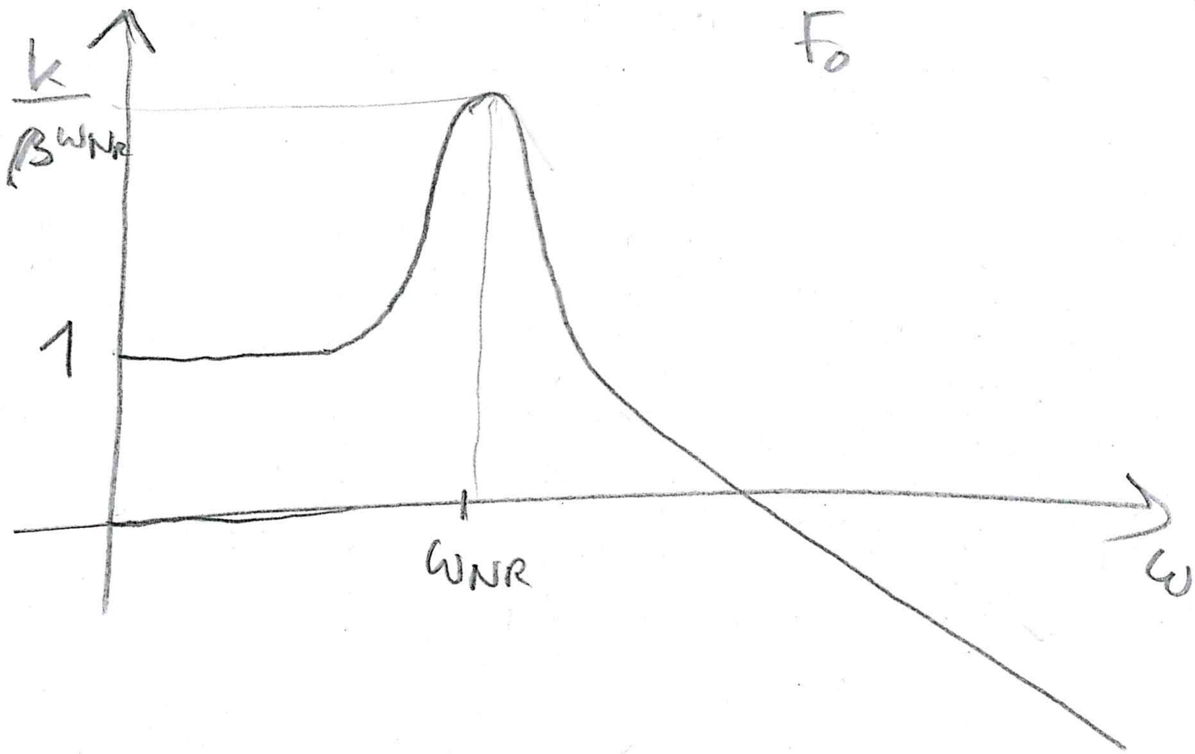
BODE DIAGRAM OF $\frac{|F_0^{SPRING}(\omega)|}{F_0}$



AMPLIFICATION FACTORS CAN BE 5-10, SO RESONANCE SHALL TYPICALLY BE AVOIDED!

AT VERY LOW FREQUENCIES, SPRING FORCE EQUALS APPLIED FORCE, I.E. STATIC ANALYSIS IS SUFFICIENT (CF. SECTION 4.2)

4.3.2 EIGENMODES

FOR SPRING-MASS-(DAMPER) SYSTEMS WITH MORE THAN ONE DEGREE OF FREEDOM, THE DISPLACEMENT CAN BE DESCRIBED BY A VECTOR $w(t) \in \mathbb{R}^n$ AND THE EQUATION OF MOTION BECOMES

$$M \cdot \ddot{w} + D \cdot \dot{w} + K \cdot w = F(t), \quad M \in \mathbb{R}^{4\times4}, \; K \in \mathbb{R}^{4\times}$$

MASS MATRIX   STIFFNESS MATRIX

IF DAMPING IS NEGLECTED $(D=0)$,
NATURAL RESONANCES MUST SATISFY, WITH
$w(t) = \bar{w} \cdot e^{j\omega t}$ $\quad \bar{w} \in \mathbb{R}^n$

$$M\ddot{w} + Kw = 0$$

I.E.

