AXIAL FORCE THRUST OF COMPLETE ANNULUS :

$$dF_A = \left(dL \cdot \cos\phi + dD \cdot \sin\phi\right) \cdot B$$

$$= \frac{1}{2} \rho w^2 B \cdot c \left(c_L \cdot \cos\phi + c_D \cdot \sin\phi\right) dr \quad (1)$$

<u>TANGENTIAL FORCE :</u>

$$dF_T = -\left(dD \cdot \cos\phi - dL \cdot \sin\phi\right) \cdot B$$

$$= \frac{1}{2} \rho w^2 B c \left(c_L \cdot \sin\phi - c_D \cos\phi\right) dr \quad (2)$$

AXIAL THRUST & TORQUE CAUSE THE INDUCTION (a & a') OF THE FLOW BY MOMENTUM BALANCE (AS BEFORE)

AXIAL:

$$dF_A = d\dot{m} \cdot \left(2 \cdot a \cdot V_\infty\right) \qquad \text{WITH } d\dot{m} = \rho \cdot 2\pi r \cdot dr \; V_\infty (1-a)$$

$$= \frac{1}{2} \rho \, 2\pi r \, dr \, V_\infty^2 \underbrace{4a(1-a)}_{= C_T(a)} \quad (3)$$

TANGENTIAL FORCE:

$$dF_T = d\dot{m}\,(2a' r \Omega)$$

$$= \frac{1}{2}\rho\, 2\pi r \cdot dr \cdot U_\infty\, 4a'\,(1-a)\; r \cdot \Omega \qquad (4)$$

WE GOT TWO EQUATIONS : (1) & (3)

AND (2) & (4)

FOR TWO UNKNOWNS : a & a'

CAN BE SOLVED WITH ROOTFINDER PROCEDURE, TYPICALLY OF NEWTON-TYPE, FOR FINDING ZEROS / ROOTS OF NONLINEAR FUNCTION: $F: \mathbb{R}^n \to \mathbb{R}^n$

$$\boxed{F(x) = 0}$$

(ITERATIVE SOLUTION OF LINEAR SYSTEMS)

LET US FIRST COLLECT & SIMPLIFY OUR EQUATIONS :

(1) & (3) YIELD:

$$\frac{1}{2}\rho W^2 B \cdot c \left(c_L \cos\phi + c_D \sin\phi\right) dr = \frac{1}{2}\rho\, 2\pi r \cdot dr \cdot U_\infty^2\, 4a(1-a)$$

$$\Longleftrightarrow \boxed{W^2 \cdot B \cdot c \left(c_L \cos\phi + c_D \sin\phi\right) = 2\pi r\, U_\infty^2\, 4a(1-a)} \quad (5)$$

"BLADE ELEMENT SIDE" "MOMENTUM SIDE"

(2) & (4) YIELD

$$\frac{1}{2}\rho W^2 B c \left(c_L \sin\phi - c_D \cos\phi\right) dr = \frac{1}{2}\rho\, 2\pi r \cdot dr \cdot U_\infty \cdot 4a'(1-a)\, r \cdot \Omega$$

$$\Longleftrightarrow \boxed{W^2 B c \left(c_L \sin\phi - c_D \cos\phi\right) = 2\pi \cdot r^2 \cdot U_\infty\, \Omega\, 4a'(1-a)} \quad (6)$$

USING SOLIDITY $\boxed{\sigma_r = \frac{Bc}{2\pi r}}$ AND LOCAL SPEED RATIO $\boxed{\lambda_r = \frac{r\Omega}{U_\infty}}$

AND

$$W = \sqrt{U_\infty^2 \lambda_r^2 (1+a')^2 + U_\infty^2 \cdot (1-a)^2} = U_\infty \sqrt{\lambda_r^2 (1+a')^2 + (1-a)^2}$$

WE GET

$$(5) \Leftrightarrow \left(\lambda_r^2(1+a')^2+(1-a)^2\right)\sigma_r\left(C_L\cos\phi + C_D\sin\phi\right) = 4a(1-a)$$

AND

$$(6) \Leftrightarrow \left(\lambda_r^2(1+a')^2+(1-a)^2\right)\sigma_r\left(C_L\sin\phi - C_D\cos\phi\right) = \lambda_r \cdot 4a'(1-a)$$

USING THE EXPRESSIONS $\sin\phi = \dfrac{(1-a)}{\sqrt{\lambda_r^2(1+a')^2+(1-a)^2}}$ AND

$$\cos\phi = \frac{\lambda_r\cdot(1+a')}{\sqrt{\ldots}}$$

WE GET THE EQUIVALENT FORMULAE

$$(5) \Leftrightarrow \sqrt{\lambda_r^2(1+a')^2+(1-a)^2}\;\sigma_r\left(C_L\cdot\lambda_r(1+a') + C_D\cdot(1-a)\right) = 4a(1-a) \qquad (7)$$

