Exercises for Lecture Course on Numerical Optimal Control (NOC)
Albert-Ludwigs-Universität Freiburg – Summer Term 2017

Exercise 8: Differential Dynamic Programming and Continuous-Time Optimal Control

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Consider the following continuous-time optimal control problem:

$$\begin{aligned} \min_{x(t),\, u(t)} \quad & \int_{t=0}^{T} L(x(t), u(t))\mathrm{d}t + M(x(T)) \\ \text{s.t.} \quad & x(0) = \bar{x}_0 \\ & \dot{x}(t) = f(x(t), u(t)), \quad t \in [0, T]. \end{aligned} \tag{1}$$

1. (a) Discretize problem (1) using the explicit Euler integrator with step-size h over N intervals. Write on paper the obtained discrete-time optimal control problem.

   (2 points)

   (b) Write the first-order optimality conditions for the discretized problem obtained at point (a). Use the Hamiltonian function defined as

   $$H(x, u, \lambda) := L(x, u) + \lambda^T f(x, u) \tag{2}$$

   for compactness.

   (2 points)

   (c) Now let $N \to \infty$ and $h \to 0$. What type of problem do the conditions derived in (b) converge to?

   (3 points)

   (d) **[Bonus]** Fix $N = 2$ and apply the Newton method to the first-order optimality conditions for the discretized optimal control obtained in (b). Derive the form of the linear systems associated with the Newton steps. Order the variables as $[\lambda_0^T,\, x_0^T,\, u_0^T, \lambda_1^T, x_1^T, u_1^T,\, \lambda_2^T,\, x_2^T]^T$.

   (2 bonus points)

   (e) **[Bonus]** The linear systems associated with the Newton steps in (d) can be solved exploiting the Riccati Difference Equation (equation 8.5 in the course's script). Derive this equation.

   (3 bonus points)

   (f) **[Bonus]** What kind of matrix ODE does the difference equation derived in (e) converge to for $N \to \infty$ and $h \to 0$? *Hint: if you have not solved the bonus point (e) you can refer to equation 8.5 from the course's script.*

   (2 bonus points)

   (g) **[Bonus]** For $N = 2$, and assuming that the Lagrange term is of the form $L(x, u) = L_x(x) + L_u(u)$, derive the expression of the blocks of the reduced Hessian obtained after eliminating the state variables through the condensing procedure from the discretized problem in (1). Eliminate the initial state x_0 from the condensed problem. Of which order in h are the terms on the diagonal? Of which order are the off-diagonal terms?

   (3 bonus points)

This sheet gives in total 7 points and 10 bonus points