State-Space Control Systems – Exam

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Page	1	2	3	4	5	6	7
Points on page (max)	0	12	14	8	10	10	6
Points obtained							
Intermediate sum							

Mark: Exam inspected on: Signature of examiner:

Surname: First name: Matriculation number:

Major: Programme: Bachelor ☐ Master ☐ Lehramt ☐ others ☐ Signature:

Please fill in your name above. For each question, give a short formula or text answer just below the question in the space provided, and, if necessary, write on the **backpage of the same sheet** where the question appears, and add a comment "see backpage". Do not add extra pages (for fast correction, all pages will be separated for parallelization). The exam is a closed book exam, i.e. no books or other material are allowed besides 2 sheets (total 4 pages) of notes and a non-programmable calculator. Some legal comments are found in a footnote.[1]

Note: Data given in the questions are enough to figure the answers. Engineers often face situations with 'vague' data, it's indeed additional freedom for them to choose reasonable parameters.

1. **Control of DC motor speed**

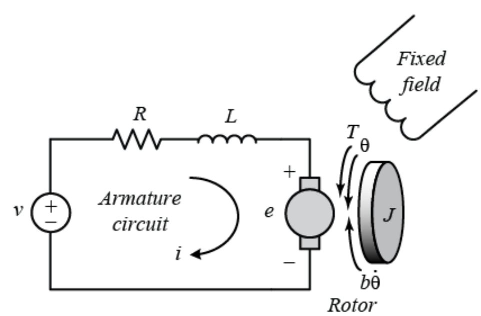

Figure 1: Schematical depiction of a DC motor.
Courtesy Control Tutorials for MATLAB and Simulink.

Given a DC motor with the following dynamical equations:

$$\ddot{\theta}(t) + 10\dot{\theta}(t) = i(t) \tag{1}$$

$$\frac{di}{dt} + 2i(t) = V(t) - 0.02\dot{\theta}(t) \tag{2}$$

[1]WITHDRAWING FROM AN EXAMINATION: In case of illness, you must supply proof of your illness by submitting a medical report to the Examinations Office. Please note that the medical examination must be done at the latest on the same day of the missed exam. In case of illness while writing the exam please contact the supervisory staff, inform them about your illness and immidiatelly see your doctor. The medical certificate must be submitted latest 3 days after the medical examination. More informations: http://www.tf.uni-freiburg.de/studies/exams/withdrawing_exam.html

CHEATING/DISTRUBING IN EXAMINATIONS: A student who disrupts the orderly proceedings of an examination will be excluded from the remainder of the exam by the respective examiners or invigilators. In such a case, the written exam of the student in question will be graded as 'nicht bestanden' (5.0, fail) on the grounds of cheating. In severe cases, the Board of Examiners will exclude the student from further examinations.

(a) Write the state-space representation of this system, taking $V(t)$ as the control input, $\dot{\theta}(t)$ as the output.

2	

Take $\mathbf{x} = \begin{bmatrix} \dot{\theta} \\ i \end{bmatrix}$, $u = V, y = \dot{\theta}$, the state-space system:

$$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}$$
$$\mathbf{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

(b) Calculate the controllability matrix of this system. Is the system fully controllable? Why?

2	

$$\mathscr{C} = [\mathbf{B}, \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$
$$det(\mathscr{C}) \neq 0$$

Therefore $\mathscr{C}$ is full rank, the system is fully controllable.

(c) Calculate the observability matrix of this system. Is the system fully observable? Why?

2	

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -10 & 1 \end{bmatrix}$$
$$det(\mathcal{O}) \neq 0$$

Therefore $\mathcal{O}$ is full rank, the system is fully observable.

(d) We want to design a full-state feedback controller $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$, so that the closed-loop system has two eigenvalues at locations: $-5 \pm j$, where j denotes the imaginary unit. Calculate the suitable matrix $\mathbf{K}$ to achieve that.