AND

$$(6) \Leftrightarrow \sqrt{}\;\sigma_r\left(C_L(1-a) - C_D\lambda_r(1+a')\right) = \lambda_r\, 4a'(1-a) \qquad (8)$$

$$\Leftrightarrow \sqrt{(1+a')^2+\frac{(1-a)^2}{\lambda_r^2}}\cdot\sigma_r\left(C_L(1-a) - C_D\lambda_r(1+a')\right) = 4a'(1-a)$$

DIVIDING BOTH EQUATIONS GIVES

$$\lambda_r\cdot\frac{C_L\lambda_r(1+a') + C_D(1-a)}{C_L(1-a) - C_D\lambda_r(1+a')} = \frac{a}{a'} \Leftrightarrow$$

$$a' = \frac{a(1-a)}{\lambda_r^2}\left[\frac{1-\frac{C_D}{C_L}\lambda_r\frac{1+a'}{1-a}}{(1+a')+\frac{C_D}{C_L}\frac{(1-a)}{\lambda_r}}\right]$$

(RECALL ROTOR DISC THEORY: $a' = \dfrac{a(1-a)}{\lambda_r^2}$)

GET QUADRATIC EQUATION IN a'

$$a'^2 + \left(\frac{c_D}{c_L}\frac{(1-a)}{\lambda_r} + 1\right) a' = \frac{a(1-a)}{\lambda_r^2}\left[1 - \frac{c_D}{c_L}\lambda_r \frac{1}{1-a} - \frac{c_D}{c_L}\lambda_r \frac{\cdots}{1-\cdots}\right.$$

$$\Leftrightarrow a'^2 + \left\{1 + \frac{c_D}{c_L}\left[\frac{1-a}{\lambda_r} + \frac{a\cancel{(1-a)}}{\lambda_r \cancel{(1-a)}}\right]\right\} a' = \frac{a(1-a)}{\lambda_r^2}\left[1 - \frac{c_D}{c_L}\lambda_r \frac{1}{(1-a)}\right]$$

$$\Leftrightarrow a'^2 + \underbrace{\left\{1 + \frac{c_D}{c_L}\frac{1}{\lambda_r}\right\}}_{=:p>0} a' - \underbrace{\left\{\frac{a(1-a)}{\lambda_r^2} - \frac{c_D}{c_L}\frac{a}{\lambda_r}\right\}}_{=:q>0}$$

SOLUTION

$$a'_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

$$= -\frac{\left(1 + \frac{c_D}{c_L}\frac{1}{\lambda_r}\right)}{2} \pm \sqrt{\frac{\left(1 + \frac{c_D}{c_L}\frac{1}{\lambda_r}\right)^2}{4} + \frac{a(1-a)}{\lambda_r^2} - \frac{c_D}{c_L}\frac{a}{\lambda_r}}$$

ONLY POSITIVE ROOT IS MEANINGFUL

FOR $c_D = 0$ IT WOULD GIVE:

$$a' = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{a(1-a)}{\lambda_r^2}}$$

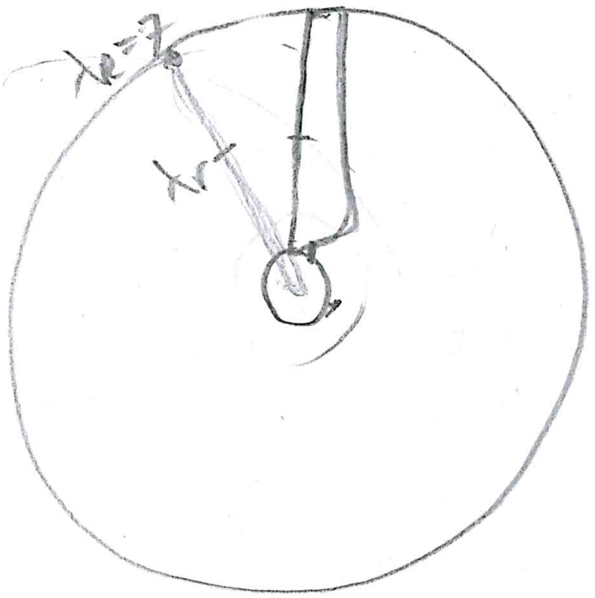
$$= \frac{1}{2}\sqrt{1 + \frac{4a(1-a)}{\lambda_r^2}} - \frac{1}{2}$$

$$= \frac{a(1-a)}{\lambda_r^2} + O(\lambda_r^{-4})$$ BY TAYLOR SERIES $\sqrt{1+x} = 1 + \frac{1}{2}x + O(x$

