3.4 Eigenvalue assignment for LTI SISO systems

Eigenvalue assignment (also known as *pole placement*) means putting the *eigenvalues of the closed-loop system* to given values. Before explaining the method in detail, some brief hints how to choose the eigenvalues for the closed loop shall be summarized.

- In order to achieve stability, all eigenvalues must be shifted to the left half plane, i.e. $\mathrm{Re}(\lambda_i) < 0$ for $i = 1, \ldots, n$.
- The location of the eigenvalues determines speed and overshooting/oscillations of the closed loop.
- For many systems, system dynamics is mainly determined by a dominant eigenvalue (or eigenvalue pair). In this situation, the focus should be put on this eigenvalue (pair).

As eigenvalues are the roots of the characteristic polynomial (CP), the CP of the closed-loop system is obtained with eigenvalues that we choose.

With the open-loop system represented by $(\mathbf{A}, \mathbf{B})$, using a linear controller $\mathbf{u} = -\mathbf{K}\mathbf{x}$, the closed-loop system is:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \tag{3.25}$$

For simple systems, in order to determine the *feedback gain* $\mathbf{K}$, we calculate the characteristic polynomial of the closed-loop system (3.25) as:

$$p_{\mathrm{cl}}(\lambda) \triangleq \det(\lambda\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) \tag{3.26}$$

and then compare the coefficients so that $p_{\mathrm{cl}}(\lambda)$ is the same as the CP that gives desired roots (eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$).

This will be demonstrated by the following simple example. Assume a system given by

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3.27}$$

The eigenvalues (poles) shall be placed at $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence the given characteristic polynomial for the closed loop is

$$p_{\mathrm{cl}}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + (-\lambda_1 - \lambda_2)\lambda + (\lambda_1\lambda_2) \tag{3.28}$$

This must be same as computed by using (3.26)

$$\begin{aligned} p_{\mathrm{cl}}(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) = \det(\begin{bmatrix} \lambda - 1 + k_1 & -3 + k_2 \\ k_1 & \lambda + 1 + k_2 \end{bmatrix}) \\ &= \lambda^2 + \underbrace{(k_1 + k_2)}_{=-\lambda_1-\lambda_2=3} + \underbrace{4k_1 - k_2 - 1}_{=\lambda_1\lambda_2=2} \end{aligned} \tag{3.29}$$

Comparing the coefficients as indicated results in

$$\mathbf{K} = [k_1, k_2] = [1.2, 1.8] \tag{3.30}$$

This method becomes more complicated when the size of $\mathbf{A}$ is big. In the following section, we study a general formula to compute the feedback controller via the controllable canonical form.

3.4.1 Eigenvalue assignment for systems in controllable canonical form

We assume that the system is transformed into the controllable canonical form

$$\begin{aligned}
\dot{\mathbf{z}}(t) &= \begin{bmatrix} -a_1 & \cdots & \cdots & -a_n \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \mathbf{z}(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t) & (3.31) \\
y(t) &= [b_1, \ldots, b_n]\, \mathbf{z}(t) & (3.32)
\end{aligned}$$

The feedback controller is given by

$$u(t) = -\tilde{\mathbf{K}}\mathbf{x}(t) \qquad \text{with} \qquad \tilde{\mathbf{K}} = \left[\tilde{k_1}, \ldots, \tilde{k_n}\right] \tag{3.33}$$

