Numerical Optimal Control with DAEs
Lecture 13: Periodic Optimal Control with DAEs

Sébastien Gros

AWESCO PhD course

Outline

Periodic OCP

Periodic OCP:

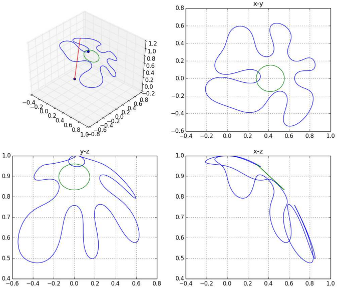
$$
\begin{aligned}
\min_{\mathbf{x}(.),\mathbf{u}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\
& \mathbf{h}\left(\mathbf{x}(t),\mathbf{u}(t),t\right) \leq 0 \\
& \mathbf{x}\left(0\right) - \mathbf{x}\left(t_{\mathrm{f}}\right) = 0
\end{aligned}
$$

- Free initial/terminal conditions...
- ... but they have to match !!

Periodic OCP

Periodic OCP:

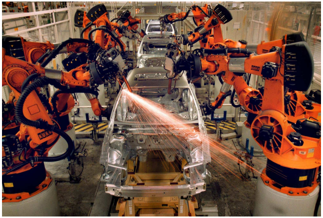
$$\begin{aligned} \min_{\mathbf{x}(.),\mathbf{u}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{h}\left(\mathbf{x}(t),\mathbf{u}(t),t\right) \leq 0 \\ & \mathbf{x}\left(0\right) - \mathbf{x}\left(t_{\mathrm{f}}\right) = 0 \end{aligned}$$



- Free initial/terminal conditions...
- ... but they have to match !!

Examples of applications:

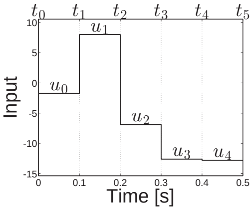
- Simulated moving bed
- Filtration processes
- Low-Density Polyethylene Process
- Cyclic motions (robotic and mobile applications)
- Airborne Wind Energy systems



NLP from Multiple-Shooting - Periodic OCP

OCP:

$$
\begin{aligned}
\min_{\mathbf{x}(.),\mathbf{u}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\
& \mathbf{h}\left(\mathbf{x}(t),\mathbf{u}(t),t\right) \leq 0 \\
& \mathbf{x}\left(0\right) - \mathbf{x}\left(t_{\mathrm{f}}\right) = 0
\end{aligned}
$$

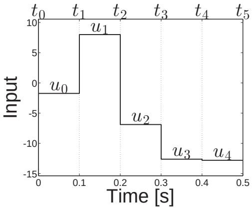


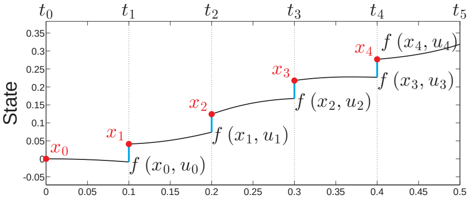
NLP from Multiple-Shooting - Periodic OCP

OCP:

$$\begin{aligned} \min_{\mathbf{x}(.),\mathbf{u}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{h}\left(\mathbf{x}(t),\mathbf{u}(t),t\right) \leq 0 \\ & \mathbf{x}\left(0\right) - \mathbf{x}\left(t_{\mathrm{f}}\right) = 0 \end{aligned}$$

$\mathbf{f}\left(\mathbf{x}_k,\mathbf{u}_k\right)$ integrates the dynamics $\mathbf{F}$ over the time interval $[t_k,\ t_{k+1}]$





NLP from Multiple-Shooting - Periodic OCP

OCP:

$$\min_{\mathbf{x}(.),\mathbf{u}(.)} \quad \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right)$$

$$\text{s.t.} \quad \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0$$

$$\mathbf{h}\left(\mathbf{x}(t),\mathbf{u}(t),t\right) \leq 0$$

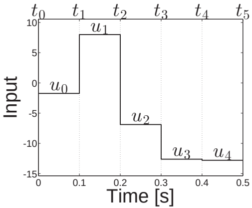
$$\color{blue}{\mathbf{x}\left(0\right) - \mathbf{x}\left(t_f\right)} = 0$$

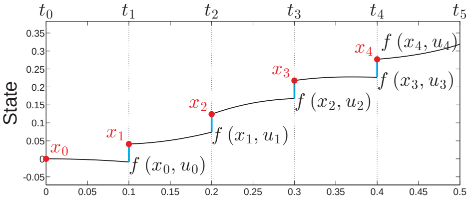
$\mathbf{f}\left(\color{red}{\mathbf{x}_k}, \mathbf{u}_k\right)$ integrates the dynamics $\mathbf{F}$ over the time interval $[t_k,\ t_{k+1}]$

NLP with $\mathbf{w} = \left\{\color{red}{\mathbf{x}_0}, \mathbf{u}_0, ..., \color{red}{\mathbf{x}_{N-1}}, \mathbf{u}_{N-1}, \color{red}{\mathbf{x}_N}\right\}$

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

s.t.





