Exercises for Lecture Course on Numerical Optimal Control (NOC)
Albert-Ludwigs-Universität Freiburg – Summer Term 2019

Exercise 8: Continuous-Time Optimal Control

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Consider the following continuous-time optimal control problem:

$$\begin{aligned} \min_{x(t),\, u(t)} \quad & \int_{t=0}^{T} L(x(t), u(t))\mathrm{d}t + E(x(T)) \\ \text{s.t.} \quad & x(0) = \bar{x}_0 \\ & \dot{x}(t) = f(x(t), u(t)), \quad t \in [0, T]. \end{aligned} \tag{1}$$

1. (a) Discretize problem (1) using the explicit Euler integrator with step-size h over N intervals. Write on paper the obtained discrete-time optimal control problem.

   (2 points)

   (b) Write the first-order optimality conditions for the discretized problem obtained at point (a). Use the Hamiltonian function defined as

   $$H(x, u, \lambda) := L(x, u) + \lambda^T f(x, u) \tag{2}$$

   for compactness.

   (2 points)

   (c) Now let $N \to \infty$ and $h \to 0$. What type of problem do the conditions derived in (b) converge to?

   (3 points)

   (d) Fix $N = 2$ and apply the Newton method to the first-order optimality conditions for the discretized optimal control obtained in (b). Derive the form of the linear systems associated with the Newton steps. Order the variables as $z = (\lambda_0,\, x_0,\, u_0, \lambda_1, x_1, u_1,\, \lambda_2,\, x_2)$ and the KKT conditions accordingly as $\nabla_z \mathcal{L}(w) = 0$, where $\mathcal{L}(z)$ is the Lagrangian of the NLP.

   For notational simplicity we suggest you use the abbreviations $Q_k := h\nabla_x^2 H(x_k, u_k, \lambda_k)$, $R_k := h\nabla_u^2 H(x_k, u_k, \lambda_k)$, $S_k := h\nabla_{ux}^2 H(x_k, u_k, \lambda_k)$, $A_k := I + h\nabla_x f(x_k, u_k)$, $B_k := h\nabla_u f(x_k, u_k)$ for $k \in \{0, \ldots, N-1\}$ and $Q_N := \nabla_x^2 E(x_N)$

   (3 points)

   (e) **[Bonus]** The linear systems associated with the Newton steps in (d) can be solved exploiting the Riccati Difference Equation (equation 8.5 in the course's script). Derive this equation.

   (3 bonus points)

   (f) **[Bonus]** What kind of matrix ODE does the difference equation derived in (e) converge to for $N \to \infty$ and $h \to 0$? *Hint: if you have not solved the bonus point (e) you can refer to equation 8.5 from the course's script.*

   (2 bonus points)

This sheet gives in total 10 points and 5 bonus points