Solutions: Statistics

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Task 1 *Consider rolling a fair dice twice. What is the chance that the sum of both roll is exactly 9 (event A)?*

Possible outcomes are

$$\begin{array}{cccccc} \{1,1\} & \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{1,6\} \\ \{2,1\} & \{2,2\} & \{2,3\} & \{2,4\} & \{2,5\} & \{2,6\} \\ \{3,1\} & \{3,2\} & \{3,3\} & \{3,4\} & \{3,5\} & \{3,6\} \\ \{4,1\} & \{4,2\} & \{4,3\} & \{4,4\} & \{4,5\} & \{4,6\} \\ \{5,1\} & \{5,2\} & \{5,3\} & \{5,4\} & \{5,5\} & \{5,6\} \\ \{6,1\} & \{6,2\} & \{6,3\} & \{6,4\} & \{6,5\} & \{6,6\} \end{array}$$

Possible outcomes contained in event A: {6,3}, {3,6}, {5,4}, {4,5}

So there are 4 elements of all 36 possible outcomes that are contained in A.

$$P(A) = \frac{\text{amount of elements } s_i \text{ in A}}{\text{amount of elements in } \Omega} = \frac{4}{36} = \frac{1}{9}$$

Task 2 *What do the axioms for probability laws mean in terms of the example above?*

- Nonnegativity: The probability that the event A (containing at least one possible outcome) occurs is always greater to zero.
- Additivity: The probability that either one of two disjoint events, e.g. that event A (all outcomes that result in 9) or event B (all outcomes that result in 8) occurs, is the sum of the probabilities P(A) and P(B).
- Normalization: The probability that any possible outcome occurs is 1 (it is sure that any possible outcome occurs). Hence there is no chance, that no possible outcome occurs and $P(\emptyset) = 0$.

Task 3 *Determine the conditional probability $P(B|A)$ with $A = \{$first roll is a $3\}$, $B = \{$sum of two rolls is $\leq 9\}$. Repeat this task for the cases that he first gets a 1, 2, 4, 5 or 6 i.e. compute the conditional probability $P(B|A_i)$, with i = 1, 2, 4, 5, 6.*

In general it the conditioonal probability can be computed as follows:

$$P(A|B) = \frac{\text{amount of elements of A} \cap \text{B}}{\text{amount of elements of B}}.$$

i = 1: $P(B|A_1) = \frac{6}{6}$

i = 2: $P(B|A_2) = \frac{6}{6}$

i = 3: $P(B|A_3) = \frac{6}{6}$

i = 4: $P(B|A_4) = \frac{5}{6}$

i = 5: $P(B|A_5) = \frac{4}{6}$

i = 6: $P(B|A_6) = \frac{3}{6}$

Task 4 *Compute $P(B)$ using the conditional probabilities $P(B|A_i)$*

For all $i \in [1, 6]$ it holds, that

$$P(A_i) = \frac{1}{6}$$

$$\begin{aligned} P(B) &= P(A_1)P(B|A_1) + \cdots + P(A_6)P(B|A_6) \\ &= 3 \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \frac{4}{6} + \frac{1}{6} \cdot \frac{3}{6} = \frac{30}{36} = \frac{5}{6} \end{aligned}$$

Task 5 *Given that Max get less than 9 within two rolls of the fair dice (event B). What is the probability that in his first roll he got a 5?*

We are looking for the probability $P(A_5|B)$.

$$P(A_5|B) = \frac{P(A_5)P(B|A_5)}{P(B)} = \frac{1/6 \cdot 4/6}{5/6} = \frac{4}{5}$$

Task 6 *Are the events $A = \{$first roll is a 3$\}$ and $B = \{$sum of the two rolls is 9$\}$ independent?*

The following must hold for independence: $P(A, B) = P(A)P(B)$.

$P(A, B) = P(A)P(B|A) = \frac{1}{6} \cdot 1$

$P(A)P(B) = \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}$

So the two events are dependent.

Task 7 *For the outcome $\{1, 3\}$, what is the value of a random variable X that describes the sum of both rolls?*

$x = 4 \qquad .$

Task 8 *What is the value of the random variable X describing the number of tails in a sequence of three tosses of a coin, for the outcome $\{THH\}$?*

$x = 1$

Task 9 *Which of the following examples can be described by a random variable?*

1. The number of false positives of a medical test
2. The percentage of students registered to MSI, which actually come to the class.

