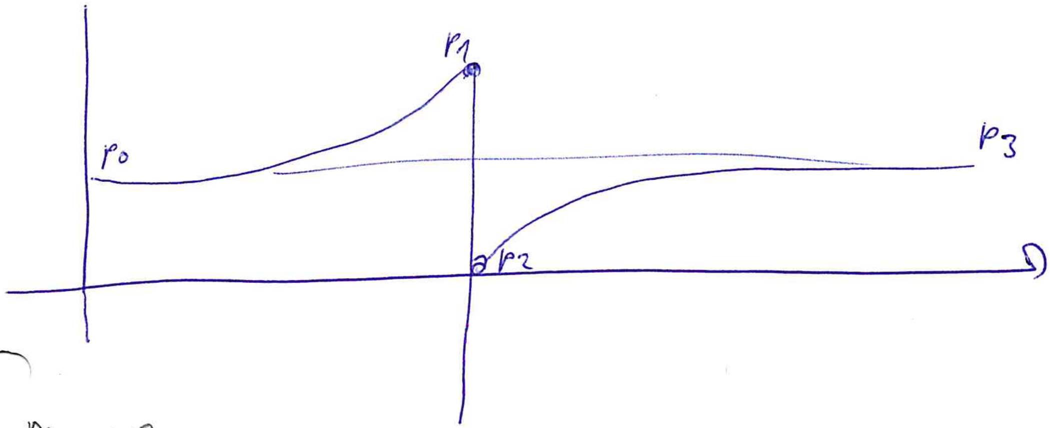
3.2 (CONTINUED) ACTUATOR DISC THEORY & BETZ LIMIT



POWER

$$P = \dot{m}\left(\frac{1}{2}u_0^2 - \frac{1}{2}u_3^2\right) \quad \text{WITH} \quad \dot{m} = \rho A u_1$$

THRUST

$$\boxed{T = A(p_1 - p_2) = \dot{m}(u_0 - u_3)} \quad (1)$$

THRUST EQUATION

BERNOULLI: BEFORE DISC:

$$\boxed{p_0 + \frac{1}{2}\rho u_0^2 = p_1 + \frac{1}{2}\rho u_1^2} \quad (2)$$

& AFTER DISC:

$$\boxed{p_2 + \frac{1}{2}\rho u_1^2 = p_0 + \frac{1}{2}\rho u_3^2} \quad (3)$$

8 numbers:

BUT:

u_0 GIVEN
p_0

$u_1 = u_2$

$p_3 = p_0$

↓

4 unknowns:
u_1, u_3, p_1, p_2

3 EQUATIONS
(1), (2), (3)

$\Rightarrow$ ONE REMAINING UNKNOWN (EG. u

$$(2) \Leftrightarrow p_1 = p_0 + \frac{1}{2}\rho\left(u_0^2 - u_1^2\right)$$

$$(3) \Leftrightarrow p_2 = p_0 + \frac{1}{2}\rho\left(u_3^2 - u_1^2\right)$$

$$\Downarrow$$

$$p_1 - p_2 = \frac{1}{2}\rho\left(u_0^2 - u_3^2\right) = \frac{1}{2}\rho(u_0 - u_3)(u_0 + u_3)$$

②

WITH (1):

$$\cancel{A}\frac{1}{2}\cancel{\rho}\,\cancel{(u_0-u_3)}(u_0+u_3) = \cancel{\rho}\cancel{A}\,u_1\cdot\cancel{(u_0-u_3)}$$

$$\Leftrightarrow u_1 = \frac{1}{2}(u_0+u_3)$$ (u_1 AVERAGE OF u_0 & u_3 !)

INTRODUCE "INDUCTION FACTOR $a = \frac{u_0-u_1}{u_0}$

OR $\boxed{u_1 = u_0\cdot(1-a)}$

THEN $u_3 = u_0(1-2a)$

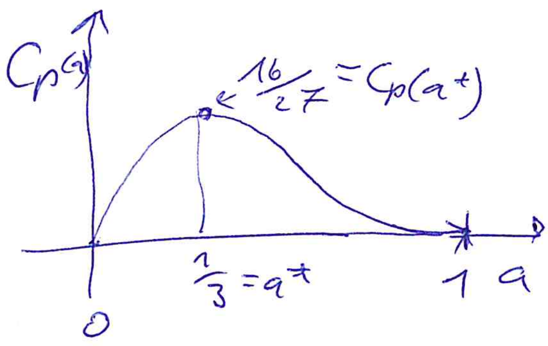
NOW CAN COMPUTE T & P AS FUNCTION OF a:

$$P = \frac{1}{2}\rho A\, u_0^3\,(1-a)\left(1-(1-2a)^2\right)$$

$$= \frac{1}{2}\rho A u_0^3\,(1-a)\,(4a-4a^2)$$

$$= \frac{1}{2}\rho A u_0^3\,\underbrace{(1-a)(1-a)(4a)}_{=:\,C_p(a)}$$

"POWER COEFFICIENT"



$$\frac{d\,C_p(a)}{da} = -2(1-a)\,4a + (1-a)^2\,4 \overset{!}{=} 0 \Leftrightarrow$$

$$(1-a) = 2a \Leftrightarrow a^* = \frac{1}{3}$$

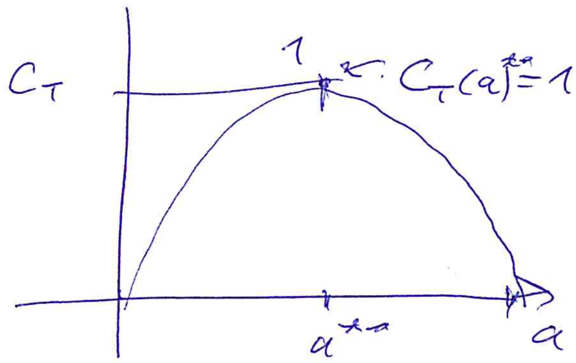
$$C_p(a^*) = (1-a^*)^2\,4\cdot a^* = \left(\frac{2}{3}\right)^2\cdot 4\cdot\frac{1}{3} = \frac{16}{27}$$ "BETZ LIMIT"

$$T(a) = \dot{m}(u_0 - u_3) = \varrho A (1-a) u_0 \; (u_0 - (1-2a) u_0)$$

$$= \frac{1}{2} \varrho A u_0^2 \underbrace{2 \cdot (1-a)\, 2a}_{=: C_T(a) = 4a(1-a)}$$

$$\frac{dT}{da} = 4(1-a) - 4a \overset{!}{=} 0 \iff 1 - a = a \iff a^{**} = \frac{1}{2}$$

$$T(a^{**}) = \cancel{4 \cdot \frac{1}{2}} \cancel{4 \cdot \frac{1}{2}} \cdot 4 \, \frac{1}{2} \, \frac{1}{2} = 1$$



COMPARE

$$C_T(a) = 4a(1-a) \quad \text{AND} \quad C_P(a) = 4a(1-a)^2$$

THIS IS CONSISTENT WITH ALTERNATIVE POWER FORMULA

$$P = T \cdot u_1 = \frac{1}{2} \varrho A u_0^2 \cdot C_T(a) \cdot u_0 (1-a) = \frac{1}{2} \varrho A u_0^3 \underbrace{(1-a)\, C_T(a)}_{= C_P(a)}$$