Nonlinear Parametric Optimization

Moritz Diehl

Systems Control and Optimization Laboratory
Department of Microsystems Engineering and Department of Mathematics
University of Freiburg, Germany

ELO-X Short Course
July 19, 2022

Optimal Control needs to solve Nonlinear Programs (NLP)

Original Optimal Control Problem in Continuous Time

$$\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T L(x,u)\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = f(x(t),u(t)) \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}$$

Assume smooth convex L, E, h, r.
Nonlinear f makes problem nonconvex.
Direct methods discretize, then optimize.
E.g. collocation or multiple shooting.

Discretized Optimal Control Problem (an NLP)

$$\begin{aligned}
\min_{x,z,u} \quad & \textstyle\sum_{k=0}^{N-1} \Phi_L(x_k,z_k,u_k) + E(x_N) \\
\text{s.t.} \quad & x_0 = \bar{x}_0 \\
& x_{k+1} = \Phi_f^{\text{dif}}(x_k,z_k,u_k) \\
& 0 = \Phi_f^{\text{alg}}(x_k,z_k,u_k) \\
& 0 \geq \Phi_h(x_k,z_k,u_k),\ k = 0,\ldots,N-1 \\
& 0 \geq r(x_N)
\end{aligned}$$

Again, smooth convex Φ_L, E, Φ_h, r.
Variables $x = (x_0,\ldots)$ and $z = (z_0,\ldots)$ and $u = (u_0,\ldots,u_{N-1})$ can be summarized in vector $w \in \mathbb{R}^{n_w}$. Linear dependence on $\bar{x}_0$.

Parametric Nonlinear Programs (pNLP) in NMPC

Newton-type methods generate a sequence $w_0, w_1, w_2, \ldots$ by linearizing and solving convex subproblems. E.g., sequential convex programming (SCP) linearizes nonconvex constraints.

Summarized pNLP

$$\begin{aligned} \min_{w \in \mathbb{R}^{n_w}} \quad & F(w) \\ \text{s.t.} \quad & 0 = G(w, \bar{x}_0) \\ & 0 \geq H(w) \end{aligned}$$

Still assume smooth convex F, H.
Nonlinear G makes problem nonconvex.

SCP subproblem at linearization point w_i

$$\begin{aligned} w_{i+1} \in \arg \min_{w \in \mathbb{R}^{n_w}} \quad & F(w) \\ \text{s.t.} \quad & 0 = G_{\mathrm{L}}(w, \bar{x}_0; w_i) \\ & 0 \geq H(w) \end{aligned}$$

First order Taylor series:
$G_{\mathrm{L}}(w, , \bar{x}_0; w_i) := G(w_i, \bar{x}_0) + \frac{\partial G}{\partial w}(w_i)(w - w_i)$

Jacobian of G does not depend on $\bar{x}_0$ so can be precomputed. SCP based algorithms work well for mildly nonlinear G, also in microsecond NMPC [cf. Zanelli 2021, Lekic 2020, Hausberger 2020]

Real-time iterations (RTI) perform only one SCP iteration per NMPC problem, i.e., change $\bar{x}_0$ during the SCP iterations.

Generic Parametric Nonlinear Programs (pNLP)

Regard arbitrary input parameter p instead of $\bar{x}_0$ and allow arbitrary dependence on p.

Generic pNLP

$$\begin{aligned} \min_{w \in \mathbb{R}^{n_w}} \quad & F(w,p) \\ \text{s.t.} \quad & 0 = G(w,p) \\ & 0 \geq H(w,p) \end{aligned}$$

pNLP after Parameter Embedding

$$\begin{aligned} \min_{w \in \mathbb{R}^{n_w},\ p \in \mathbb{R}^{n_p}} \quad & F(w,p) \\ \text{s.t.} \quad & 0 = p - \bar{p} \\ & 0 = G(w,p) \\ & 0 \geq H(w,p) \end{aligned}$$

Linear dependence on $\bar{p}$ can be achieved by making p an optimization variable, a process called "parameter embedding" ("initial value embedding" for $p = \bar{x}_0$), which eases implementation of parametric pathfollowing methods.