Lecture Energy Systems: Hardware and Control - Control Part
University of Freiburg – Winterterm 2017/2018

Exercise Sheet 1 with solutions

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Within the control part of the "Energy Systems: Hardware and Control" course there will be 45 min. exercise sessions after each lecture. The exercises are guided by tutors and will contain some **MATLAB**-based tasks. Therefore, a **MATLAB** installation including the Control System Toolbox is needed.

Getting started

1. Except **MATLAB** is not yet installed on your computer, the first thing you need to do is install it. Detailed installation and licensing instructions can be found at
   `https://www.rz.uni-freiburg.de/services-en/beschaffung-em/software-en/matlab-license`
   Remember that the Control System Toolbox is required.
2. If you are new to **MATLAB**, the first thing you will appreciate is the extensive help system. You can simply type `doc` into the console and the documentation opens. If you type `doc plot`, you will find a detailed description of function `plot`.
3. Here are some useful commands for the exercises:
   ```
   hold on/off
   figure
   close all
   clear
   clc
   ```

Problem 1: Dynamical System, ODE, Simulation and Solution

A simple pendulum is sketched in figure 1. The point-mass m is fixed to a solid, massless rod of length l, which is connected to a frictionless hinge on the other side. All movements take place in the vertically oriented x-y-plane and the gravitation g acts in y-direction.

(a) Derive the equation of motion for the pendulum and note it in the shape $\ddot{\alpha} = f(\alpha)$. How do the mass or the length determine the motion of the pendulum?

$$E = \frac{1}{2}ml^2\dot{\alpha}^2 + mgl(1-\cos(\alpha))$$
$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \quad \text{conservation of energy}$$
$$\ddot{\alpha} = -\frac{g}{l}\sin(\alpha)$$

(b) What are the states x, which are needed to completely describe the system?

$$x = \begin{bmatrix}\alpha \\ \dot{\alpha}\end{bmatrix} = \begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

(c) Convert the ODE to the system of equations $\dot{\mathbf{x}} = f(\mathbf{x})$.

$$\dot{x} = \begin{bmatrix}\dot{\alpha} \\ \ddot{\alpha}\end{bmatrix} = \begin{bmatrix}\dot{\alpha} \\ -\frac{g}{l}\sin(\alpha)\end{bmatrix} = \begin{bmatrix}x_2 \\ -\frac{g}{l}\sin(x_1)\end{bmatrix}$$

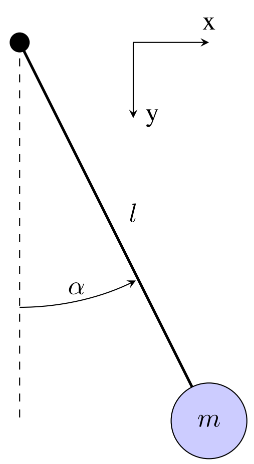


Figure 1: Sketch of a simple pendulum

(d) Simulate the motion of the pendulum for 10 seconds using different initial values x_0. Therefore write a function `function dx = nonlin_pendel(t, x)` which implements the system of equations. For the simulation use `lsode` and the following constants: $l = 1\,\text{m}$, $g = 9.81\,\frac{\text{m}}{\text{s}^2}$. Solution see `problem1.m`.

(e) What characterizes steady states? Calculate the steady states for the pendulum.

$$\dot{x} \stackrel{!}{=} 0$$
$$0 = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{bmatrix}$$
$$\Rightarrow x_{2\text{ss}} = 0, \quad x_{1\text{ss}} = n \cdot \pi \quad \text{with } n \in \mathbb{Z}$$
$$x_{\text{ss}} = \begin{bmatrix} n \cdot \pi \\ 0 \end{bmatrix}$$

(f) Linearize the system at the steady state $x_{\text{ss}} = [0, 0]^\top$ and write the system of equations in the shape $\dot{\mathbf{x}} = A\mathbf{x} + Bu$ and $y = \alpha = C\mathbf{x}$. Compute the state space matrices A,B and C.

$$\dot{x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_{\text{ss}}) & \frac{\partial f_1}{\partial x_2}(x_{\text{ss}}) \\ \frac{\partial f_2}{\partial x_1}(x_{\text{ss}}) & \frac{\partial f_2}{\partial x_2}(x_{\text{ss}}) \end{bmatrix} x$$
$$= \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_{1\text{ss}}) & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(0) & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} x$$

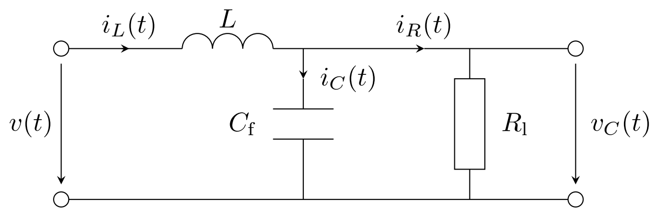
(g) Compare the linear and the nonlinear system via simulations using increasing initial values for $\alpha(0)$ ranging from $\pi/8$ to π.

