Modeling and System Identification – Microexam 1

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Surname: Name: Matriculation number:

Study: Programm: Bachelor ☐ Master ☐

Please fill in your name above and tick exactly **ONE** box for the right answer of each question below.
You can get a maximum of **10 points** on this microexam.

1. Which of the following functions is NOT convex on $x \in [-1, 1]$

(a) ☐ $x + 42$	(b) ☐ $\exp(-x)$	(c) [x] $\sin^{-1}(x)$	(d) ☐ $-\cos(x)$

2. What is the probability density function (PDF) $p_X(x)$ for a normally distributed random variable X with mean -6 and standard deviation 3? The answer is $p_X(x) = \frac{1}{\sqrt{18\pi}} \dots$

(a) ☐ $e^{-\frac{(x+6)^2}{3}}$	(b) ☐ $e^{\frac{(x-6)^2}{9}}$	(c) ☐ $e^{-\frac{(x-6)^2}{18}}$	(d) [x] $e^{-\frac{(x+6)^2}{18}}$

3. Which of the following statements does NOT hold for all PDFs $p(x)$ of a scalar random variable?

(a) ☐ $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$	(b) ☐ $p(x) \geq 0$
(c) [x] $p(x) < 1$	(d) ☐ $\int_{-1}^{1} p(x)\mathrm{d}x \geq 0$

4. What is the PDF of a random variable Y with uniform distribution on the interval $[0, \sqrt{3}]$? For $z \in [0, \sqrt{3}]$ it has the value:

(a) ☐ $p_Z(y) = \frac{\sqrt{3}}{y}$	(b) ☐ $p_Z(y) = \frac{1}{\sqrt{3}}$	(c) [x] $p_Y(z) = \frac{1}{\sqrt{3}}$	(d) ☐ $p_Y(z) = 1$

5. Consider a multi-dimensional random variable $X \in \mathbb{R}^n$ with mean value μ. What is the covariance? $\mathrm{cov}(X) = \dots$

(a) ☐ $\mathbb{E}\{(X-\mu)\}^2$	(b) ☐ $\mathbb{E}\{(X-\mu)^2\}$
(c) [x] $\mathbb{E}\{(X-\mu)(X-\mu)^\top\}$	(d) ☐ $\mathbb{E}\{(X-\mu)^\top(X-\mu)\}$

6. Consider a multi-dimensional random variable $X \in \mathbb{R}^d$. What are the dimensions of the covariance? $\mathrm{cov}(X) \in \dots$

(a) [x] $\mathbb{R}^{d\times d}$	(b) ☐ $\mathbb{R}^{d\times d^2}$	(c) ☐ $\mathbb{R}^{d^2\times d}$	(d) ☐ $\mathbb{R}^{d\times 1}$

7. Regard a random variable $X \in \mathbb{R}^n$ with mean $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$. For a fixed $b \in \mathbb{R}^n$ and $D, A \in \mathbb{R}^{m\times n}$, regard another random variable $Y \in \mathbb{R}^m$ defined by $Y = AX + Db$. The mean of Y is given by $\mu_Y = \dots$

(a) ☐ $A\Sigma^{-1}A^\top$	(b) ☐ A^+Db	(c) [x] $A\mu + Db$	(d) ☐ $A\mu$

8. In the Question above, what is the covariance matrix of Y?

(a) ☐ $D\Sigma^{-1}D^\top$	(b) ☐ $A^\top\Sigma A$	(c) ☐ $D^+\Sigma$	(d) [x] $A\Sigma A^\top$

9. Regard a random variable $X \in \mathbb{R}^n$ with zero mean and covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$. Given a vector $c \in \mathbb{R}^n$, what is the mean of $Z = c^\top XX^\top c$?

(a) ☐ $\det(\Sigma)$	(b) ☐ $c^\top \mathrm{trace}(\Sigma)\, c$	(c) [x] $c^\top\Sigma\, c$	(d) ☐ $c^\top c\, \mathrm{trace}(\Sigma)$

10. Regard a random variable $\lambda \in \mathbb{R}$ with zero mean and standard deviation d. What is the mean of the random variable $Y = \lambda^2$?

(a) ☐ 0	(b) ☐ d	(c) [x] d^2	(d) ☐ $2\lambda d$

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11. Regard another scalar random variable that has variance (d^2-2). What is its standard deviation?

(a) [ ] 0	(b) [ ] d	(c) [x] $\sqrt{d^2-2}$	(d) [ ] $(d^2-2)^2$

12. Regard the function $f:\mathbb{R}^n \to \mathbb{R}$, $\boxed{f(x)=\frac{1}{2}\|-b+Dx\|_W^2}$ (with D of rank n and W positive definite). What is the gradient of $f(x)$?

