Modeling and System Identification (Modellbildung und Systemidentifikation)– Exam

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Page	0	1	2	3	4	5	6	7	8	9	sum
Points on page (max)	3	10	9	9	5	5	11	5	10	0	67
Points obtained											
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Intermediate sum (max)	3	13	22	31	36	41	52	57	67	67	

Mark: Exam inspected on: Signature of examiner:

Surname: First name: Matriculation number:

Subject: Programme: Bachelor ☐ Master ☐ Lehramt ☐ others ☐ Signature:

Please fill in your name above. For the multiple choice questions, which give exactly one point, tick exactly one box for the right answer. For the text questions, give a short formula or text answer just below the question in the space provided, and, if necessary, write on the **backpage of the same sheet** where the question appears, and add a comment "see backpage". Do not add extra pages (for fast correction, all pages will be separated for parallelization). The exam is a closed book exam, i.e. no books or other material are allowed besides 2 sheet (with 4 pages) of notes and a non-programmable calculator. Some legal comments are found in a footnote.[1]

1. Give the probability density functions (PDF) $p_1(x)$ and $p_2(x)$ for normally distributed random variables with means $\mu_1 = -1$ cm and $\mu_2 = 1$ cm and variance $\sigma_1^2 = 4$ cm^2 and standard deviation $\sigma_2 = 3$ cm. Add a sketch of both in the same plot including numbers and units on the x-axis:

$$p_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \dots \qquad\qquad p_2(x) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \dots$$

Sketch:

3	

[1] WITHDRAWING FROM AN EXAMINATION: In case of illness, you must supply proof of your illness by submitting a medical report to the Examinations Office. Please note that the medical examination must be done at the latest on the same day of the missed exam. In case of illness while writing the exam please contact the supervisory staff, inform them about your illness and immidiatelly see your doctor. The medical certificate must be submitted latest 3 days after the medical examination. More informations: http://www.tf.uni-freiburg.de/studies/exams/withdrawing_exam.html
CHEATING/DISTRUBING IN EXAMINATIONS: A student who disrupts the orderly proceedings of an examination will be excluded from the remainder of the exam by the respective examiners or invigilators. In such a case, the written exam of the student in question will be graded as 'nicht bestanden' (5.0, fail) on the grounds of cheating. In severe cases, the Board of Examiners will exclude the student from further examinations.

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2. Given the uniform distribution $p_X(x)$ of the random variable X on the interval $[1, 4]$, calculate the mean of the the random variable $Y = 1 + 6X$

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3. Regard a random variable $X \in \mathbb{R}^n$ with mean $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$. For a fixed $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times n}$, with C of full rank, regard another random variable Y defined by $Y = Ab + C^{-1}X$. What is the mean and covariance matrix of Y?

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4. Given two functions $f(x) = \log(10x) + 3x^2$ and $g(x) = -\log(10x) + 2x^2$. Which of the two functions is convex on $(0, \infty)$ and why? Select the convex function and calculate its minimizer x^*, sketch the function and highlight the minimum.

$x^* =$. **Sketch:**

3	

5. (a) Give a formula for the minimizer x^* of the function $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x) = \frac{1}{2}\|Ax - b\|_2^2$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given. You can assume that A has rank n.

2	

(b) Assume now that b is a random variable with mean μ_b and covariance matrix Σ_b, while A remains fixed. This makes x^* also a random variable. What is the mean and what is the covariance matrix of x^*?

2	

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6. Given a set of measurements y_N following the model $y_N = \Phi_N \theta_0 + \epsilon$, where Φ_N is a regression matrix, θ_0 a vector with true parameter values and $\epsilon(k) \sim \mathcal{N}(0, \sigma_\epsilon^2)$ the noise contribution for $k = 1, ..., N$, we can compute the LLS estimator of the parameters θ as $\hat{\theta}_{\mathrm{LS}}$. Defining the covariance of $\hat{\theta}_{\mathrm{LS}}$ as $\Sigma_{\hat{\theta}}$ and Φ^+ is the pseudo-inverse of Φ, which of the following is NOT true?

(a) ☐ $\hat{\theta}_{\mathrm{LS}}$ is a random variable	(b) ☐ $\hat{\theta}_{\mathrm{LS}} = \Phi_N^+ y_N$
(c) ☐ $\Sigma_{\hat{\theta}} = \sigma_\epsilon^2 (\Phi_N^{+\mathrm{T}} \Phi_N^+)$	(d) ☐ $\hat{\theta}_{\mathrm{LS}} \sim \mathcal{N}(\theta_0, \Sigma_{\hat{\theta}})$

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7. In the case given in the previous question, if the measurements y_N come from a single experiment, which condition on the noise do we need to require in order to be able to compute an estimate of σ_ϵ^2?