Task 10 *Compute the PDF of the random variable X describing the sum of two rolls of a fair dice.*

$$p_X(x) = \begin{cases} 1/36 & \text{if } x = 2 \\ 2/36 & \text{if } x = 3 \\ 3/36 & \text{if } x = 4 \\ 4/36 & \text{if } x = 5 \\ 5/36 & \text{if } x = 6 \\ 6/36 & \text{if } x = 7 \\ 5/36 & \text{if } x = 8 \\ 4/36 & \text{if } x = 9 \\ 3/36 & \text{if } x = 10 \\ 2/36 & \text{if } x = 11 \\ 1/36 & \text{if } x = 12 \\ 0 & otherwise \end{cases}$$

Task 11 *Compute the expected value $\mathbb{E}\{X\}$ of the random variable X describing the result of rolling a fair dice.*

$$E\{X\} = \sum_x x p_X(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} + = \frac{21}{6} = 3.5$$

Task 12 *Compute the variance σ_X^2 and the standard deviation σ_X of the random variable X describing the result of rolling a fair dice.*

$$\sigma_X^2 = \mathbb{E}\{(X - \mathbb{E}\{X\})^2\} = \sum_x (x - \mathbb{E}\{X\})^2 p_X(x) = \frac{(1-3.5)^2}{6} + \frac{(2-3.5)^2}{6} + \cdots + \frac{(6-3.5)^2 \cdot}{6} = \frac{35}{12} = 2.91\overline{6}$$

Task 13 *Given the random variables X and Y, with $Y = aX + b$ (affine function) and scalar parameters a and b, compute the expected value and the variance of Y.*

$$\mathbb{E}\{Y\} = \sum_y y p_Y(y) = \sum_x (a\,x + b) p_X(x) = (a \cdot \sum_x x p_X(x)) + b = a\mathbb{E}\{X\} + b$$

$$\sigma_Y^2 = \mathbb{E}\{(Y - \mathbb{E}\{Y\})^2\} = \mathbb{E}\{(aX + b - a\mathbb{E}\{X\} - b)^2\} = \mathbb{E}\{(a(X - \mathbb{E}\{X\}))^2\} = a^2\sigma_X^2$$

Task 14 *Consider rolling the fair dice twice again. Let the random variables X and Y describe the outcome of the first and second roll respectively.*

1. *Compute the joint PDF $p_{X,Y}(x, y)$ of the random variables X and Y.*
2. *What is the chance that a 6 occured once?*

As every outcome occurs only once, and every outcome is equally likely the joint PDF follows as

$$p_{X,Y}(x, y) = \begin{cases} 1/36 & \forall\,(x, y) : x, y \in [1, 6] \\ 0 & otherwise \end{cases}$$

$$P(X, Y \in A) = \sum_{(x,y)\in A} p_{X,Y}(x, y) = \frac{10}{36}$$

Task 15 *Compute the marginal PDFs $p_X(x)$ and $p_Y(y)$ from the joint PDF $p_{X,Y}(x, y)$ you have just computed.*

$$p_X(x) = \sum_y p_{X,Y}(x, y) = \begin{cases} 6 \cdot 1/36 & \forall x : x \in [1, 6] \\ 0 & otherwise \end{cases}$$

$$p_Y(y) = \sum_x p_{X,Y}(x, y)) = \begin{cases} 6 \cdot 1/36 & \forall y : y \in [1, 6] \\ 0 & otherwise \end{cases}$$

Task 16 *Compute the conditional PDF $p_{X|Y}(x|y) = P(X = x|Y = y)$ for the random variables X and Y describing the result of the first, respectively the second roll of a fair dice. Assume that the second roll of the fair dice was a 4 ($y = 4$). Use the joint PDF of the random variables X and Y and the related marginal PDF.*

$$p_{X|Y}(x|4) = P(X = x|Y = 4) = \frac{p_{X,Y}(x, 4)}{p_y(4)} = \frac{1/36}{1/6} = 1/6$$

Task 17 *Check whether the random variables X and Y describing the result of the first, respectively the second roll of a fair dice are independent.*

The random variables X and Y are independent, iff the following holds

$$P(X = x, Y = y) = p_X(x)p_Y(y), \quad \forall x, y$$

We have

$$P(X = x, Y = y) = p_{X,Y}(x, y) = \begin{cases} 1/36 & \forall\,x, y : x, y \in [1, 6] \\ 0 & otherwise \end{cases}$$

and

$$p_X(x)p_Y(y) = \begin{cases} (1/6)^2 & \forall\,x, y : x, y \in [1, 6] \\ 0 & otherwise \end{cases}$$

which are equal. Hence the two random variables are independent.

Task 18 *Continuous Random Variables*

See MATLAB Solution.

