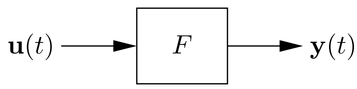
Chapter 1

Dynamical Systems in State Space

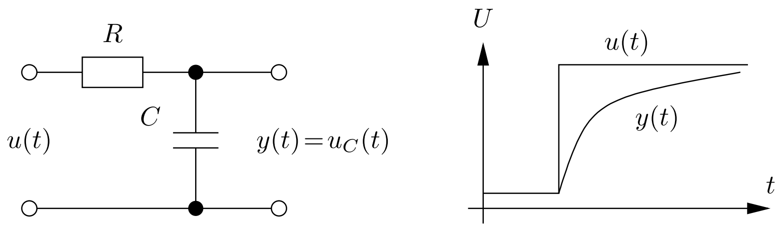
A dynamic system responds to an input signal $\mathbf{u}(t)$ with an output signal $\mathbf{y}(t)$ as depicted in the following block diagram



This behavior could be regarded as a "mapping in time domain" of a function $\mathbf{u} : t \mapsto \mathbf{u}(t)$ to a function $\mathbf{y} : t \mapsto \mathbf{y}(t)$,

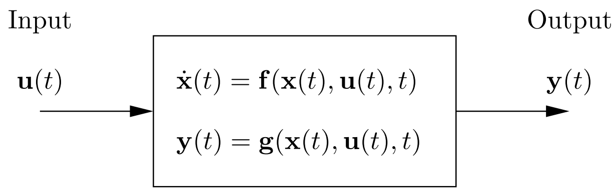
$$\mathbf{u} \mapsto \mathbf{y} = F\{\mathbf{u}\} \tag{1.1}$$

An example is a RC-lowpass circuit and its response to a step input signal



1.1 System dynamics given by Ordinary Differential Equations

If the system dynamics is given by ordinary differential equations (ODE), the system can be represented in the following "state space":



- $\mathbf{x}$ is the n-dimensional internal state of the system. It can be regarded as 'memory' of the system.
- The dynamics is given by the equations of motion in form of an ODE

  $$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{1.2}$$

  called "state equation" (or "system equation"). It determines the time evolution of the state $\mathbf{x}(t)$ by an ordinary differential equation.

- The second equation

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{1.3}$$

  is called "output equation" and maps the state (and input) to the output vector $\mathbf{y}(t)$. Note that the output, state and input vectors can have a different dimensions.

1.2 Linear Time-Invariant (LTI) System

When $\mathbf{f}, \mathbf{g}$ are linear functions and function parameters do not change over time, we have a LTI system written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) && (1.4) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) && (1.5) \end{aligned}$$

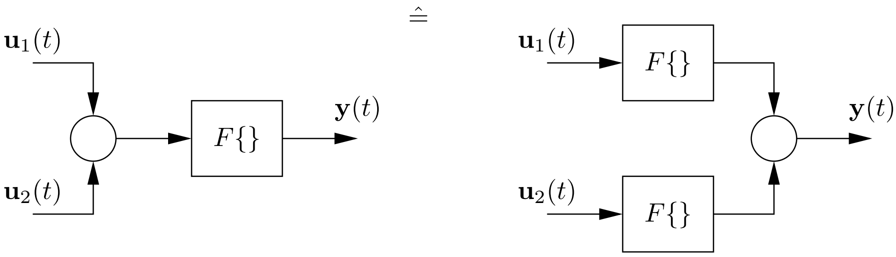
in which the vectors and matrices can be multi-dimensional: the state $\mathbf{x} \in \mathbb{R}^n$ has dimension n, the input vector $\mathbf{u} \in \mathbb{R}^p$ has dimension p and the output vector $\mathbf{y} \in \mathbb{R}^q$ has dimension q. Accordingly, the state space matrices have dimensions: $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times p}$, $\mathbf{C} \in \mathbb{R}^{q \times n}$ and $\mathbf{D} \in \mathbb{R}^{q \times p}$.