Then, the ODE for the feedback system read

$$\begin{aligned}
\dot{\mathbf{z}}(t) &= \tilde{\mathbf{A}}\mathbf{z}(t) + \tilde{\mathbf{B}}u(t) = (\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{K}})\mathbf{z}(t) \\
&= \underbrace{\begin{bmatrix} (-a_1 - \tilde{k_1}) & \cdots & \cdots & (-a_n - \tilde{k_n}) \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}}_{\mathbf{A}_{\mathrm{cl}} \triangleq} \mathbf{z}(t) & (3.34)
\end{aligned}$$

with the thereby defined matrix for the closed loop $\mathbf{A}_{\mathrm{cl}}$. The idea is now to give the eigenvalues $\bar{\lambda}_i$ for $i = 1, \ldots, n$ in order to obtain a certain dynamical behavior of the system. Hence, the characteristic polynomial reads

$$p_{\mathrm{cl}}(\lambda) = \prod_{i=1}^{n} (\lambda - \bar{\lambda}_i) = \lambda^n + p_1 \lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n \tag{3.35}$$

and defines the coefficients $p_1, \ldots, p_n$. As the system is given in control canonical form, the coefficients of $p_{\mathrm{cl}}(\lambda)$ determine the last row of $\mathbf{A}_{\mathrm{cl}}$, hence

$$\mathbf{A}_{\mathrm{cl}} = \begin{bmatrix} -p_1 & \cdots & \cdots & -p_n \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \tag{3.36}$$

Comparison with (3.34) yields $-a_i - \tilde{k_i} = -p_i$, and hence we get the coefficients of the controller $\tilde{k_i} = p_i - a_i$. In summary, we get the simple rule for eigenvalue assignment:

Eigenvalue assignment

Assume a system in <u>controllable canonical</u> form with characteristic polynomial (CP)

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{3.37}$$

A given CP (calculated from given eigenvalues) for the closed loop

$$\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 \tag{3.38}$$

is implemented by the state feedback controller with vector

$$\tilde{\mathbf{K}} = [(p_1 - a_1), \ldots, (p_n - a_n)] \tag{3.39}$$

3.4.2 Eigenvalue assignment for general LTI SISO systems

In the following, we will apply the eigenvalue assignment to systems given in another representation than controllable canonical form. The procedure is to consider the transformation to control canonical form first and then derive a general formula for $\mathbf{K}$.

With a controllable LTI SISO system given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \tag{3.40}$$

We use the transformation $\mathbf{T}$, defining the new state vector $\mathbf{z}(t) = \mathbf{T}\mathbf{x}(t)$ in order to get the controllable canonical form (3.31).

We have derived the formula of this transformation matrix, as in (3.24):

$$\mathbf{T} = \tilde{\mathscr{C}}\mathscr{C}^{-1} \tag{3.41}$$

Note that the system must be controllable for calculation of the inverse $\mathscr{C}^{-1}$.

Using the feedback gain $\tilde{\mathbf{K}}$ in (3.39) that is designed for the transformed system, we can obtain the relation of such controller w.r.t. the original state:

$$u(t) = -\tilde{\mathbf{K}}\mathbf{z}(t) = -\underbrace{\tilde{\mathbf{K}}\mathbf{T}}_{\mathbf{K}}\mathbf{x}(t) \tag{3.42}$$

The state feedback can be calculated as

$$\mathbf{K} = [(p_1 - a_1), \dots, (p_n - a_n)]\,\tilde{\mathscr{C}}\mathscr{C}^{-1} \tag{3.43}$$

(*) Ackermann's formula Besides the formula (3.43), we can also calculate the state feedback gain using the Ackermann's formula, to be derived as follow.

With $p(\cdot)$ the characteristic polynomial of the open-loop system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, Cayley-Hamilton theorem says that:

$$p(\mathbf{A}) = 0 \tag{3.44}$$

This result also holds with a transformation $\mathbf{z} = \mathbf{T}\mathbf{x}$, i.e. $p(\tilde{\mathbf{A}}) = 0$. Let T be the matrix that transforms the system into the controllable canonical form $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$. We now consider two polynomials:

$$p(\tilde{\mathbf{A}}) = \tilde{\mathbf{A}}^n + a_1\tilde{\mathbf{A}}^{n-1} + + \cdots + a_{n-1}\tilde{\mathbf{A}} + a_n\mathbf{I} = 0 \tag{3.45}$$

$$p_{\mathrm{cl}}(\tilde{\mathbf{A}}) = \tilde{\mathbf{A}}^n + p_1\tilde{\mathbf{A}}^{n-1} + \cdots + p_{n-1}\tilde{\mathbf{A}} + p_n\mathbf{I} \neq 0 \tag{3.46}$$

where $p_1, \dots, p_n$ obtained from the coefficients of the closed-loop characteristic polynomial with desired eigenvalues.