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8. Imagine that the condition asked in the previous question is NOT met. We know that the noise has zero mean and covariance Σ_{ϵ_N}. What would be the covariance matrix $\Sigma_{\hat{\theta}}$ of the unweighted LLS estimate?

(a) ☐ $\Phi_N^{\mathrm{T}} \Sigma_{\epsilon_N}^{-1} \Phi_N$	(b) ☐ $\Sigma_{\epsilon_N} \Phi_N^{+\mathrm{T}} \Phi_N^+$
(c) ☐ $\Sigma_{\epsilon_N}^{-1} \Phi_N^{\mathrm{T}} \Phi_N$	(d) ☐ $\Phi_N^+ \Sigma_{\epsilon_N} \Phi_N^{+\mathrm{T}}$

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9. Given a one step ahead prediction model $\boxed{y(k) = \theta_2 u(k)^3 + \theta_1 y(k-1) + \epsilon(k)}$ with unknown parameter vector $\theta = (\theta_1, \theta_2)^T$, and assuming i.i.d. noise $\epsilon(k)$ with zero mean, and given a sequence of N scalar input and output measurements $u(1), \ldots, u(N)$ and $y(1), \ldots, y(N)$, we want to compute the linear least squares (LLS) estimate $\hat{\theta}$ by minimizing a function $f(\theta) = \|y_N - \Phi_N \theta\|_2^2$. How do we need to choose the matrix Φ_N and vector y_N?

$\Phi_N =$ $\qquad\qquad$ $y_N =$

3	

10. Recall: given a linear model $y = \varphi^\top \theta + \epsilon$, with $y \in \mathbb{R}, \varphi \in \mathbb{R}^m$ and $\theta \in \mathbb{R}^m$, from which we obtain a new measurement $y(k)$ at times $k\Delta t$, with $k = 0, 1, \ldots, N, \ldots$, where ϵ is i.i.d. and Gaussian noise. The Recursive Least Squares algorithm with forgetting factor α is:

$$Q_{N+1} = \alpha Q_N + \varphi(N+1)\varphi(N+1)^\top$$
$$\hat{\theta}(N+1) = \hat{\theta}(N) + Q_{N+1}^{-1}\varphi(N+1)[y(N+1) - \varphi(N+1)^\top \hat{\theta}(N)]$$

Consider now a scalar model $y = \theta + \epsilon$, where $y, \theta \in \mathbb{R}$, and where new measurements $y(k)$ (with additive i.i.d. Gaussian noise) are obtained every $\Delta t = 2[\mathrm{s}]$. If we use RLS with a forgetting factor $\alpha = \frac{1}{2}$, which is the value that Q_k reaches when $k \to \infty$?

$$\lim_{k\to\infty} Q_k =$$

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11. Consider the same problem as in the previous question. Which of the following represents the update equation on the estimate $\hat{\theta}(k+1)$ when $k \to \infty$?

(a) ☐ $\hat{\theta}(k+1) = \hat{\theta}(k) - y(k+1)$	(b) ☐ $\hat{\theta}(k+1) = \frac{\hat{\theta}(k)+y(k+1)}{2}$
(c) ☐ $\hat{\theta}(k+1) = \frac{3\hat{\theta}(k)+y(k+1)}{4}$	(d) ☐ $\hat{\theta}(k+1) = \hat{\theta}(k)$

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12. Given the probability density function of the exponential distribution, $p_X(x) = \theta e^{-\theta x}$, with an unknown parameter θ, and a set of measurements $y_N = [y(1), y(2), \ldots, y(N)]^T$, what is the right minimisation problem you need to solve for a maximum likelihood estimate of θ? The problem is: $\min\limits_{\theta} \ldots$?

(a) ☐ $\|\theta e^{-\theta y(k)}\|_2^2$	(b) ☐ $\sum_{k=1}^{N} \theta e^{-\theta y(k)}$
(c) ☐ $\theta^N e^{-\theta \sum_{k=1}^{N} y(k)}$	(d) ☐ $-N \log(\theta) + \theta \sum_{k=1}^{N} y(k)$

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13. In the MLE framework, the Fisher information matrix M together with the Cramer-Rao-Inequality can be used to obtain information about the covariance $\Sigma_{\hat{\theta}}$ of an unbiased estimator $\hat{\theta}$. Define $L(\theta, y_N) := -\log p(y_N|\theta)$ as the negative log likelihood function of an MLE problem.

