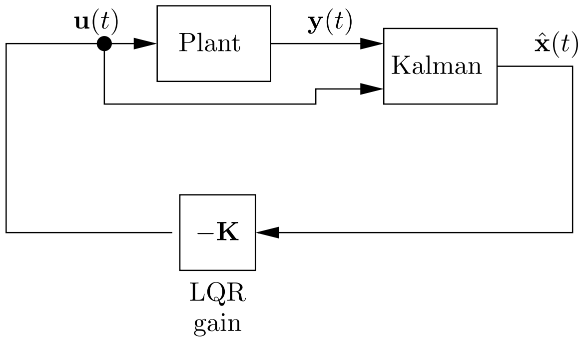
4.6 Linear Quadratic Gaussian control

Linear Quadratic Gaussian (LQG) is the combination of LQR for the control task and Kalman filter (also named Linear Quadratic Estimation - LQE) for the estimation task. The principle of LQG is expressed in the following diagram:



We consider LTI systems with noises in the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t) \tag{4.43}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t) \tag{4.44}$$

where $\mathbf{w}$ and $\mathbf{v}$ are independent zero-mean, Gaussian white noise, with covariances: $\mathbb{E}(\mathbf{w}\mathbf{w}^T) = \mathbf{Q_w}$, $\mathbb{E}(\mathbf{v}\mathbf{v}^T) = \mathbf{R_v}$.

Recall that for LTI systems without noises, there is the *separation principle*, see (4.18). This principle also holds for the case with noise, when we use a linear state feedback controller and a Luenberger observer:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} (\mathbf{A}-\mathbf{B}\mathbf{K}) & \mathbf{B}\mathbf{K} \\ \mathbf{0} & (\mathbf{A}-\mathbf{L}\mathbf{C}) \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix} \tag{4.45}$$

in which $\mathbf{e}(t) \triangleq \mathbf{x}(t) - \hat{\mathbf{x}}(t)$.

For the tracking problem where there is a reference gain (prefilter) $\mathbf{K}_f$, the control law is then $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{K}_f\mathbf{r}$, we have the combined closed loop system:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} (\mathbf{A}-\mathbf{B}\mathbf{K}) & \mathbf{B}\mathbf{K} \\ \mathbf{0} & (\mathbf{A}-\mathbf{L}\mathbf{C}) \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}\mathbf{K}_f & \mathbf{I} & \mathbf{0} \\ \mathbf{B}\mathbf{K}_f & \mathbf{I} & -\mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{w} \\ \mathbf{v} \end{bmatrix} \tag{4.46}$$

Note that $\mathbf{K}_f$ can be determined to achieve output tracking as discussed in section 3.5.

We design the LQG controller for this system by:

- using LQR to obtain the state feedback gain $\mathbf{K}$ (with chosen weighting matrices $\mathbf{Q}, \mathbf{R}$), and
- using Kalman filter to obtain the observer gain $\mathbf{L}$ (with covariance matrices $\mathbf{Q_w}, \mathbf{R_v}$).

In summary, LQG is an optimal controller design method that aims for both optimal state estimation (with Kalman filter) and optimal linear feedback control (with LQR). Depending on either continuous-time or discrete-time setting, we employ corresponding instruments from LQR and Kalman filter design.