A dynamical system F is called *linear* if the following conditions are fulfilled:

1. Superposition principle

$$F\{\mathbf{u}_1 + \mathbf{u}_2\} = F\{\mathbf{u}_1\} + F\{\mathbf{u}_2\} \tag{1.6}$$

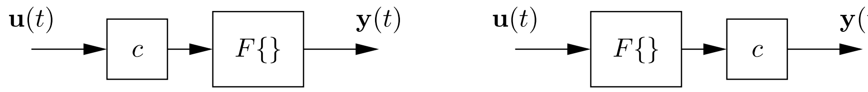
   which can be illustrated as follows



2. Principle of amplification

$$F\{c\mathbf{u}\} = cF\{\mathbf{u}\} \tag{1.7}$$

   depicted as follows



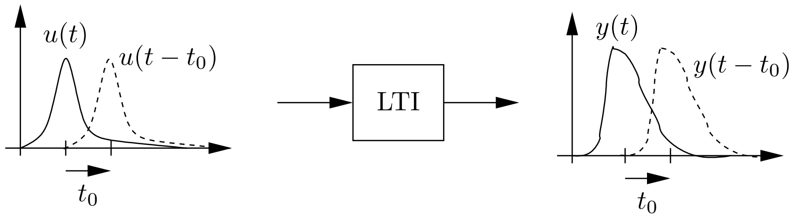
A dynamical system F is called *time-invariant*, if for any function $\mathbf{u}(t)$

$$\mathbf{y} \triangleq F\{\mathbf{u}\} \tag{1.8}$$

the equation

$$\mathbf{y}_0 = F\{\mathbf{u}_0\} \tag{1.9}$$

is valid for all t_0, where the function definitions $\mathbf{u}_0 : t \mapsto \mathbf{u}_0(t) \triangleq \mathbf{u}(t - t_0)$ and $\mathbf{y}_0 : t \mapsto \mathbf{y}_0(t) \triangleq \mathbf{y}(t - t_0)$ are introduced. This can be illustrated by



1.3 Solving the state-space ODEs

In the following, the equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1.10}$$

with $\mathbf{x}(t_0) = \mathbf{x}_0$ as initial condition will be solved.

Homogeneous solution Consider $\dot{x}(t) = ax(t)$ in which $x(t)$ is a scalar, the solution is:

$$x(t) = \mathrm{c}.e^{at} \tag{1.11}$$

where c is a constant. We want to generalize this result for the multi-dimensional case.

We use the *matrix exponential function*, which is defined by

$$e^{\mathbf{A}(t-t_0)} \triangleq \sum_{\nu=0}^{\infty} \frac{\mathbf{A}(t-t_0)^{\nu}}{\nu!} \tag{1.12}$$

Then, the function:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 \tag{1.13}$$

is the solution for

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{1.14}$$

which is the homogeneous part of (1.10).

General solution The general solution of (1.10) is

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{1.15}$$

with

$$\mathbf{\Phi}(t, t_0) \triangleq e^{\mathbf{A}(t-t_0)} \tag{1.16}$$

Note that the first term is the homogeneous solution due to the initial condition $\mathbf{x}_0$ and the second term is a convolution integral of input $\mathbf{u}(t)$.

Numerical solution Computing the integral for constructing (1.15) is not convenient. We usually use numerical solution of ODEs in order to simulate the system, numerical simulation also works for nonlinear systems.

Tools for solving ODEs numerically are available in computational software, such as functions *ode45* in MATLAB, *lsode* in Octave.

1.4 Transforming higher order ODE to state-space equations

In this section, we consider a SISO system. The dynamics is assumed to be given by a high degree differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = u \tag{1.17}$$

The superscript $^{(n)}$ denotes the n^{th} time derivative, the $a_i \in \mathbb{R}$ are constant real coefficients.

This n^{th} order ODE is transformed into a 1^{st} order system by introduction of the state $\mathbf{x} = [x_1, \ldots, x_n]^\top$ and the definitions

$$\begin{aligned} x_1 &\doteq y^{(n-1)} && (1.18)\\ x_2 &\doteq y^{(n-2)} && (1.19)\\ &\vdots && (1.20)\\ x_n &\doteq y && (1.21) \end{aligned}$$

The ODE (1.17) can then be written as

$$\begin{aligned} \dot{x}_1 &= \frac{\mathrm{d}}{\mathrm{d}t} y^{(n-1)} = y^{(n)} = -a_1 y^{(n-1)} - a_2 y^{(n-2)} - \cdots - a_n y + u \\ &= -a_1 x_1 - a_2 x_2 - \cdots - a_n x_n + u \qquad (1.22) \end{aligned}$$

or in matrix representation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & & 0 & \vdots \\ \vdots & & \ddots & & \\ 0 & 0 & & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} u(t) \qquad (1.23)$$