3.4.1 BEM EXAMPLE

LET US NOW ASSUME A FEW TYPICAL VALUES :

$\lambda_r \in [1,7]$ FOR $\lambda_R = 7$, NOTING THAT INNER $\frac{1}{7}$ PART OF BLADE IS NOT AERODYNAMICALLY IMPORTANT ($\frac{1}{49}$ OF TOTAL AREA)



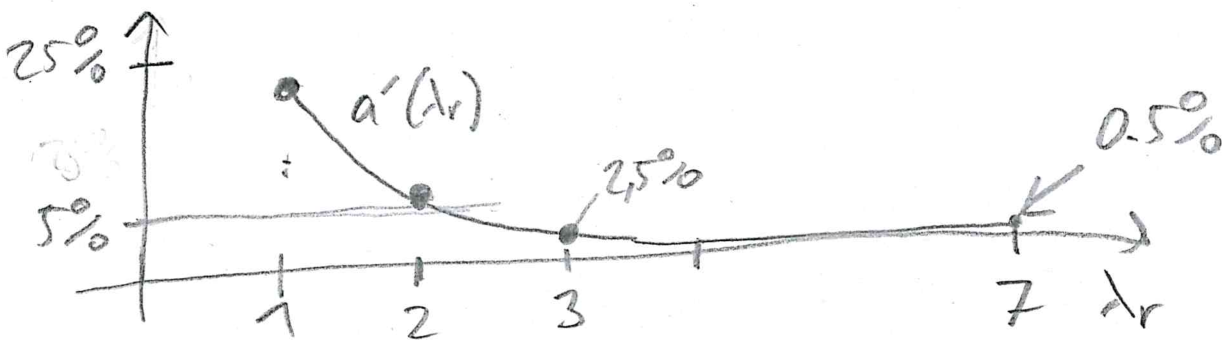
$B = 3$

$a = \frac{1}{3}$ (BETZ LIMIT EXTRACTS MAXIMUM POWER)

$\frac{C_L}{C_D} = 100$, $C_L = 1$ (TWIST / SET PITCH OPTIMALLY CHOSEN SUCH THAT ANGLE OF ATTACK IS CONSTANT AT $\alpha = 5°$)

$\Rightarrow \frac{C_D}{C_L} = 0.01 \approx 0$

$$a' \approx \frac{a(1-a)}{\lambda_r^2} = \frac{2}{9} \cdot \frac{1}{\lambda_r^2}$$



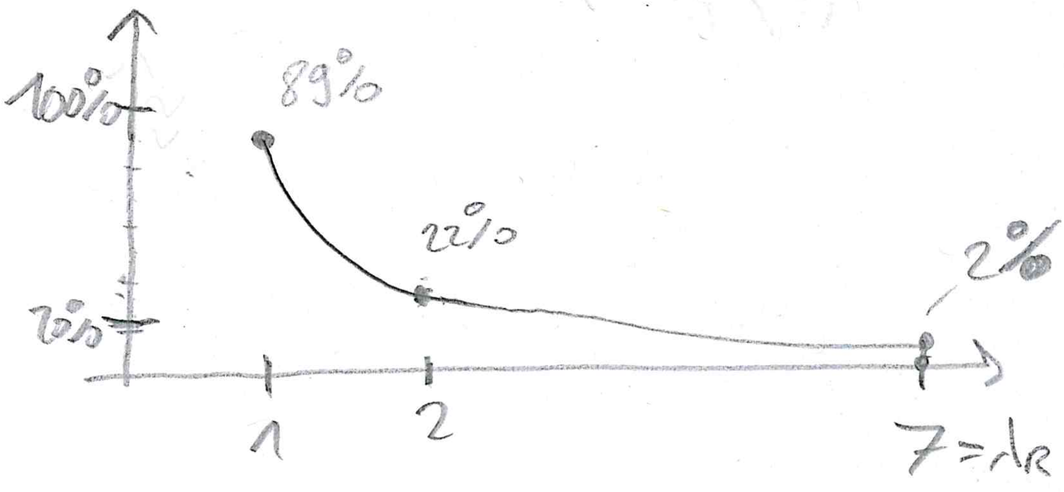
GET LOCAL SOLIDITY FROM EQ. (7)

$$\sigma_r = \frac{4a(1-a)}{C_L \lambda_r (1+a') + C_D (1-a)} \cdot \frac{1}{\sqrt{\lambda_r^2 (1+a')^2 + (1-a)^2}}$$

$$= \frac{8}{9} \frac{1}{\lambda_r (1+a') + \underbrace{0.006}_{\approx 0}} \cdot \frac{1}{\lambda_r (1+a') \sqrt{1 + \underbrace{\frac{4}{9} \frac{1}{(1+a')\lambda_r}}_{\approx 0}}}$$

$$\approx \frac{8}{9} \frac{1}{\lambda_r^2 (1+a')^2}$$

$$\approx \frac{8}{9} \frac{1}{\lambda_r^2}$$



WHAT DOES THIS MEAN FOR CHORD LENGTH C ?