   (a) Define the Fisher information matrix M.

1	

   (b) What is the relation between $\Sigma_{\hat{\theta}}$ and M?

1	

   (c) In general, computing an explicit expression for M is rather difficult. State briefly the reason for this problem:

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   (d) Give an example of a MLE problem for which M can be easily computed.

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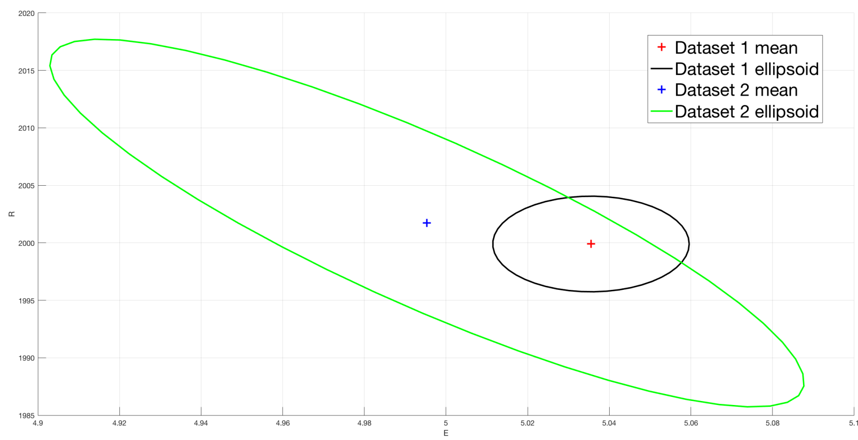
14. We want to estimate the resistance $R\,[\Omega]$ of a new metal and we found in the only existing previous article that an estimate of R is given by $10\,\Omega$ with standard deviation $0.5\,\Omega$. Our own measurement apparatus sets a current $I\,[\mathrm{A}]$ as a noise-free input, and measures the output voltage $V\,[\mathrm{V}]$ which has Gaussian errors with a standard deviation of $0.1\,\mathrm{V}$. Given a set of N measurements, $[V(1),\ldots,V(N)]$ obtained from a set $[I(1),\ldots,I(N)]$, what function is minimized by the Bayesian Maximum-A-Posteriori (MAP) estimator in this context? To simplify notation we assume that all variables are made unitless.

(a) ☐ $\frac{(R-10)^2}{0.5}+\sum_{i=1}^{N}\frac{(V(i)/I(i)-R)^2}{0.1}$	(b) ☐ $\frac{(R-10)^2}{0.5}+\sum_{i=1}^{N}\frac{(V(i)-I(i)R)^2}{0.1}$
(c) ☐ $4(R-10)^2+\sum_{i=1}^{N}100(V(i)-I(i)R)^2$	(d) ☐ $0.5(R-10)^2+\sum_{i=1}^{N}0.1(V(i)-I(i)R)^2$

| 1 | |

15. Consider an experiment where you are trying to estimate two parameters $\theta=[E,R]^\top$ by means of two different measurement setups. Given the covariance matrix Σ_θ of the estimated parameters $\hat{\theta}$ for both setups, you decide to plot the confidence ellipsoids in order to assess the quality of the estimation. If you obtain as an output:



(a) Which experimental setup is more trustworthy and why?

| 2 | |

(b) What is the relation between the eigenvalues and eigenvectors of the matrix Σ_θ and the confidence ellipsoids?

| 2 | |

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16. Exponential smoothing is a technique often used in economics for low-pass filtering time series data $x(k) \in \mathbb{R}$. In its simplest form, exponential smoothing is given by the following:

$$s(k) = \alpha \cdot x(k) + (1 - \alpha) \cdot s(k-1), \quad k = 1, 2, \ldots, \tag{1}$$

with the smoothed signal $s(k) \in \mathbb{R}$ and smoothing factor $\alpha \in \mathbb{R}$. The time series consists of noisy measurements, $x(k) = \overline{x}(k) + \epsilon$, with real-valued random i.i.d. noise $\epsilon \sim \mathcal{N}(0, \sigma^2)$.

Characterize the system given by Equation (1). Is the system:

(a) linear or nonlinear?

(b) time-varying or time-invariant?

(c) an IIR or FIR filter?

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17. What should be in practice a good sampling rate for a acquiring a periodic signal with $f = 255\text{Hz}$?

(a) ☐ 256Hz	(b) ☐ 510Hz	(c) ☐ 1020Hz	(d) ☐ 1200MHz

1	

18. You measure a signal where the signal-to-noise-ratio (SNR) at a certain frequency f_0 is given by 60 dB. How accurately can you estimate the amplitude of this frequency component (approximately)?