$$-\omega^2 M\bar{w} + K\bar{w} = 0 \iff (M^{-1}K - \omega^2 I)\bar{w} =$$

THIS IS AN EIGENVALUE EQUATION FOR
MATRIX $(M^{-1}K) \in \mathbb{R}^{n \times n}$, AND WE KNOW THERE
ARE n EIGENVALUES WITH n EIGENVECTORS $\bar{w}$
("EIGENMODES")

AS BOTH M AND K ARE POSITIVE
DEFINITE, EIGENVALUES OF $M^{-1}K$ ARE REAL & POSITIVE

OFTEN, WE ARE ONLY INTERESTED IN THE
EIGENMODES WITH LOWEST EIGENFREQUENCY.

4.3.3 RAYLEIGHS METHOD

IF WE HAVE A GUESS OF $\underbrace{\bar{w}}_{\text{AN EIGENMODE}}$,

WE CAN USE THE EQUATION

$$K\bar{w} = \omega^2 M\bar{w} \qquad (1)$$

TO FIND THE CORRESPONDING ω.

(1) IS OVERDETERMINED, BUT MULTIPLICATION BY $\left(\frac{1}{2}\bar{w}^T\right)$ GIVES

$$\underbrace{\frac{1}{2}\bar{w}^T K\bar{w}}_{\text{ENERGY STORED IN SPRING AT MAXIMAL DISPLACEMENT}} = \underbrace{\omega^2 \cdot \frac{1}{2}\bar{w}^T M\bar{w}}_{\text{KINETIC ENERGY AT ZERO DISPLACEMENT (BUT MAXIMUM VELOCITY)}}$$

IF THE GUESS OF $\bar{w}$ WAS GOOD, THIS METHOD CAN GIVE SURPRISINGLY ACCURATE ESTIMATES OF ω

(TO CHECK, ONE CAN INSERT ω & $\bar{w}$ IN (1))

BACKGROUND FOR INTERESTED READERS:

WHAT IS THE ERROR OF RAYLEIGHS METHOD?

ASSUME $\omega_0 \in \mathbb{R}$ AND $w_0 \in \mathbb{R}^4$ ARE THE TRUE EIGEN-PAIR, I.E. SATISFY

$$K w_0 = \omega_0^2 M w_0$$ EXACTLY.

$\bar{w} = w_0 + \Delta w$ WITH Δw THE ERROR OF OUR GUESS

WE THEN GET

$$\omega^2 = \underbrace{\frac{\frac{1}{2}\bar{w}^T K \bar{w}}{\frac{1}{2}\bar{w}^T M \bar{w}}}_{=:f(\bar{w})} \approx \underbrace{\frac{\frac{1}{2} w_0^T K w_0}{\frac{1}{2} w_0^T M w_0}}_{=f(w_0)=\omega_0^2} + \nabla f(w_0)^T \Delta w + O(\|\Delta w\|^2)$$

BUT HERE $\nabla f(w_0) = \dfrac{\left(\frac{1}{2} w_0^T M w_0\right) K w_0 - \left(\frac{1}{2} w_0^T K w_0\right) M w_0}{\left(\frac{1}{2} w_0^T M w_0\right)^2}$

$$= \frac{K w_0 - \omega_0^2 M w_0}{\left(\frac{1}{2} w_0^T M w_0\right)} = 0$$

THUS,

$\omega^2 = \omega_0^2 + O(\|\Delta w\|^2)$

ERROR IS OF SECOND ORDER

RAYLEIGHS METHOD CAN BE EXTENDED EVEN TO "DISTRIBUTED PARAMETER SYSTEMS" DESCRIBED BY "PARTIAL DIFFERENTIAL EQUATIONS" (PDE) AS THEY APPEAR IN OSCILLATING BEAMS

4.3.4 DYNAMIC BEAM EQUATION

EULER BERNOULLI & LAGRANGE GIVE THE FOLLOWING PDE

$$\boxed{\frac{d^2}{dx^2}\left(E\,I(x)\frac{d^2w}{dx^2}\right) = -\mu(x)\frac{d^2w}{dt^2} + q(x,t)}$$

WITH $\mu(x)$ THE MASS DENSITY AND $q(x,t)$ THE DISTRIBUTED LOAD, AND THE TIME VARYING SOLUTION $w(x,t)$ (THERE IS NO DAMPING).