$$\sigma_r = \frac{B \cdot c}{2\pi r} = \frac{8}{9} \frac{1}{\lambda_r^2} \Longleftrightarrow c = \frac{2\pi r}{B} \frac{8}{9} \frac{R^2}{r^2} \frac{1}{\lambda_R^2}$$

$$= \frac{2\pi R}{B \lambda_R^2} \cdot \frac{8}{9} \cdot \frac{1}{\mu}$$

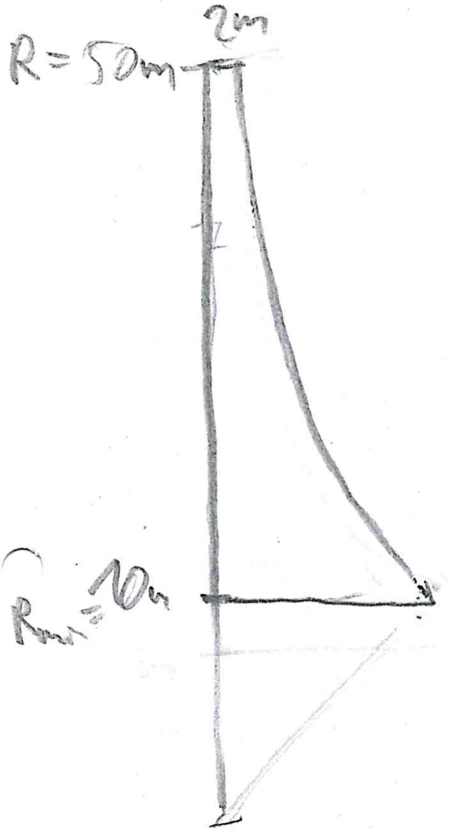
$$\approx \frac{2\pi R}{B} \cdot 2\% \cdot \frac{1}{\mu} \approx 4\% \frac{R}{\mu}$$

FOR $R = 50\,m$ WE GET

$$c(R) \approx 2\,m$$

AND

$$c(10m) = 10\,m$$



TOTAL BLADE AREA

$$A_B = \int_{R_{min}}^{R} B \cdot c(r)\,dr = \int \frac{2\pi R^2}{\lambda_R^2} \cdot \frac{8}{9} \frac{1}{r}\,dr$$

$$= \frac{2\pi R^2}{\lambda_R^2} \frac{8}{9} \left[\operatorname{Log} r \right]_{R_{min}}^{R}$$

$$= \pi R^2 \cdot \frac{16}{9} \frac{1}{\lambda_R^2} \log\left(\frac{R}{R_{min}}\right) \approx \pi R^2 \cdot 6\%$$

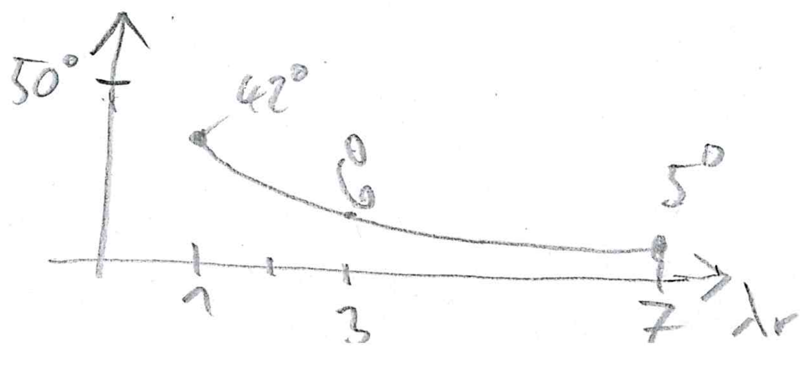
WHAT IS THE FLOW ANGLE ϕ ?

$$\sin \phi = \frac{1-a}{\lambda_r (1+a') \sqrt{1 + \underbrace{\frac{(1-a)^2}{\lambda_r^2 (1+a')^2}}_{\approx 0}}}$$