4	

The characteristic polynomial of closed-loop system:

$$p_{\mathrm{cl}}(\lambda) = (\lambda - \bar{\lambda}_1)(\lambda - \bar{\lambda}_2) = (\lambda + 5 - j)(\lambda + 5 + j) = \lambda^2 + 10\lambda + 26 \quad (3)$$

Using the controller $\mathbf{u} = -\mathbf{K}\mathbf{x}$ with $\mathbf{K} = [k_1, k_2]$, the closed-loop characteristic polynomial is:

$$\begin{aligned} p_{\mathrm{cl}}(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{A} + \mathbf{BK}) = \det\left(\begin{bmatrix} \lambda + 10 & -1 \\ k_1 + 0.02 & \lambda + 2 + k_2 \end{bmatrix}\right) \\ &= \lambda^2 + \underbrace{(k_2 + 12)}_{=10}\lambda + \underbrace{10k_2 + 20 + k_1 + 0.02}_{=26} \end{aligned} \quad (4)$$

Comparing the coefficients as indicated results in

$$\mathbf{K} = [k_1, k_2] = [25.98, -2] \quad (5)$$

(e) We want the output $\mathbf{y}$ of the controlled system to track a reference $\mathbf{r}$, using a prefilter $\mathbf{K}_f$ such that $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{K}_f\mathbf{r}$. Write the formula for computing $\mathbf{K}_f$ based on $\mathbf{K}$ computed from step (1d).

2	

$$\mathbf{K}_f = -\left(\mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}\right)^{-1} \quad (6)$$

points on page: 12	

2. **LQR**

Given a state-space continuous system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

with the following parameters:

$$\mathbf{A} = \begin{bmatrix} -10 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(a) Determine from the state-space system: which state is controllable, which state is observable.

2	

x_1 is uncontrollable and observable. x_2 is controllable and unobservable. (Students would get full score if they just write x_2 is controllable, x_1 is observable. However the info on uncontrollable / unobservable state would be helpful to deal with later questions.)

(b) Calculate the eigenvalues of the closed-loop system using a state-feedback controller $\mathbf{u} = -\bar{\mathbf{K}}\mathbf{x}$ with $\bar{\mathbf{K}} = \begin{bmatrix} 0 & 2 \end{bmatrix}$.

2	

$$\mathbf{A}_{cl} = \mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} -10 & 0 \\ 0 & -5 \end{bmatrix}$$

So the eigenvalues are: $\lambda_1 = -10$, $\lambda_2 = -5$.

(c) Explain why the output signal of the closed-loop system always exhibits a stable behaviour, regardless of the state-feedback gain $\mathbf{K}$ that is used in the controller $\mathbf{u} = -\mathbf{K}\mathbf{x}$.

2	

The state x_1 is uncontrollable, and the output $y = x_1$, hence we always have:

$$\dot{y} = \dot{x}_1 = -10x_1 = -10y, \tag{7}$$

this state has eigenvalue at $-10 < 0$, hence it is always stable.

(d) Write the algebraic Riccati equation to be solved, in order to design a state-feedback controller using linear quadratic regulator (LQR) with the cost function:

$$J(\mathbf{x}, \mathbf{u}) = \int_0^\infty \left(\mathbf{x}^\top(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\top(t)\mathbf{R}\mathbf{u}(t)\right)\, \mathrm{d}t \tag{8}$$

2	

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{9}$$

We solve (9) to obtain a $\mathbf{P} \succ 0$, with $\mathbf{Q} \succcurlyeq 0$ and $\mathbf{R} \succ 0$.

(e) **(*)** Lily the Queen of Regulation claims that $\bar{\mathbf{K}} = \begin{bmatrix} 0 & 2 \end{bmatrix}$ is an optimal controller in some sense. Your task is to trace back her claim, by finding which weighting matrices $\mathbf{Q}, \mathbf{R}$ used in the LQR formulation, such that $\bar{\mathbf{K}}$ is the solution.

6	

First, we just choose $\mathbf{R} = 1$ (we only need $\mathbf{R} > 0$, another number is just a scaling factor, as we look at (8)). Using:

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}, \tag{10}$$

we calculate $\mathbf{P}$ to obtain:

$$\mathbf{P} = \begin{bmatrix} p_{11} & 0 \\ 0 & 1 \end{bmatrix}$$

with p_{11} free, we can let it be 1.

Then using (9), we substitute values of $\mathbf{A}, \mathbf{B}, \mathbf{P}, \mathbf{K}$ to calculate $\mathbf{Q}$, and obtain:

$$\mathbf{Q} = \begin{bmatrix} 20 & 0 \\ 0 & 6 \end{bmatrix}$$

Note that a different R comes with a different Q, they are also correct as long as the ARE is satisfied.

points on page: 14	

(f) Later the Queen found that her 'optimal' state-feedback controller with $\bar{\mathbf{K}} = \begin{bmatrix} 0 & 2 \end{bmatrix}$ is indeed impossible to be implemented in practice. Point out and explain the trouble she would face when implementing this controller.