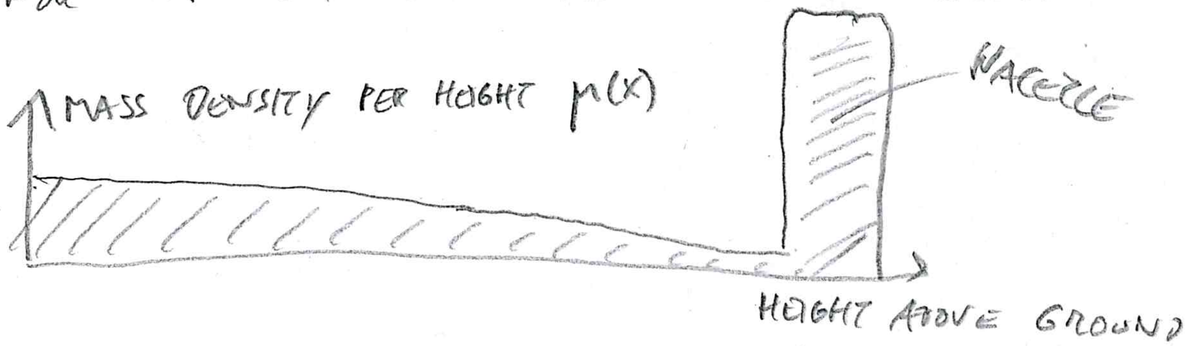
NOTE THAT THIS IS A LINEAR PDE, WHICH AFTER SPACE DISCRETIZATION WOULD RESULT IN A LINEAR ODE [illegible]

4.3.5 TOWER EIGENMODES

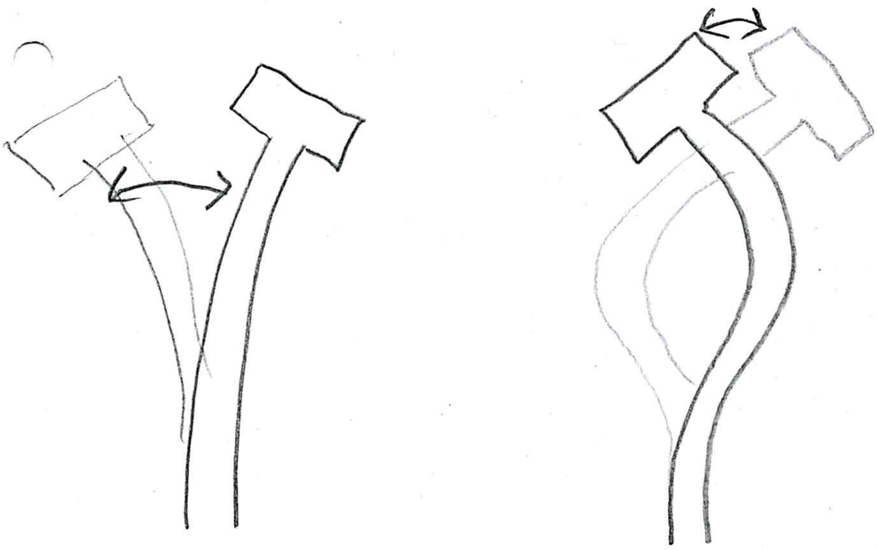
BOTH NACELLE AND TOWER HAVE MASS, E.G. MHI-VESTAS V164:

NACELLE : 400 t $= m_{\text{NACELLE}}$
TOWER : 400 t $= m_{\text{TOWER}}$
HUB HEIGHT : 100 m $= L$

SO THE EIGENMODES NEED TO BE COMPUTED FOR A VERY UNEQUAL MASS DISTRIBUTION



THE LOWEST TWO EIGENMODES LOOK APPROXIMATELY AS FOLLOWS:



LOWEST 2ND-LOWEST FREQUENCY

ONLY FINITE ELEMENT METHOD (FEM) OR OTHER NUMERICAL TECHNIQUES ARE ABLE TO COMPUTE EIGENMODES

KINETIC ENERGY:

$$E_{KIN} = \frac{1}{2}\int_0^L \mu(x)\left(\frac{\partial w(x,t)}{\partial t}\right)^2$$

