Exercise Sheet 3 with solutions

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Problem 5: Observer

Consider a simple LC-circuit with the state vector is defined as $\mathbf{x} := \begin{bmatrix} v_C & i_L \end{bmatrix}^\intercal$ and the following dynamics:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & \frac{1}{C_\mathrm{f}} \\ \frac{-1}{L_\mathrm{f}} & 0 \end{bmatrix} \mathbf{x}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

To reduce the component costs in a device that follows the given dynamics, only a voltage sensor on the capacitor voltage is employed. The current in the circuit is not measured. We want to implement an Luenberger observer to get knowledge of both states. Recall that the estimated state has the following dynamics:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = \mathbf{C}\hat{\mathbf{x}}$$

(a) Is the system observable?

$$\det\left(\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{C_\mathrm{f}} \end{bmatrix}\right) = \frac{1}{C_\mathrm{f}} \neq 0$$

$\Rightarrow$ Observable because observability matrix has full rank.

(b) Calculate the matrix $\mathbf{A}_\mathrm{obs}$ that defines the dynamics of the observer error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, $\dot{\mathbf{e}}(t) = \mathbf{A}_\mathrm{obs}\mathbf{e}(t)$

$$\begin{aligned} \dot{\mathbf{e}} &= \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}x + \mathbf{B}u - \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}u - \mathbf{L}(y - \hat{y}) \\ &= \mathbf{A}\mathbf{x} - \mathbf{A}\hat{\mathbf{x}} - \mathbf{L}(\mathbf{C}\mathbf{x} - \mathbf{C}\hat{\mathbf{x}}) = (\mathbf{A} - \mathbf{LC})\mathbf{x} - (\mathbf{A} - \mathbf{LC})\hat{\mathbf{x}} \\ &= (\mathbf{A} - \mathbf{LC})(\mathbf{x} - \hat{\mathbf{x}}) = (\mathbf{A} - \mathbf{LC})\mathbf{e} \\ \Rightarrow \mathbf{A}_\mathrm{obs} &= (\mathbf{A} - \mathbf{LC}) \end{aligned}$$

(c) To simulate the whole system we need the dynamics of $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$. Fill in the matrix:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$
$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{L}(y - \hat{y}) = \mathbf{A}\hat{\mathbf{x}} + \mathbf{L}(\mathbf{C}\mathbf{x} - \mathbf{C}\hat{\mathbf{x}}) = (\mathbf{A} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{LC}\mathbf{x}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{LC} & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

(d) Assume $L_\mathrm{f} = 5$ mH and $C_\mathrm{f} = 10$ mF and use the `place()` function to calculate $\mathbf{L}$ if the eigenvalues of the observer are $\lambda_{1,2} = -30\frac{1}{\mathrm{s}}$ (slow observer $\mathbf{L}_1$) or $\lambda_{1,2} = -1000\frac{1}{\mathrm{s}}$ (fast observer $\mathbf{L}_2$).
(Hint: Use the transposed system.)

(e) Simulate the whole system for both cases given in (d) with initial state $\begin{bmatrix} \mathbf{x}_0 & \hat{\mathbf{x}}_0 \end{bmatrix}^\top = \begin{bmatrix} 10 & 0 & 0 & 0 \end{bmatrix}^\top$. Assume to have no input ($u = 0$).

(f) What happens if the observer does not know the real system dynamics exactly? Change L_f to 7 mH in $\mathbf{A}$ and keep $L_\mathrm{f,obs} = 5$ mH in $\mathbf{A}_\mathrm{obs}$ and repeat the simulation.