4	

With $\bar{\mathbf{K}} = \begin{bmatrix} 0 & 2 \end{bmatrix}$, we need to use the information of x_2. But x_2 is unobservable, hence there is no way to provide value of x_2 to the controller.

3. **Ball on beam**

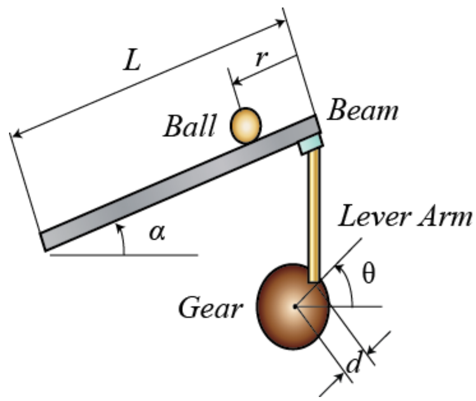

Figure 2: Schematical depiction of a ball on beam.
Courtesy Control Tutorials for MATLAB and Simulink.

Consider a ball on a controlled beam under some approximation, having the following differential equation:

$$0 = \left(\frac{J}{R^2} + m\right)\ddot{r}(t) + mg\sin\alpha(t) \tag{11}$$

and the relation $\alpha(t) = \frac{d}{L}\theta(t)$, in which J, R, m, g, d, L are positive constants.

(a) Let θ be the control input, r be the output. Write the nonlinear state-space system, using the state vector $\mathbf{x} = \begin{bmatrix} r \\ \dot{r} \end{bmatrix}$.

2	

$$\ddot{r}(t) = -\frac{mg}{\left(\frac{J}{R^2} + m\right)}\sin\left(\frac{d}{L}\theta(t)\right)$$

and the nonlinear state-space system:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{mg}{\left(\frac{J}{R^2}+m\right)}\sin\left(\frac{d}{L}u\right) \end{bmatrix}$$
$$y = x_1$$

(b) Find the equilibrium points of this system.

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Equilibrium is where $\dot{\mathbf{x}} = 0$:

$$\dot{x}_1 = 0 \Leftrightarrow x_2 = 0$$
$$\dot{x}_2 = 0 \Leftrightarrow \frac{d}{L}u = k\pi, k \in \mathbb{Z} \Leftrightarrow u = \frac{dk\pi}{L}, k \in \mathbb{Z}$$

The equilibrium can happen with any $x_1 \in \mathbb{R}$ (physical meaning: when the beam is horizontal and the ball's velocity is zero, then it doesn't move, regardless of its position).

points on page: 8	

(c) Linearize the original system around the equilibrium $\mathbf{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{u}_e = 0$. You can simplify the exposition by denoting $H = \frac{mgd}{L\left(\frac{J}{R^2}+m\right)}$.

2	

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_e,\mathbf{u}_e)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$\mathbf{B} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_e,\mathbf{u}_e)} = \begin{bmatrix} 0 \\ -\frac{mgd}{\left(\frac{J}{R^2}+m\right)L}\cos\left(\frac{d}{L}u\right)\Big|_{(\mathbf{x}_e,\mathbf{u}_e)} \end{bmatrix} = \begin{bmatrix} 0 \\ -H \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\mathbf{D} = 0$$

(d) Suppose we want to control the system around $(\mathbf{x}_e, \mathbf{u}_e)$, using the linearized model in the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t) \quad (12)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t) \quad (13)$$

where $\mathbf{w}$ and $\mathbf{v}$ are independent zero-mean, Gaussian white noises, with covariances:

$$\mathbb{E}(\mathbf{w}\mathbf{w}^T) = \mathbf{Q}_\mathbf{w}, \quad \mathbb{E}(\mathbf{v}\mathbf{v}^T) = \mathbf{R}_\mathbf{v}. \quad (14)$$

We want to design a Kalman filter for estimating states of this system.
Write the formula that would be used to obtain the state estimation, and how to find the gain of the filter.

2	

Let $\hat{\mathbf{x}}(t)$ denote the estimated state, it is a dynamical process with differential equation:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}), \quad \hat{\mathbf{x}}(0) = \mathbb{E}(\mathbf{x}(0))$$

with the gain

$$\mathbf{L} = \mathbf{P}\mathbf{C}^T\mathbf{R}_\mathbf{v}^{-1}, \text{where } \mathbf{P} = \mathbb{E}\left((\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T\right)$$

and the covariance $\mathbf{P} = \mathbf{P}^T \succ 0$ is the solution to the *algebraic Riccati equation*:

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \mathbf{P}\mathbf{C}^T\mathbf{R}_\mathbf{v}^{-1}\mathbf{C}\mathbf{P} + \mathbf{Q}_\mathbf{w} = 0 \quad (15)$$

Note: we need to use continuous Kalman filter in this context, not discrete Kalman filter.