Task 19 *Given a random variable X with a normal propability density $p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$ with μ the mean and σ^2 the variance of X, write down the probability density function $p_Y(y)$ of the random variable $Y = 5X + 3$, using μ and σ.*

With $\mu_Y = 5\,\mu_X + 3$ and $\sigma_Y^2 = 5^2\,\sigma^2$ the PDF follows as

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(\frac{-(y - \mu_Y)^2}{2\sigma_Y^2}\right) \tag{1}$$

$$= \frac{1}{\sqrt{2\pi\,5^2\sigma_X^2}} \exp\left(\frac{-(y - (5\mu_X + 3))^2}{2\,5^2\sigma_X^2}\right). \tag{2}$$

Task 20 *Compute the covariance of the scalar valued variable $Y = aX + b$ with constants $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and the random variable $X \in \mathbb{R}$ with mean $\mathbb{E}\{X\} = \mu_X$.*

$$\begin{aligned}
\mathbb{E}\{Y\} &= \mathbb{E}\{aX + b\} \\
&= a\,\mathbb{E}\{X\} + b \\
\sigma(X, Y) &= \mathbb{E}\{(X - \mu_X)(Y - \mu_Y)\} \\
&= \mathbb{E}\{(X - \mu_X)(aX + b - (a\mathbb{E}\{X\} + b))\} \\
&= \mathbb{E}\{(X - \mu_X)(a(X - \mu_X))\} \\
&= a\,\mathbb{E}\{(X - \mu_X)(X - \mu_X)\} \\
&= a\,\sigma_X^2
\end{aligned}$$

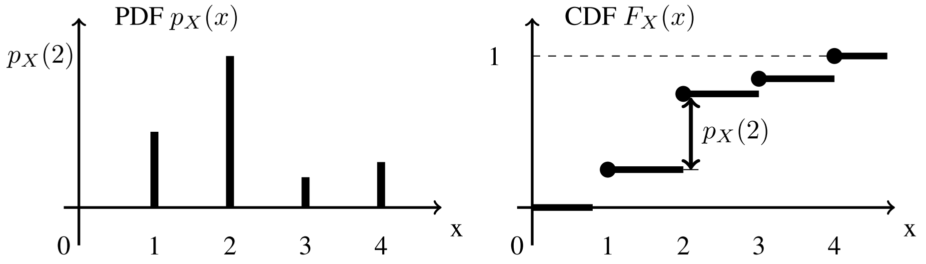

Figure 1: CDF of a discrete random variable.

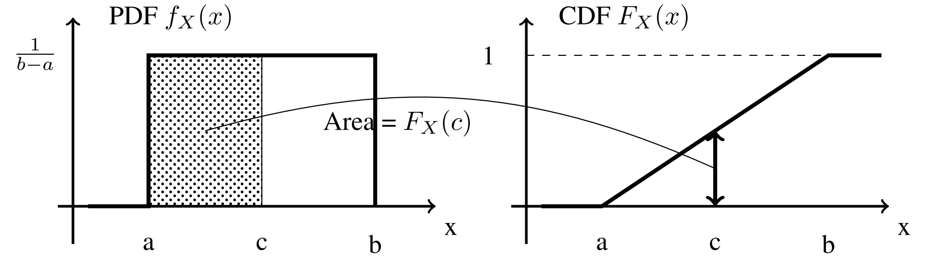

Figure 2: CDF of a continuous random variable.

Task 21 *(Discrete RV) Give the PDF of the discrete random variable shown in figure 1 and compute $P(X \leq 3)$. Assume that $p_X(1) = 0.25,\ p_X(2) = 0.5,\ p_X(3) = 0.1$ and $p_X(4) = 0.15$.*

$$p_X(x) = \begin{cases} 0.25 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2 \\ 0.1 & \text{if } x = 3 \\ 0.15 & \text{if } x = 4 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$\tag{4}$$

$$P(X \leq 3) = p_X(1) + p_X(2) + p_X(3) \tag{5}$$

$$= 0.85 \tag{6}$$

Task 22 *(Continuous RV) Compute $P(X \leq c)$ with X being the continuous random variable shown in figure 2.*

$$p_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

$$\tag{8}$$

$$\begin{aligned} P(X \le c) &= \int_{-\infty}^{c} p_X(x)dx && (9)\\ &= \int_{a}^{c} p_X(x)dx && (10)\\ &= \int_{a}^{c} \frac{1}{b-a}dx && (11)\\ &= \frac{c-a}{b-a} && (12) \end{aligned}$$

Task 23 *Regard a volleyball team and their 3 opponent teams. A_i is the event of getting team 'i' as opponent. The probabilities are given as*

$$P(A_1) = 0.3, \qquad P(A_2) = 0.4, \qquad P(A_3) = 0.3. \tag{13}$$

The event B is the event of winning the game, and

$$P(B|A_1) = 0.1, \qquad P(B|A_2) = 0.6, \qquad P(B|A_3) = 0.3. \tag{14}$$

Given the knowledge that your team wins. What is the probability $P(A_2|B)$ that the opponent was team 2?

Using Bayes Rule and the total probability theorem we get

$$\begin{aligned} P(A_2|B) &= \frac{P(A_2 \cap B)}{P(B)} && (15)\\ &= \frac{P(A_2)P(B|A_2)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)} \qquad = \frac{2}{3}. && (16) \end{aligned}$$

Task 24 *Give the PDF of a random variable X which is uniformly distributed between a and b, and zero otherwise. Compute the expected value of X.*

$$p_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Task 25 *Discrete Random Variables:*

See MATLAB solution.