# Exercise 5: Linear Quadratic Regulator

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## Exercises

1. **Discrete-time LQR.** Consider the discrete-time infinte horizon linear quadratic regulator problem. The optimal linear quadratic regulator given by the feedback law $\kappa_\infty(x) = -K_\infty x$ minimizes the cost function

$$V(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} x_k^\top Q x_k + u_k^\top R u_k$$

where $x_k$ is the solution at time $k$ of

$$x_{k+1} = Ax_k + Bu_k$$

where the initial state is $x_0$ and the input sequence is $\mathbf{u}$.

Suppose that $Q, R \succ 0$ and $(A, B)$ controllable. We show that the infinite horizon regulator $\kappa_\infty(x)$ asymptotically stabilizes the origin $x_e = \mathbf{0}$ for the closed-loop system. To this end, we proceed as follows.

- Show that the optimal cost $V^*(x_0)$ defined as

$$V^*(x_0) = \min_{\mathbf{u}} \; V(x_0, \mathbf{u})$$

is finite for any $x_0$.

- Show that the cost-to-go along the closed-loop trajectory defined as

$$V_k(x_k) = \sum_{k'=k}^{\infty} x_{k'}^\top Q x_{k'} + \kappa_\infty(x_{k'})^\top R \kappa_\infty(x_{k'})$$

is monotonically decreasing for $x_k \neq \mathbf{0}$.

- Use the previous results to conclude that $x_k \to \mathbf{0}$ and $u_k \to \mathbf{0}$ as $k \to \infty$.

2. **Continuous-time LQR.** Consider the normalized, linearized inverted pendulum model which is described by

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

- Find a state feedback $u = -Kx$ that minimizes the quadratic cost function

$$J = \int_0^\infty \left(q_1 x_1^2 + q_2 x_2^2 + q_u u^2\right) \mathrm{d}t$$

where $q_2 \geq 0$ is the penality on the position, $q_1 \geq 0$ is the penalty on the velocity, and $q_u > 0$ is the penalty on the control actions.

- Compute the characteristic polynomial of the closed-loop system.
- Does $K$ change, if we replace $q_1, q_2, q_u$ by $\tilde{q}_1 = cq_1, \tilde{q}_2 = cq_2, \tilde{q}_u = cq_u$ for some constant $c > 0$.
- Simulate the closed-loop system and compare the trajectories you obtain for different values of $q_1, q_2, q_u$.