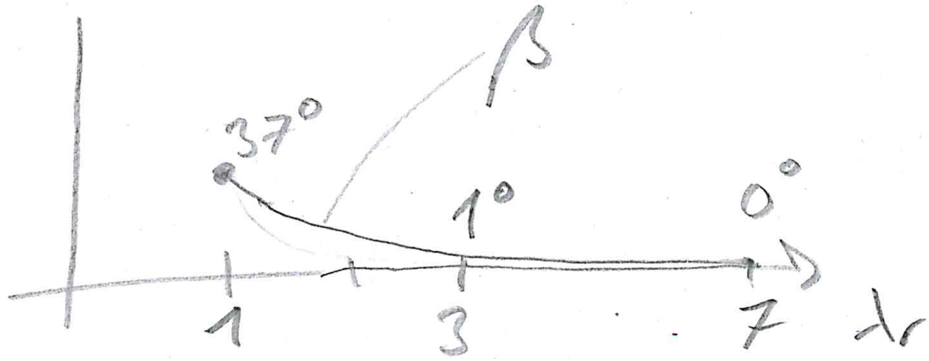
$$\approx \frac{1-a}{\lambda_r (1+a')}$$

$$\approx \frac{2}{3} \frac{1}{\lambda_r}$$

$$\Leftrightarrow \phi \approx \arcsin \frac{2}{3} \frac{1}{\lambda_r}$$



CORRESPONDING PITCH ANGLE (FOR $\alpha \equiv 5^0$)



WHAT IS THE POWER OUTPUT ?

$$dP = dF_T \cdot \Omega \cdot r$$

$$= \frac{1}{2}\rho 2\pi\, r\, dr\, U_\infty 4a'(1-a)\, r \cdot \Omega\, r\Omega$$

$$= \frac{1}{2}\rho\, 2\pi\, U_\infty^3 \Omega^2\, \underbrace{4a'}_{=\frac{a(1-a)}{\lambda^2}}\underbrace{(1-a)}_{\frac{2}{3}}\, \lambda_r^2\, r \cdot dr$$

$$= \frac{16}{27} \cdot \frac{1}{2}\rho U_\infty^3 \cdot 2\pi r \cdot dr \qquad \text{(CF. BETZ-LIMIT)}$$

$$P = \int_{R_{min}}^{R} dP = \frac{16}{27} \cdot \frac{1}{2}\rho U_\infty^3 \cdot \pi \left[R^2 - R_{min}^2\right]$$

POWER PER BLADE AREA:

$$\frac{P}{A_B} = \frac{\frac{16}{27} \cdot \frac{1}{2} \rho U_\infty^3 \cdot \pi \left[R^2 - R_{min}^2\right]}{\pi R^2 \frac{16}{9} \frac{1}{\lambda_R^2} \log\left(\frac{R}{R_{min}}\right)}$$

$$= \frac{1}{3} \cdot \frac{1}{2} \rho U_\infty^3 \cdot \lambda_R^2 \frac{\left[1 - \frac{R_{min}^2}{R^2}\right]}{\log\left(R/R_{min}\right)}$$

FOR $\frac{R}{R_{min}} = 5$ GET E.G. $\frac{\left[1 - \frac{1}{25}\right]}{1.6} = \frac{\frac{24}{25}}{1.6} \approx 0.6$

I.E. $\boxed{\frac{P}{A_B} \approx 0.2 \cdot \frac{1}{2} \rho U_\infty^3 \cdot \lambda_R^2 \approx 10 \cdot \frac{1}{2} \rho U_\infty^3}$

$\xi \approx 10$

POWER HARVESTING FACTOR

OUTER 20% OF RADIUS

FOR $\frac{R_1}{R_{min}} = 4$ GET $\frac{\left[1 - \frac{R_1^2}{R^2}\right]}{\log\left(\frac{R}{R_1}\right)} = \frac{\left[1 - \frac{16}{25}\right]}{0.22} = \frac{\frac{9}{25}}{0.22} \approx 1.6$

14% OF AREA, 38% OF POWER : 2,7 TIMES MORE EFFICIENT

OUTER 40% OF RADIUS

$R_1 = 3$ $\quad \left[1 - \frac{R_1^2}{R^2}\right] = \left[1 - \frac{9}{25}\right] = \frac{16}{25}$

$$\log\left(R/R_1\right) = \log\left(5/3\right) \approx 0.51$$

$\frac{0.51}{1.6} = 0.32 \mathrel{\hat{=}} 32\%$ OF AREA $\quad \frac{2}{3} \mathrel{\hat{=}} 66\%$ OF POWER

30%

51% OF POWER

22% OF AREA