Exercises for Course on State-Space Control Systems (SSC)
Albert-Ludwigs-Universität Freiburg – Summer Term 2019

Exercise 1: General Information and Introduction to the State-Space Formulation

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The lecture course on State-Space Control Systems (SSC) provides basic understandings about control systems, using a state-space approach with the focus on linear dynamical systems. Students will learn how to represent dynamical systems with mathematical models, analyze them, and design feedback controllers and observers.

Organization of the Course

The course is based on two pillars, lectures and exercises, and accompanied by written material for self-study. Course language is English and all course communication is via the course homepage:

`https://www.syscop.de/teaching/ss2019/state-space-control-systems`

Lectures are on Mondays, 14:00-16:00. Recordings of the lecture will be uploaded to the course webpage.

Exercises are on Thursdays, 8:30 to 10:00, starting 2.5. Exercise sheets are published on Mondays on the course homepage. The exercises are based both on pen-and-paper exercises and on programming exercises using MATLAB. You *don't* have to hand in solutions to the exercise sheets. The solutions are presented and discussed during the following exercise session on Friday. We won't upload any solutions to the course homepage!

Course material that accompanies the lecture course comprises:

- *Feedback Systems* by Karl J. Åström and Richard M. Murray.
- *Model Predictive Control: Theory, Computation, and Design* by James B. Rawlings, David Q. Mayne, and Moritz M. Diehl.
- *Lecture Notes* by Michael Erhard, Gianluca Frison, Moritz Diehl
- Control Tutorials for MATLAB and Simulink

You can find the links to the corresponding pdfs on the course webpage.

Final Evaluation

The final grade of the course is based solely on a final written exam at the end of the semester. The **final exam** is a closed book exam, only pencil, paper, and a calculator, and two double-sided A4 pages of self-chosen formulae are allowed.

Theoretical Exercises

(1) Prove the following statement: If $y(t)$ is the output of an LTI system corresponding to input $u(t)$, then $\dot{y}(t)$ is the output corresponding to input $\dot{u}(t)$.

(2) Suppose that $x(t)$ is a solution of the following initial value problem (IVP):

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t)),\ x(t_0) = x_0.$$

Show that $\tilde{x}(\tau) = x(\tau + t_0)$ is a solution of the IVP

$$\frac{\mathrm{d}\tilde{x}(\tau)}{\mathrm{d}\tau} = F(\tilde{x}(\tau)),\ \tilde{x}(0) = x_0.$$

(3) Show that the ODE

$$\frac{\mathrm{d}^4 y}{\mathrm{d}t^4} + a_1\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} + a_2\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a_3\frac{\mathrm{d}y}{\mathrm{d}t} + a_4 y = u$$

can be rewritten as

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -a_4 & -a_3 & -a_2 & -a_1\end{bmatrix},\ B = \begin{bmatrix}0\\0\\0\\1\end{bmatrix},\ C = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$$

Matlab

(1) Read the documentation of the MATLAB functions `ode45` and `ss`.

(2) A cylindrical tank has cross section $A\mathrm{m}^2$, effective outlet area $a\mathrm{m}^2$ and inflow q in m^3/s. An energy balance shows that the outlet velocity is $v = \sqrt{2gh}$m/s, where g in m/s^2 is the acceleration of gravity and h is the distance between the outlet and the water level in the tank (in meters). Show that the system can be modeled by

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{q_{\mathrm{in}}}{A} - \frac{a}{A}\sqrt{2gh},\ q_{\mathrm{out}} = a\sqrt{2gh}$$

Use the parameters $A = 0.2$, $a = 0.01$. Simulate the system when the inflow is zero and the initial level is $h = 0.2$. Do you expect any difficulties in the simulation?

(3) Consider Example 4.4 (Inverted Pendulum) from *Feedback Systems*, Second Edition.

- Show that $x_e = (\pm n\pi, 0)^\top$, $n \in \mathbb{N}$, is an equilibrium point of (5.5) for the open-loop system, i.e. $u = 0$.
- Assume $u = 0$ and $c = 0.2$. Use the MATLAB function `ode` to simulate the state trajectory on the time interval $[0, 20]$ for several initial conditions $x_0 \in [-5, 5] \times [-2, 2]$. Plot the trajectories in one figure.
- Use the MATLAB function `quiver` to plot the flow field in the same figure.

(4) Use the function `ss` to set up a state space model in Matlab where the state and model equations are given as

$$\begin{aligned}\dot{x}_1 &= -0.25x_1 + 0.25x_2,\\ \dot{x}_2 &= -0.2x_2 + 0.4x_3,\\ \dot{x}_3 &= -0.1x_3 + 0.2u,\\ y &= x_1.\end{aligned}$$

Use the functions `step` and `impulse` to plot the step and impulse response.

(5) Consider the following continuous time dynamic system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Discretize the system with $\Delta t = 1$ using the matrix exponential where you can assume that $u(t) = u_k$ for $t \in [k \cdot \Delta t, (k+1) \cdot \Delta t[$. Calculate the matrix exponential by hand using the power series definition of the exponential function.

Set up the continuous time system in Matlab using `ss` and discretize it using `c2d`. Simulate the discrete time system using `lsim` to validate your results.