With increasing $\alpha(0)$ the linear approximation gets worse. But for control tasks we want to keep our system close to a desired state, so the linearization is a good simplification and thus an important and very useful tool. For plot run `problem1.m`

Problem 2: Buck-converter, Modelling and Stabilization

1. The electrical circuit sketched below shows a simplified buck-converter with a constant load at the output. The system can be described in state-space representation as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad y = \mathbf{C}\mathbf{x}, \quad \mathbf{D} = [0],$$
$$u := v, \quad y := v_C.$$



(a) Derive the the I/O-ODE (Input/Output-Ordinary Differential Equation) for the given circuit using equations

$$i_C = C_\text{f}\frac{dv_C}{dt}, \quad v_L = L\frac{di_L}{dt} \quad \text{and} \quad i_R = \frac{v_C}{R_\text{l}}$$

(Hint: Use Kirchhoff's voltage law for inductors and current law for capacitors)

$$v = v_C + v_L = v_C + L \cdot (\dot{i}_C + \dot{i}_R)$$
$$\ddot{v}_C + \frac{1}{R_\text{l}C_\text{f}}\dot{v}_C + \frac{1}{LC_\text{f}}v_C = \frac{1}{LC_\text{f}}v$$

(b) Convert the I/O-ODE to state space representations, i.e. set up the $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$-matrices for

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \quad (1)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) \quad (2)$$

using

(i) the control canonical form.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC_\mathrm{f}} & -\frac{1}{R_\mathrm{l}C_\mathrm{f}} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}$$
$$y = \begin{bmatrix} \frac{1}{LC_\mathrm{f}} & 0 \end{bmatrix} \mathbf{x}$$

(ii) the observer canonical form.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -\frac{1}{LC_\mathrm{f}} \\ 1 & -\frac{1}{R_\mathrm{l}C_\mathrm{f}} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{LC_\mathrm{f}} \\ 0 \end{bmatrix} \mathbf{u}$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}$$

(c) Now derive matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ for the state vector given as $\mathbf{x} := \begin{bmatrix} i_L & v_C \end{bmatrix}^\mathsf{T}$.

Inductor current:

$$\frac{di_L}{dt} = \frac{v_L}{L}$$

Express v_C by utilizing Kirchhoff's voltage law:

$$v_L = v - v_C$$
$$\Rightarrow \frac{di_L}{dt} = \frac{v}{L} - \frac{v_C}{L}$$

Capacitor voltage:

$$\frac{dv_C}{dt} = \frac{i_C}{C_\mathrm{f}}$$

Express i_C by utilizing Kirchoff's current law:

$$i_C = i_L - i_R = i_L - \frac{v_C}{R_\mathrm{l}}$$
$$\Rightarrow \frac{dv_C}{dt} = \frac{i_L}{C_\mathrm{f}} - \frac{v_C}{C_\mathrm{f}R_\mathrm{l}}$$

State-space representation:

$$\dot{\mathbf{x}} = \begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_C}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C_\mathrm{f}} & -\frac{1}{R_\mathrm{l}C_\mathrm{f}} \end{bmatrix}}_{\mathbf{A}} \mathbf{x} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{v}_{u}$$
$$\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

(d) Derive the characteristic polynomial. Evaluate the eigenvalues of the system for $L = 4.7$ mH, $C_\mathrm{f} = 100\ \mu$F and

i. $R_\mathrm{l} = \infty\ \Omega$
ii. $R_\mathrm{l} = 100\ \Omega$

Is the system BIBO-stable in both cases?

Characteristic polynomial:

$$\begin{aligned} p(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{A}) \\ &= \det \begin{bmatrix} \lambda & \frac{1}{L} \\ -\frac{1}{C_\mathrm{f}} & \lambda + \frac{1}{R_\mathrm{l}C_\mathrm{f}} \end{bmatrix} \\ &= \lambda^2 + \frac{\lambda}{R_\mathrm{l}C_\mathrm{f}} + \frac{1}{LC_\mathrm{f}} \end{aligned}$$

Roots of the characteristic polynomial:

- $R_\mathrm{l} = \infty\ \Omega$:

$$\lambda^2 + \frac{1}{LC_\mathrm{f}} \overset{!}{=} 0$$
$$\Rightarrow \lambda = \pm \frac{i}{\sqrt{LC_\mathrm{f}}} \approx \pm\, i\, 1.46 \cdot 10^3$$

$\mathrm{Re}(\lambda_i) = 0$ for all eigenvalues.
$\Rightarrow$ system is undamped and therefore not BIBO-stable.

