Exercises for Course on State-Space Control Systems (SSC)
Albert-Ludwigs-Universität Freiburg – Summer Term 2019

Exercise 8: LQG

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Theoretical Exercises

1. Consider a dynamic system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + w, \quad & \mathbb{E}(w(s)w^\top(t)) &= Q_w\delta(t-s) \\ y &= Cx + v, & \mathbb{E}(v(s)v^\top(t)) &= R_v\delta(t-s), \end{aligned}$$

(a) If the system is LQG controlled, the closed loop system can be described by

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \tilde{A} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \tilde{B} \begin{bmatrix} w \\ v \end{bmatrix}.$$

Derive matrices $\tilde{A}$ and $\tilde{B}$.

(b) How do the poles of the closed loop system change if we describe it by

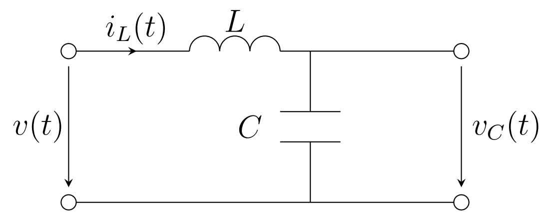
$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \tilde{A}_e \begin{bmatrix} x \\ e \end{bmatrix} + \tilde{B}_e \begin{bmatrix} w \\ v \end{bmatrix}$$

with $e := x - \hat{x}$.

2. Write down the recursive Kalman filter equations for continuous time and explain them.

MATLAB/Simulink: LC resonant circuit

The electrical circuit sketched below shows a resonant LC circuit.



The system can be described in state-space representation by

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

with

$$A = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

The state vector is defined as $x := \begin{bmatrix} i_L & v_C \end{bmatrix}^\top$ and the input as $u := v$.

The initial state of the simulation used in this exercise is $x_0 = [0, 5]$. Our aim is to design an LQG controller that stabilizes the system.

1. Open the Simulink template `ex08_sim.slx` and implement a continuous Luenberger observer with poles $[-1, -1]$. Use the `ex08_init.m` file to define the observer gain L. Compare x and $\hat{x}_{\text{luen}}$.

2. Add process disturbance w and measurement noise v, such that the system becomes stochastic:

$$\begin{aligned} \dot{x} &= Ax + Bu + w, & \mathbb{E}(w(s)w^\top(t)) &= Q_w\delta(t-s) \\ y &= Cx + v, & \mathbb{E}(v(s)v^\top(t)) &= R_v\delta(t-s), \end{aligned}$$

with

$$Q_w = \begin{bmatrix} r_{w1} & 0 \\ 0 & r_{w2} \end{bmatrix} = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \quad R_v = \begin{bmatrix} r_{v1} & 0 \\ 0 & r_{v2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hint: Use the `Band-Limited White Noise` block with noise powers $r_{w1}, r_{w2}, r_{v1}, r_{v2}$ and a sampling time $t = 10\ \mu$s. Also use different integer seeds for all random signals.

3. Implement the continuous **recursive** Kalman filter for the stochastic system. Use

$$\hat{x}_0 = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \quad \text{and} \quad P_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

as the initial states.

4. Observe the trajectory of the error covariance matrix of the Kalman filter. How can we simplify the filter structure?

5. Implement an LQR controller which feeds back

   (a) the noisy measured states $y = x + v$

   (b) the estimated state $\hat{x}$ from the Kalman filter

   Use weighting matrices

$$Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad R = 0.1$$

6. Now we will remove the voltage measurement of our system. Copy the plant and the Kalman filter in a new subsystem and replace the output matrix C by an appropriate matrix C_2. You will also have to adapt the measurement noise in your plant as well as in the Kalman filter.

7. Change the initial state of the Kalman filter to

$$\hat{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and explain the behavior of the LQG controlled closed loop system.