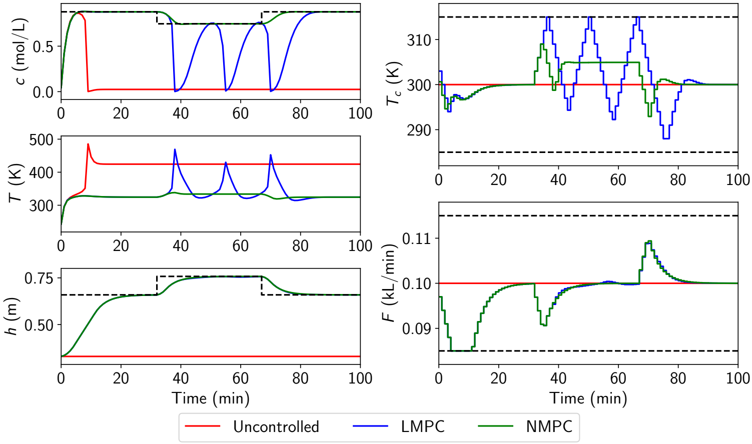


Figure 15: Solution for Reactor Startup Exercise.

Ball Maze

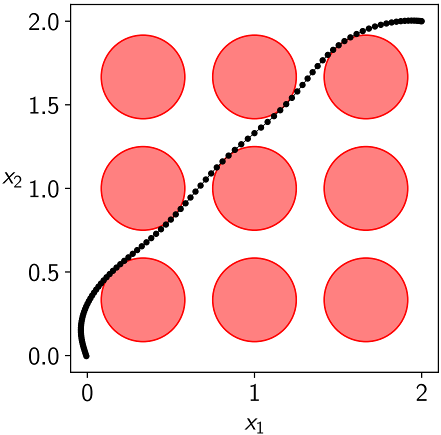


Figure 16: MPC navigating a ball maze. Although the constraints are nonconvex, we can still find a local solution.

Airplane Descent

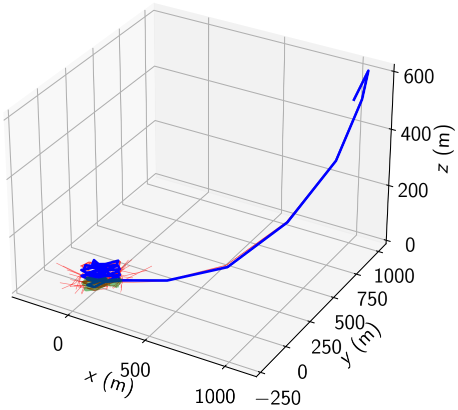


Figure 17: MPC for guiding a descending plane. While the goal is to reach a periodic holding pattern, the optimizer does not find that solution due to nonconvexity.