We then get:

$$p_{\mathrm{cl}}(\tilde{\mathbf{A}}) = p_{\mathrm{cl}}(\tilde{\mathbf{A}}) - \underbrace{p(\tilde{\mathbf{A}})}_{0} = (p_1 - a_1)\tilde{\mathbf{A}}^{n-1} + \cdots + (p_{n-1} - a_{n-1})\tilde{\mathbf{A}} + (p_n - a_n)\mathbf{I}$$

$$\Rightarrow \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} p_{\mathrm{cl}}(\tilde{\mathbf{A}}) = (p_1 - a_1)\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}\tilde{\mathbf{A}}^{n-1} + \cdots + (p_n - a_n)\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \tag{3.47}$$

Observe that due to the structure of $\tilde{\mathbf{A}}$ in the controllable canonical form, we have:

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}\tilde{\mathbf{A}} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}\tilde{\mathbf{A}}^k = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{\text{pos. n}-\text{k}} & 0 & \cdots & 0 \end{bmatrix}$$

Substituting these matrices into (3.47), we would obtain:

$$\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} p_{\text{cl}}(\tilde{\mathbf{A}}) = [(p_1 - a_1), \dots, (p_n - a_n)] = \tilde{\mathbf{K}} \tag{3.48}$$

Therefore, in order to obtain $\tilde{\mathbf{K}}$ as in (3.39), we can use the computation: $\tilde{\mathbf{K}} = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} p_{\text{cl}}(\tilde{\mathbf{A}})$. Furthermore, we can find the feedback gain for the original open-loop system using

$$\begin{aligned}
\mathbf{K} = \tilde{\mathbf{K}}\mathbf{T} &= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} p_{\text{cl}}(\tilde{\mathbf{A}})\mathbf{T} \\
&= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} p_{\text{cl}}(\mathbf{T}\mathbf{A}\mathbf{T}^{-1})\mathbf{T} \\
&= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \mathbf{T} p_{\text{cl}}(\mathbf{A}) \\
&= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \tilde{\mathscr{C}}\mathscr{C}^{-1} p_{\text{cl}}(\mathbf{A}) \\
&= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \mathscr{C}^{-1} p_{\text{cl}}(\mathbf{A})
\end{aligned}$$

The last equation is obtained due to the upper triangular structure of $\tilde{\mathscr{C}}$.

This controller realization is called Ackermann's formula, which shall be summarized

Ackermann's formula for eigenvalue assignment

Given the characteristic polynomial $p_{\text{cl}}(\lambda)$ for the closed-loop system, the control feedback has to be chosen as $\mathbf{K} = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \mathscr{C}^{-1} p_{\text{cl}}(\mathbf{A})$, i.e. we take the last row of the inverse controllability matrix, and multiply to the sum of matrix-exponentials of $\mathbf{A}$ with the coefficients of the desired CP.