(a) ☐ $0.1\ \%$	(b) ☐ $0.3\ \%$	(c) ☐ $1\ \%$	(d) ☐ $3\ \%$

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19. Consider a vector of real numbers $u(0), \dots, u(N-1)$. We recall that the DFT is defined by:

$$U(m) := \sum_{k=0}^{N-1} u(k) e^{-j\frac{2\pi mk}{N}}.$$

Compute and plot the real parts of the DFT signal U corresponding to the following input signals u with length $N = 8$:

(a) $u_1 = [2, 1, 0, 1, 2, 1, 0, 1]^\top$:

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(b) $u_2 = [1, 0, 0, 0, 0, 0, 0, 0]^\top$:

3

(c) Assume now that we sample new measurements u_3 at $f_s = 100\,\text{Hz}$ and we take $N = 1000$ samples. What is the frequency resolution of the DFT signal, i.e. what is the smallest nonzero frequency on the frequency axis?

2

20. A system is excited with a periodic excitation signal $u(t)$ of period T that is for $t \in [0, T/2)$ given by $u(t) = -7$ and for $t \in [T/2, T]$ by $u(t) = 1$. What is the **crest factor** of this signal?

(a) ☐ $1/7$	(b) ☐ $7/\sqrt{7}$	(c) ☐ $7/5$	(d) ☐ $7/4$

1

21. Explain in four lines, in your own words, the phenomenon of *leakage* arising in frequency domain identification.

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22. What is multisine identification? Why do we choose different phases for the sinusoids of different frequencies present in the multisine signal?

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23. Assuming you want to estimate the xy-position and the xy-velocity of a car driving on the streets of Freiburg in real time. The car has a GPS sensor onboard which gives you a position estimate in the desired coordinate system with a frequency of 10 Hz. In the following tasks we will solve that problem using the Kalman filter.

Recall a general linear system with state $x(k) \in \mathbb{R}^n$ and measurement $y(k) \in \mathbb{R}^m$ given by the following equations:

$$\begin{aligned} x_{k+1} &= A_k x_k + b_k + w_k \\ y_k &= C_k x_k + v_k \end{aligned}$$

with w_k, v_k as i.i.d. zero mean noises with covariances W_k, V_k. Then, the steps of the Kalman Filter to for estimate the state $x(k)$ are:

Prediction Step :

$$\begin{aligned} \hat{x}_{[k|k-1]} &= A_{k-1} \cdot \hat{x}_{[k-1|k-1]} + b_{k-1} \\ P_{[k|k-1]} &= A_{k-1} \cdot P_{[k-1|k-1]} \cdot A_{k-1}^\top + W_{k-1} \end{aligned}$$

Innovation Update Step :

$$\begin{aligned} P_{[k|k]} &= \left(P_{[k|k-1]}^{-1} + C_k^\top V_k^{-1} C_k \right)^{-1} \\ \hat{x}_{[k|k]} &= \hat{x}_{[k|k-1]} + P_{[k|k]} \cdot C_k^\top V_k^{-1} (y_k - C_k \hat{x}_{[k|k-1]}) \end{aligned}$$

(a) We assume that the car moves with a constant velocity. Develop the continuous model equations for the positions $p_x(t)$ $[\mathrm{m}]$ and $p_y(t)$ $[\mathrm{m}]$ using the corresponding velocities s_x $[\mathrm{m/s}]$ and s_y $[\mathrm{m/s}]$.

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(b) Discretize the model equations using the measurements on the time grid. Change the continuous index $t \in \mathbb{R}$ to a discrete index $k \in \mathbb{Z}$.

1	

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(c) Now assume that the velocity is not constant, but can change with velocity changes $a(k)$ modeled by a Gaussian distribution with zero mean and yet unspecified standard deviation. Note that this is physically due to the accelerations and braking of the car. What are the new discretized model equations?

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(d) Using the developed model equations, define the expressions for the state vector x, the process model matrix A and the measurement model matrix C.

2	

(e) Considering the dynamic car model used, your general knowledge about cars driving on the street, and the fact that the sensor gives you position estimates with a standard deviation of $\sigma_w = 0.5$ m, give the values of the process and measurement noise covariance matrices W and V, including units. Justify your choise for the covariance matrices.

4	

(f) Imagine that the car drives in to a tunnel where you have definitely no possibility to receive any new GPS measurements. Propose a procedure for this scenario and describe the effects of this procedure on the covariance matrix of the state $P(k)$.

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Additional material

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