

**Exercise 3: Optimality Conditions and Linear Least Squares**  
(to be returned on November 15th, 10:00 a.m.)

Prof. Dr. Moritz Diehl, Katrin Baumgärtner, Yizhen Wang, David Berrazueta, Mohammed Hababeh

---

The aim of this sheet is to strengthen your knowledge in least squares estimation, optimality conditions and convexity.

Please hand in the MATLAB tasks through Grader (<http://grader.mathworks.com>). Pen-and-paper exercises can be uploaded on the Ilias course page as a pdf or handed in during the lecture.

**Exercise Tasks**

1. ON PAPER: Given the function  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $f(x) = x^\top Q x + c^\top x$  and fixed  $c \in \mathbb{R}^n$ .
  - (a) Consider the not necessarily symmetric matrix  $Q \in \mathbb{R}^{n \times n}$  and compute the gradient  $\nabla f(x) \in \mathbb{R}^n$  and the Hessian  $\nabla^2 f(x) \in \mathbb{R}^{n \times n}$  of this function for any  $x$ .
  - (b) If  $Q$  is symmetric, what properties does it have to fulfil such that the unique minimizer  $x^*$  can be computed?
  - (c) Compute the unique minimizer and the minimum function value  $f(x^*)$  under the correct assumptions. (3 points)

*Hint: You can re-write a matrix-vector product  $b = Qx$  as  $b_i = \sum_{j=1}^n Q_{ij} x_j$ , for  $i = 1, \dots, n$ .*

2. ON PAPER: Consider the function  $f(x) : \mathbb{R}^2 \rightarrow \mathbb{R}$  with

$$f(x) = x^\top \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} x + x^\top \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7.$$

- (a) Find all points that satisfy the first order necessary conditions (FONC). Which of them is the global minimizer and why? (1 points)

3. ON PAPER: In the lecture notes, the sample variance  $S^2$  is defined as

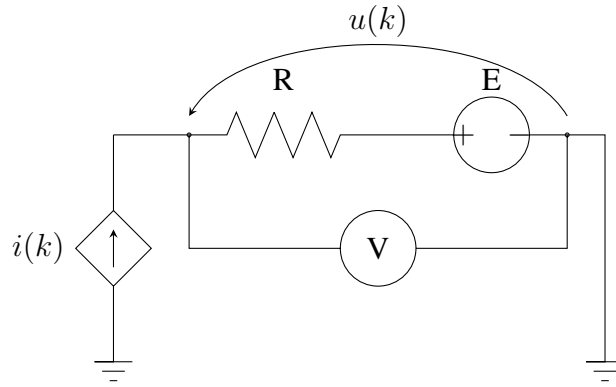
$$S^2 = \frac{1}{N-1} \sum_{n=1}^N (Y(n) - M(Y_N))^2,$$

where  $M(Y_N)$  is the sample mean (see lecture notes ch. 2.4 p. 19). Explain, why the division by  $N-1$  is preferable over  $N$ . (2 points)

*Hint: Calculate the expected value of the sample variance and compare it to the expected value of the mean squared deviations estimator. Also remember that for independent measurements  $x_j$  and  $x_k$  the following holds:*

$$\mathbb{E}\{x_j x_k\} = \begin{cases} \mathbb{E}\{x_j\} \mathbb{E}\{x_k\} = \mu^2, & \text{if } j \neq k \\ \mathbb{E}\{x_j^2\} = \sigma^2 + \mu^2, & \text{if } j = k \end{cases}$$

4. Consider the following experimental set up to estimate the values of  $E$  and  $R$ .



You obtain two datasets each containing  $N$  measurements of the voltage  $u(k)$  for different values of  $i(k)$ . The first dataset contains  $\{u_1(k)\}_{k=1}^N$  and  $\{i_1(k)\}_{k=1}^N$  and the second dataset contains  $\{u_2(k)\}_{k=1}^N$  and  $\{i_2(k)\}_{k=1}^N$ . For cleaner and simpler notation, we omit the dataset indices, e.g. instead of  $u_1(k)$  and  $u_2(k)$  we write  $u(k)$  but mean both.

We assume that the input measurement  $i(k)$  is not affected by noise, but that the measurements  $u(k)$  are affected by i.i.d. additive noise  $n_u(k)$ . Under these assumptions the measurement model is given by:

$$u(k) = m(k) + n_u(k) \text{ where } m(k) = E + R \cdot i(k).$$

Tasks:

(4 points)

- MATLAB:** Load the datasets provided on the website containing the measurements into MATLAB. Plot each dataset in a corresponding plot using the `subplot` command.
- ON PAPER:** Formulate the problem as a least squares problem where  $\theta = \begin{bmatrix} E \\ R \end{bmatrix} \in \mathbb{R}^2$  and define  $\Phi \in \mathbb{R}^{N \times 2}$  and  $y \in \mathbb{R}^N$  such that the optimizer is given by  $\theta^* = \arg \min_{\theta} \|y - \Phi\theta\|_2^2$ .
- MATLAB:** Use the least squares estimator formulated in the previous subtask to find the experimental values of  $R$  and  $E$  for each of the two datasets individually. Plot the linear fits through the respective measurement data.
- MATLAB:** For each dataset plot a histogram of the residuals defined as  $r(k) = u(k) - m(k)$ , where  $m(k) = E + R \cdot i(k)$  is the voltage determined by the model, and  $u(k)$  are the obtained measurements.
- ON PAPER:** Which dataset is noisier? Give an educated guess of the type of noise.

*This sheet gives 10 points in total.*