(e) We want to use Linear Quadratic Gaussian method to control this linearized system, with an optimal state-feedback gain $\mathbf{K}$ and an optimal observer gain $\mathbf{L}$.
Draw the block diagram of the LQG control system.

2	

A diagram that expresses a control block using LQR and an observer block using Kalman filter: 1 point.
With detailed blocks showing the linear system: + 0.5 point.
With input of $\mathbf{w}$ and $\mathbf{v}$: + 0.5 point.

(f) Provided that the closed-loop system is stable using $\mathbf{K}$ and $\mathbf{L}$ from the LQG design in step (3e). Explain why we don't need to redesign $\mathbf{K}$ for every time we use a different state estimator (i.e. $\mathbf{L}$ changes), as long as the new observer guarantees that the estimated state converges asymptotically to the real state.

4	

For LTI systems, there is the *separation principle* when we use a linear state feedback controller and a Luenberger observer, as the closed-loop dynamics (denoting $\mathbf{e}(t) \triangleq \mathbf{x}(t) - \hat{\mathbf{x}}(t)$)

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} (\mathbf{A}-\mathbf{B}\mathbf{K}) & \mathbf{B}\mathbf{K} \\ \mathbf{0} & (\mathbf{A}-\mathbf{L}\mathbf{C}) \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix} \quad (16)$$

shows that eigenvalues of new closed-loop system's $\mathcal{A}$ compose of eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$.
With a stable closed-loop system, we already have $\mathbf{A} - \mathbf{B}\mathbf{K}$ stable. Hence, as long as a new observer is stable ($\equiv$ the estimated state converges asymptotically to the real state), the closed-loop system is also stable, thus we don't need to redesign $\mathbf{K}$.

points on page: 10	

(g) Uno the King of Filtering has an idea to first discretize the nonlinear continuous dynamics to obtain a discrete-time system in the form:

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k \quad (17)$$
$$\mathbf{y}_k = h(\mathbf{x}_k) + \mathbf{v}_k, \quad (18)$$

and then linearize it around $(\mathbf{x}_e, \mathbf{u}_e)$ to have the discrete-time linear system:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{w}_k \quad (19)$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{v}_k, \quad (20)$$

then he uses discrete LQR to design the state-feedback controller, and a discrete nonlinear observer for state estimation.
Propose to Uno one type of nonlinear observer that he can use, describe the steps in the iteration of such observer, with general formula.

4	

Any nonlinear observer would be accepted, e.g. those covered in the lectures: Extended Kalman Filter, Unscented Kalman Filter, Moving Horizon Estimation.

4. **MPC for discrete-time system**
Consider a discrete linear time-invariant system:

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k$$
$$y_k = \mathbf{C}x_k$$

with $\mathbf{A} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Given an arbitrary initial state $\bar{x}_0 = \begin{bmatrix} \bar{x}_0^1 \\ \bar{x}_0^2 \end{bmatrix}$ in the region: $\bar{x}_0^1 \in [-2; 2], \bar{x}_0^2 \in [-1; 1]$. We want to use state-feedback Model Predictive Control (MPC) to regulate the system, i.e. we aim to drive the states to zero. During control, we want following conditions to be satisfied:

$$-3 \leq y \leq 3 \quad (21)$$
$$-1 \leq u \leq 1 \quad (22)$$

(a) Formulate the optimization problem to be solved at each sampling time, using quadratic cost MPC with horizon N and zero terminal state constraint.

2	

The optimization problem of MPC is:

$$\min_{\substack{u_0,\ldots,u_{N-1},\\ x_0,\ldots,x_N}} \sum_{k=1}^{N-1} (x_k^T \mathbf{Q} x_k + u_k^T \mathbf{R} u_k) \quad (23)$$

$$\text{subject to} \quad (24)$$
$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k, \quad k = 0, \ldots, N-1 \quad (25)$$
$$-3 \leq x_k^1 \leq 3, \quad k = 0, \ldots, N-1 \quad (26)$$
$$-1 \leq u_k \leq 1, \quad k = 0, \ldots, N-1 \quad (27)$$
$$x_N = 0 \quad (28)$$
$$x_0 = \bar{x}_0 \quad (29)$$

(b) **(*)** Suppose we choose a short horizon: $N = 2$. Prove that there is some value of x_0 in $[-2; 2] \times [-1; 1]$ such that the MPC problem formulated at step (4a) cannot be solved (it is infeasible).