(a) [ ] Wb	(b) [ ] $W(Dx-b)$
(c) [x] $(D^\top WD)x - D^\top Wb$	(d) [ ] $\frac{1}{2}x^\top(D^\top WD)x - b^\top WDx$

13. Regard the function f from the previous question and the optimization problem $x^\star = \arg\min f(x)$. The solution is $x^\star =$

(a) [ ] $(DWD^\top)^{-1}DWb$	(b) [ ] $(DD^\top)^{-1}DWb$
(c) [ ] $(D^\top D)^{-1}D^\top b$	(d) [x] $(D^\top WD)^{-1}D^\top Wb$

14. What is the minimizer $x^\star$ of $f:\mathbb{R}^n \to \mathbb{R}$, $f(x)=\frac{1}{2}\|Ax-b\|_2^2$ if $\operatorname{rank}(A)=n$? The solution is $x^\star = \ldots$

(a) [ ] A^+	(b) [x] $(A^\top A)^{-1}A^\top b$
(c) [ ] A^+x	(d) [ ] $(A^\top A)^{-1}Ab$

15. Given a sequence of numbers $y(1),\ldots,y(N)$, what is the minimizer θ^* of the function $\boxed{f(\theta)=\sum_{k=1}^N (y(k)-2\theta)^2}$? The answer is $\theta^* = \ldots$

(a) [ ] $\frac{1}{2N}\sum_{k=1}^N y(k)^2$	(b) [x] $\frac{\sum_{k=1}^N y(k)}{2N}$	(c) [ ] $\frac{1}{4N}\sum_{k=1}^N y(k)^2$	(d) [ ] $\sum_{k=1}^N \frac{y(k)}{4N}$

16. Consider the model $\boxed{y(k) = 2\theta_1 + \frac{\theta_2}{3}x(k)^2 + \frac{\theta_3}{4}x(k)^3 + \epsilon(k)}$ and the vector of unknown parameters $\theta = (\theta_1,\theta_2,\theta_3)^\top$. The additive noise $\epsilon(k)$ is assumed to have zero mean and to be i.i.d. For a given sequence of N scalar input and output measurements $x(1),\ldots,x(N)$ and $y(1),\ldots,y(N)$, we want to compute the linear least squares (LLS) estimate $\hat{\theta}_N$ by minimizing the function $f(\theta)=\|y_N-\Phi_N\theta\|_2^2$. If $y_N=(y(1),\ldots,y(N))^\top$, how do we need to choose the matrix $\Phi_N \in \mathbb{R}^{N\times 3}$? $\Phi_N = \ldots$

(a) [ ] $\begin{bmatrix} 2 & 3x(1)^2 & 4x(1)^3 \\ \vdots & \vdots & \vdots \\ 2 & 3x(N)^2 & 4x(N)^3 \end{bmatrix}$	(b) [x] $\begin{bmatrix} 2 & \frac{x(1)^2}{3} & \frac{x(1)^3}{4} \\ \vdots & \vdots & \vdots \\ 2 & \frac{x(N)^2}{3} & \frac{x(N)^3}{4} \end{bmatrix}$	(c) [ ] $\begin{bmatrix} \frac{x(1)^2}{3} & 2 & \frac{x(1)^3}{4} \\ \vdots & \vdots & \vdots \\ \frac{x(N)^2}{3} & 2 & \frac{x(N)^3}{4} \end{bmatrix}$	(d) [ ] $\begin{bmatrix} 2 & x(1)^2 & x(1)^3 \\ \vdots & \vdots & \vdots \\ 2 & x(N)^2 & x(N)^3 \end{bmatrix}$

17. Which of the following formulas computes the covariance for a least squares estimator and a single experiment? $\hat{\Sigma}_{\hat{\theta}} = \ldots$ (all solutions were wrong in the exam, question taken out)

(a) [x] $\frac{\|y_N-\Phi_N\hat{\theta}\|_2^2}{N-d}(\Phi_N^\top\Phi_N)^{-1}$ corrected solution	(b) [ ] $\frac{\|y_N-\Phi_N\hat{\theta}\|_2}{N-d}(\Phi_N\Phi_N^\top)^{-1}$
(c) [ ] $\frac{\|y_N-\Phi_N\hat{\theta}\|_2}{N-d}(\Phi_N^\top+\Phi_N)$	(d) [ ] $\Phi_N^+\sigma_{\epsilon_N}$

18. What is the minimizer x^* of the convex function $f:\mathbb{R}_{>0}\to\mathbb{R}$, $f(x)=-4\log(x)-\frac{2}{x}$? (function not convex, question taken out)

(a) [ ] $x^*=-2$	(b) [ ] $x^*=-1/2$	(c) [ ] $x^*=e^{1/4}-2$	(d) [x] $x^*=1/2$

19. Which of the following is NOT a name of a probability distribution?

(a) [ ] Uniform	(b) [ ] Gaussian	(c) [x] Newton	(d) [ ] Laplace

20. Given a uniformly distributed random variable X on the interval $[-1,1]$, regard the following X - dependent random variables Y. For one of the examples X and Y are uncorrelated. Which one?

(a) [ ] $y=\sin(x)$	(b) [x] $y=\cos(x)$	(c) [ ] $y=x^3$	(d) [ ] $y=e^x$

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