- $R_\mathrm{l} = 100\ \Omega$

$$\lambda^2 + \frac{\lambda}{R_\mathrm{l}C_\mathrm{f}} + \frac{1}{LC_\mathrm{f}} \overset{!}{=} 0 \Rightarrow \lambda = -\frac{1}{2R_\mathrm{l}C_\mathrm{f}} \pm i\sqrt{-\frac{1}{4R_\mathrm{l}^2C_\mathrm{f}^2} + \frac{1}{LC_\mathrm{f}}} \approx -50 \pm\ i\, 1.46 \cdot 10^3$$

$\mathrm{Re}(\lambda_i) < 0$ for all eigenvalues.
$\Rightarrow$ system is damped and therefore BIBO-stable.

(e) Write down the time constant τ in seconds and the resulting oscillating frequency in Hz for both values of R_l.

(Hint: In this example, the time constant τ is a measure for the amplitude decay (damping) and is defined as $\tau = -\frac{1}{\mathrm{Re}(\lambda)}$. The oscillating frequency is defined as $f_0 = \frac{\omega_0}{2\pi} = \frac{|\mathrm{Im}(\lambda)|}{2\pi}$)

- $R_\mathrm{l} = \infty\ \Omega$:
Time constant τ:

$$\tau = -\frac{1}{\mathrm{Re}(\lambda)} = -\infty \text{ ms}$$

Oscillating Frequency:

$$\omega_0 = |\mathrm{Im}(\lambda)| = \frac{1}{\sqrt{LC_\mathrm{f}}}$$
$$f_0 = \frac{\omega_0}{2\pi} \approx 232.15 \text{ Hz}$$

- $R_\mathrm{l} = 100\ \Omega$
Time constant τ:

$$\tau = -\frac{1}{\mathrm{Re}(\lambda)} = 2C_\mathrm{f}R_\mathrm{l} = 20 \text{ ms}$$

Oscillating Frequency:

$$\omega_0 = |\mathrm{Im}(\lambda)| = \sqrt{-\frac{1}{4R_\mathrm{l}^2C_\mathrm{f}^2} + \frac{1}{LC_\mathrm{f}}}$$
$$f_0 = \frac{\omega_0}{2\pi} \approx 232.01 \text{ Hz}$$

(f) Create a new MATLAB script and define variables L $= 4.7$ mH, Cf $= 100\ \mu$F and Rl $= \infty$. Also define matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D} = 0$ according to task (1c).

(g) Use the `ss(A,B,C,D)` command to create a state-space model `sys_ol` and evaluate the systems step response with the `step(sys,Tfinal)` function (Tfinal = 0.1 s) for

i. $R_\mathrm{l} = \infty\ \Omega$
ii. $R_\mathrm{l} = 100\ \Omega$

(h) Is the system controllable and/or stabilizable?

$$\mathcal{C} = \begin{bmatrix}\mathbf{B} & \mathbf{AB}\end{bmatrix} = \begin{bmatrix}\frac{1}{L} & 0 \\ 0 & \frac{1}{C_\mathrm{f}L}\end{bmatrix}$$
$$\det(\mathcal{C}) = \frac{1}{C_\mathrm{f}L^2} \neq 0$$

The system is fully controllable and therefore also stabilizable.

2. Now we want to introduce a state feedback with gain K to stabilize the system in the case where no load is connected ($R_\text{l} = \text{inf}$).

   (a) Where do the two poles have to be shifted to obtain the following characteristics for the closed-loop system?: $\tau = 10\text{ ms}$, $f_0 = 100\text{ Hz}$

   $$\text{Re}(\lambda_{1/2}) = -\frac{1}{\tau} = -100$$
   $$\text{Im}(\lambda_{1/2}) = \pm\, 2 f_0 \pi \approx \pm\, 628.3185$$

   (b) Use the MATLAB function `place(A,B,p)` to calculate the corresponding feedback vector K and implement the system matrix $\mathbf{A}_\text{cl}$ as well as the closed-loop model `sys_stable` for the stabilized system.

   $$\mathbf{A}_\text{cl} = \mathbf{A} - \mathbf{B}K$$
   $$(\mathbf{B}_\text{cl} = \mathbf{B}, \mathbf{C}_\text{cl} = \mathbf{C})$$

   (c) Simulate `sys_stable` with the `step()` command and verify frequency and damping is as desired.