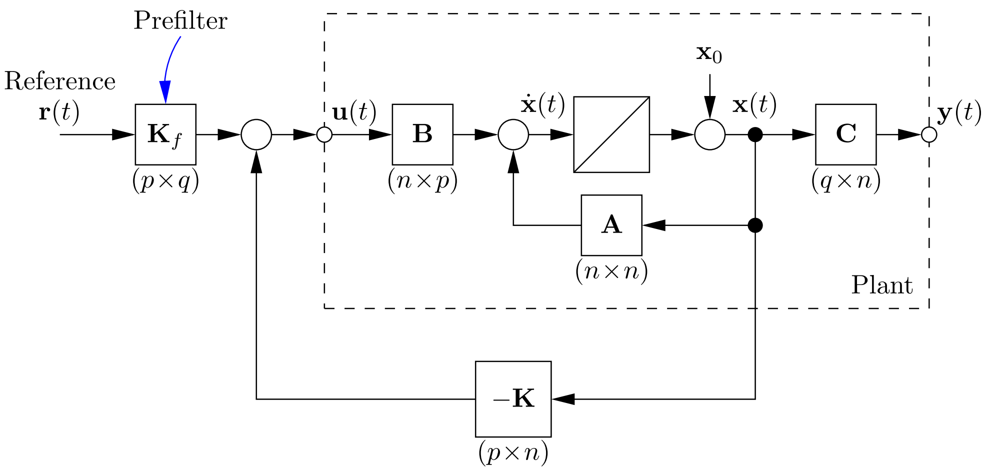
In MATLAB, Ackermann's formula is implemented with command `acker`, but it could have numerical issues. Instead, we often use command `place` that is more robust.

3.5 Prefilter

In the following, the prefilter will be discussed in order to achieve a certain set-point $\mathbf{r}$. It should be noted that most of the discussed control issues in the subsequent sections and chapters will be simplified to a zero set-point $\mathbf{x} \to \mathbf{0}$ controller for clarity of concepts. The reader should keep in mind that adding a prefilter as presented in this section will extend those to arbitrary set-points.

Requirement: we introduce a reference input $\mathbf{r}$ and demand that the output vector $\mathbf{y}(t) \to \mathbf{r}$ for $t \to \infty$.

This requirement can be achieved by adding a prefilter to the control feedback loop as follows



The adding of $\mathbf{K}_f$ does not affect the dynamics of the closed-loop system. Hence, in order to obtain a stable closed-loop system, we could design the feedback gain $\mathbf{K}$ using eigenvalue assignment as described in the previous section.

The control law reads

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{K}_f\mathbf{r} \tag{3.49}$$

For determination of the prefilter we insert (3.49) into the ODE

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) &(3.50)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) &(3.51)\end{aligned}$$

and obtain for a constant (or at least step-wise constant) $\mathbf{r}(t) = \mathbf{r}_0$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) - \mathbf{B}\mathbf{K}_f\mathbf{r}_0 \tag{3.52}$$

The system is assumed to be stable (with $\mathbf{A} - \mathbf{B}\mathbf{K}$ has all eigenvalues on the left half plane, and thus this matrix is invertible). The state evolution would lead to $\dot{\mathbf{x}}(t) \to \mathbf{0}$ for $t \to \infty$ and hence the state converges to the equilibrium point of the closed-loop system $\mathbf{x}(t) \to \bar{\mathbf{x}}_e$. Inserting these two relations into (3.52) yields

$$\mathbf{0} = (\mathbf{A} - \mathbf{B}\mathbf{K})\bar{\mathbf{x}}_e + \mathbf{B}\mathbf{K}_f\mathbf{r}_0 \tag{3.53}$$

and with (3.51)

$$\mathbf{y}_e = -\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\mathbf{K}_f\mathbf{r}_0 \tag{3.54}$$

As we demand for $\mathbf{y}_e = \mathbf{r}_0$, we need

$$-\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\mathbf{K}_f = \mathbf{I} \tag{3.55}$$

and for the prefilter

$$\mathbf{K}_f = -\left(\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\right)^{-1} \tag{3.56}$$

Without going further into detail, a final remark on the number of control variables shall be given. Regarding the dimensions of the matrices in (3.56)

$$\mathbf{K}_f = (\underbrace{\mathbf{C}}_{(q\times n)}\underbrace{(\mathbf{B}\mathbf{K} - \mathbf{A})}_{(n\times n)}{}^{-1}\underbrace{\mathbf{B}}_{(n\times p)})^{-1} \tag{3.57}$$

we get $(q \times p)$ for $\mathbf{K}_f$. Hence, it is invertible for $p = q$, i.e. for control of q output variables, q control variables (or more) are necessary.