ELASTIC POTENTIAL ENERGY:

$$E_{ELA} = \frac{1}{2}\int_0^L E(x)\,I(x)\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2$$

ASSUMING FOR EXAMPLE $w(x,t) = \bar{w}(x) \cdot e^{j\omega t}$

WITH $\bar{w}(x) = A_0 \cdot \frac{x^2}{L^2}$ FOR A ROUGH APPROXIMATION OF THE LOWEST EIGENMODE, AND ASSUMING CONSTANT MASS $\mu(x)$ AND $E(x)$ AND $I(x)$ THROUGHOUT THE TOWER, WE WOULD GET THE FOLLOWING ESTIMATE BY USING RAYLEIGHS METHOD

$$E_{KIN} = \omega^2 \cdot \left(\frac{1}{2} \int_0^L \frac{m_{TOWER}}{L} \left(\frac{A_0}{L^2}\right)^2 (x^2)^2 \, dx + \frac{1}{2} m_{NACELLE} \, A_0^2 \right)$$

$$= \frac{\omega^2}{2} A_0^2 \left(\frac{m_{TOWER}}{L^5} \int_0^L x^4 \, dx + m_{NACELLE} \right)$$

$$= \omega^2 \frac{A_0^2}{2} \left(\frac{1}{5} m_{TOWER} + m_{NACELLE} \right)$$

AND

$$E_{ELA} = \frac{1}{2} \int_0^L EI \left(\frac{A_0 2}{L^2} \right)^2 dx$$

$$= \frac{A_0^2}{2} EI \left(\frac{4}{L^4} \right) L = \frac{A_0^2}{2} EI \frac{4}{L^3}$$

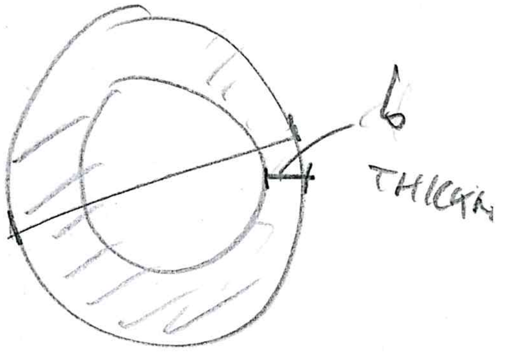
EQUATING $E_{KIN} = E_{ELA}$ GIVES

$$\omega^2 \left(\frac{m_{TOWER}}{5} + m_{NACELLE} \right) = \frac{4 EI}{L^3}$$

DENSITY OF STEEL: $\rho = 8\ t/m^3$

$100\ m = L$

CROSS SECTION SURFACE A_{CS}



$$L \cdot A_{CS} \cdot \rho = m_{TOWER}$$

$$\Leftrightarrow A_{CS} = \frac{m_{TOWER}}{L \cdot \rho} = \frac{400\ t}{100\ m} \cdot \frac{m^3}{8t} = \frac{1}{2}\ m^2$$

WE KNOW TOWER BASE DIAMETER $D = 6\ m$

$$A_{CS} = \pi D \cdot b \Leftrightarrow b = \frac{A_{CS}}{\pi D} = \frac{\frac{1}{2} m^2}{19\ m} = \frac{1}{38}\ m$$

$$= 0.025\ m$$

$$I = \frac{\pi}{4} R^4 - \frac{\pi}{4}(R-b)^4 \simeq \pi R^3 \cdot b = \pi \cdot 27\ m^3 \cdot 0.025\ m$$

$$\approx 2{,}5\ m^4$$

$E = 200\ GPa$

$$\omega^2 = \frac{4EI}{L^3 \left(\frac{m_{TOWER}}{5} + m_{NACELLE}\right)} = \frac{4 \cdot 200 \cdot 10^9 \frac{N}{m^2} \cdot 2{,}5\ m^4}{10^6\ m^3\ (480 \cdot 10^3\ kg)}$$

$$= \frac{2000 \cdot 10^9\ N}{480 \cdot 10^9\ m \cdot kg} \approx 4\ \frac{1}{s^2} \Leftrightarrow \omega \simeq 2\ \frac{rad}{s}$$

$$\omega = \frac{2\pi}{T} \Leftrightarrow T = \frac{2\pi}{\omega} = \pi\,s \quad 73$$

$$\approx 3\,s$$