4	

Notice that the control input has little affect on state x^1, the main idea is to give an example such that with any u satisfying the constraint (27), the first state cannot be driven to zero (as required by constraint (28)).
We can use the dynamical equation to eliminate state variable x_1:

$$x_1 = \begin{bmatrix} x_0^1 + 0.1x_0^2 \\ x_0^2 + u_0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} x_0^1 + 0.1x_0^2 + 0.1(x_0^2 + u_0) \\ x_0^2 + u_0 + u_1 \end{bmatrix} \quad (30)$$

We test for a value $x_0 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, from (30) we have: $x_2^1 = -2 + 0.1u_0$. The zero terminal state constraint (28) with $N = 2$ requires $x_2^1 = 0$. However due to (27), with $u_0 \leq 1 \Rightarrow x_2^1 \leq -2 + 0.1 = -1.9$, hence (28) could not be satisfied with any permissible u_0.

points on page: 10	

(c) **(*)** We want to formulate a less restrictive MPC problem, where we replace the zero terminal state constraint by:

- a terminal cost term $x_N^T \mathbf{P} x_N$, where $\mathbf{P}$ is the solution of the discrete algebraic Riccati equation:

$$\mathbf{P} = \mathbf{Q} + \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P}\mathbf{B}\left(\mathbf{R} + \mathbf{B}^T\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A} \tag{31}$$

  in which $\mathbf{Q}$ and $\mathbf{R}$ are respectively the weighting matrices for state and control input in the stage cost; and
- a terminal constaint $x_N \in X_f$, where X_f is an invariant set for the closed-loop system using the state-feedback controller $\mathbf{u} = -\mathbf{K}\mathbf{x}$, with

$$\mathbf{K} = \left(\mathbf{R} + \mathbf{B}^T\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A}, \tag{32}$$

  'invariant' means if we have a state $\bar{x} \in X_f$, then we will have $(\mathbf{A} - \mathbf{B}\mathbf{K})\bar{x} \in X_f$.

Prove that such MPC closed-loop system is stable, using Lyapunov stability theory.

6	

For each time step t, let us denote the optimal solution of the MPC optimization problem as $(\mathbf{u}_t^*, \mathbf{x}_t^*)$ with $\mathbf{u}_t^* = [u_{t|t}, u_{t+1|t}, \ldots, u_{t+N-1|t}]^T$ and $\mathbf{x}_t^* = [x_t, x_{t+1|t}, \ldots, x_{t+N|t}]^T$. We also denote V_t^* as the optimal cost function for the problem at time step t, i.e.

$$V_t^* = \sum_{k=1}^{N-1} \left(x_{t+k|t}^T \mathbf{Q} x_{t+k|t} + u_{t+k|t}^T \mathbf{R} u_{t+k|t}\right)$$

To prove that the MPC closed-loop system is stable, we aim to **show that $V_{t+1}^* \leq V_t^*$ for any t, and use V_t^* as the Lyapunov function**. To obtain the desired inequality, we construct a feasible value for problem at time step $t+1$:

$$\tilde{\mathbf{u}}_{t+1} = [u_{t+1|t}, \ldots, u_{t+N-1|t}, u_{t+N}]^T$$
$$\tilde{\mathbf{x}}_{t+1} = [x_{t+1|t}, \ldots, x_{t+N|t}, x_{t+N+1}]^T$$

where $u_{t+N} = -\mathbf{K}x_{t+N|t}, x_{t+N+1} = \mathbf{A}x_{t+N|t} + \mathbf{B}u_{t+N} = (\mathbf{A} - \mathbf{B}\mathbf{K})x_{t+N|t}$. This means we reuse the computed result $(\mathbf{u}_t^*, \mathbf{x}_t^*)$ in step t, shift the sequence of $\mathbf{u}_t^*$ one step to the right (omit $u_{t|t}$) and append the last control input using linear controller $u = -\mathbf{K}x$, the state vector $\tilde{\mathbf{x}}_{t+1}$ is constructed just by using the discrete linear dynamics.
We denote $\tilde{V}_{t+1}$ for the cost function associated with $(\tilde{\mathbf{u}}_{t+1}, \tilde{\mathbf{x}}_{t+1})$. Next, **we use $\tilde{V}_{t+1}$ as a bridge to prove the following inequalities**:

$$V_{t+1}^* \leq \tilde{V}_{t+1} \leq V_t^* \tag{33}$$

- The first inequality of (33) can be verified by proving that $(\tilde{\mathbf{u}}_{t+1}, \tilde{\mathbf{x}}_{t+1})$ is a feasible solution of the optimization problem at time step $t+1$, while V_{t+1}^* is the optimal solution for the same problem, hence it should be lower or equal to $\tilde{V}_{t+1}$). About the feasibility of $(\tilde{\mathbf{u}}_{t+1}, \tilde{\mathbf{x}}_{t+1})$: in this setting there is no uncertainty, hence the initial condition at time step $t+1$ is $x_{t+1} = x_{t+1|t}$, the dynamical constraints at time step $t+1$ should be satisfied because they are enforced from time step t (most of values of $(\mathbf{u}_t^*, \mathbf{x}_t^*)$ are reused in $(\tilde{\mathbf{u}}_{t+1}, \tilde{\mathbf{x}}_{t+1})$); moreover, by using the linear controller $u = -\mathbf{K}x$, and the terminal constraint $x_N \in X_f$ in which X_f is an invariant set, the new terminal state constraint $x_{t+N} \in X_f$ is also satisfied. Hence $(\tilde{\mathbf{u}}_{t+1}, \tilde{\mathbf{x}}_{t+1})$ **is a feasible solution of the MPC problem at time step $t+1$, therefore $V_{t+1}^* \leq \tilde{V}_{t+1}$**.

- The second inequality of (33) can be verified by analyzing:

$$\tilde{V}_{t+1} - V_t^* = x_{t+N|t}^T\mathbf{Q}x_{t+N|t} + u_{t+N}^T\mathbf{R}u_{t+N} + x_{t+N+1}^T\mathbf{P}x_{t+N+1} - x_{t+N|t}^T\mathbf{P}x_{t+N|t} - x_{t|t}^T\mathbf{Q}x_{t|t} - u_{t|t}^T\mathbf{R}u_{t|t} \tag{34}$$

$$= x_{t+N|t}^T \underbrace{\left(\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K} + (\mathbf{A} - \mathbf{B}\mathbf{K})^T\mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}) - \mathbf{P}\right)}_{\alpha} x_{t+N|t} - x_{t|t}^T\mathbf{Q}x_{t|t} - u_{t|t}^T\mathbf{R}u_{t|t} \tag{35}$$

  We will show that $\alpha = 0$. Note that we have the formula for $\mathbf{K}$ from (32) and can substitute it to α:

$$\begin{aligned}
\alpha &= \mathbf{Q} + \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P}\mathbf{B}\mathbf{K} - \mathbf{K}^T\mathbf{B}^T\mathbf{P}\mathbf{A} + \mathbf{K}^T\mathbf{B}^T\mathbf{P}\mathbf{B}\mathbf{K} + \mathbf{K}^T\mathbf{R}\mathbf{K} - \mathbf{P} \\
&= \mathbf{Q} + \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P}\mathbf{B}\mathbf{K} - \mathbf{K}^T\mathbf{B}^T\mathbf{P}\mathbf{A} + \mathbf{K}^T(\mathbf{B}^T\mathbf{P}\mathbf{B} + \mathbf{R}) \underbrace{\mathbf{K}}_{(\mathbf{R}+\mathbf{B}^T\mathbf{P}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A}} - \mathbf{P} \\
&= \mathbf{Q} + \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P}\mathbf{B} \underbrace{\mathbf{K}}_{(\mathbf{R}+\mathbf{B}^T\mathbf{P}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A}} - \mathbf{K}^T\mathbf{B}^T\mathbf{P}\mathbf{A} + \mathbf{K}^T\mathbf{B}^T\mathbf{P}\mathbf{A} - \mathbf{P} \\
&= \mathbf{Q} + \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P}\mathbf{B}\left(\mathbf{R} + \mathbf{B}^T\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A} - \mathbf{P}
\end{aligned}$$

  And by using the Riccati equation (31), we verify that $\alpha = 0$. Thus

$$\tilde{V}_{t+1} - V_t^* = -x_{t|t}^T\mathbf{Q}x_{t|t} - u_{t|t}^T\mathbf{R}u_{t|t} \leq 0 \tag{36}$$

  due to positive (semi)definiteness of $\mathbf{Q}$ and $\mathbf{R}$.

As both inequalities of (33) are proved, we see that V_t^* is a Lyapunov function (it is positive and decreasing with t), hence the MPC closed-loop system is stable.

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