NLP from Multiple-Shooting - Periodic OCP

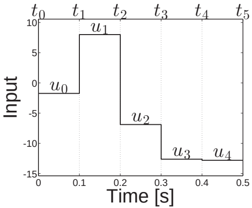
OCP:

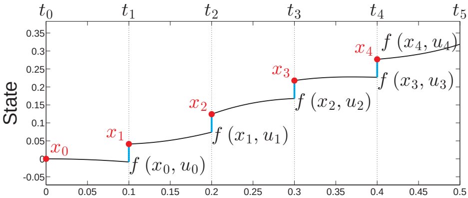
$$\begin{aligned}\min_{\mathbf{x}(.),\mathbf{u}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{h}\left(\mathbf{x}(t),\mathbf{u}(t),t\right) \leq 0 \\ & \mathbf{x}\left(0\right) - \mathbf{x}\left(t_\mathrm{f}\right) = 0\end{aligned}$$

$\mathbf{f}\left(\mathbf{x}_k,\mathbf{u}_k\right)$ integrates the dynamics $\mathbf{F}$ over the time interval $[t_k,\ t_{k+1}]$

NLP with $\mathbf{w} = \left\{\mathbf{x}_0,\mathbf{u}_0,\ldots,\mathbf{x}_{N-1},\mathbf{u}_{N-1},\mathbf{x}_N\right\}$

$$\begin{aligned}\min_{\mathbf{w}} \quad & \Phi\left(\mathbf{w}\right) \\ \text{s.t.} \quad & \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \mathbf{x}_0 - \mathbf{x}_N \\ \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}_0\right) - \mathbf{x}_1 \\ \ldots \\ \mathbf{f}\left(\mathbf{x}_N,\mathbf{u}_{N-1}\right) - \mathbf{x}_{N-1} \end{bmatrix} = 0\end{aligned}$$





NLP from Multiple-Shooting - Periodic OCP

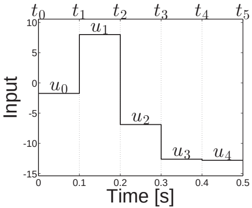
OCP:

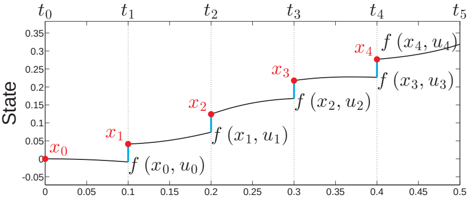
$$\min_{\mathbf{x}(.),\mathbf{u}(.)} \quad \phi\left(\mathbf{x}(.),\mathbf{u}(.)\right)$$
$$\text{s.t.} \quad \mathbf{F}\left(\dot{\mathbf{x}}(t),\mathbf{x}(t),\mathbf{u}(t)\right)=0$$
$$\mathbf{h}\left(\mathbf{x}(t),\mathbf{u}(t),t\right)\leq 0$$
$$\mathbf{x}(0)-\mathbf{x}(t_f)=0$$

$\mathbf{f}(\mathbf{x}_k,\mathbf{u}_k)$ integrates the dynamics $\mathbf{F}$ over the time interval $[t_k,\ t_{k+1}]$

NLP with $\mathbf{w}=\{\mathbf{x}_0,\mathbf{u}_0,\ldots,\mathbf{x}_{N-1},\mathbf{u}_{N-1},\mathbf{x}_N\}$

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{g}(\mathbf{w})=\begin{bmatrix}\mathbf{x}_0-\mathbf{x}_N\\ \mathbf{f}(\mathbf{x}_0,\mathbf{u}_0)-\mathbf{x}_1\\ \ldots\\ \mathbf{f}(\mathbf{x}_N,\mathbf{u}_{N-1})-\mathbf{x}_{N-1}\end{bmatrix}=0$$
$$\mathbf{h}(\mathbf{w})=\begin{bmatrix}\mathbf{h}(\mathbf{x}_0,\mathbf{u}_0)\\ \ldots\\ \mathbf{h}(\mathbf{x}_{N-1},\mathbf{u}_{N-1})\end{bmatrix}\leq 0$$





Periodic OCP & Multiple-shooting - Sparsity pattern

Equality constraints...

$$\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \mathbf{x}_0 - \mathbf{x}_N \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{bmatrix}$$

Periodic OCP & Multiple-shooting - Sparsity pattern

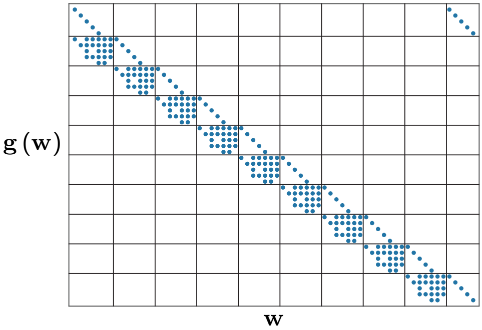
Equality constraints...

$$\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \mathbf{x}_0 - \mathbf{x}_N \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{bmatrix}$$

... pattern of the constraints Jacobian

Ordering $\mathbf{w} = \{\mathbf{x}_0, \mathbf{u}_0, \dots, \mathbf{x}_{N-1}, \mathbf{u}_{N-1}, \mathbf{x}_N\}$

$\nabla \mathbf{g}(\mathbf{w})^\mathsf{T}$



Periodic OCP & Multiple-shooting - Sparsity pattern

Equality constraints...

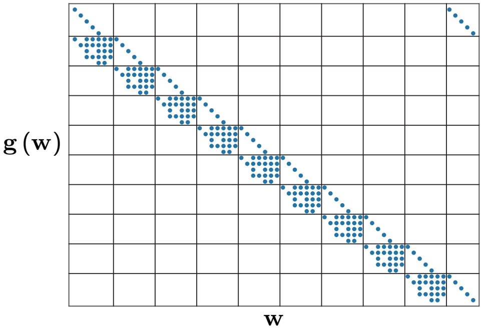
$$\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \mathbf{x}_0 - \mathbf{x}_N \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{bmatrix}$$

Observe that we have lost the "classical" banded structure of multiple-shooting !! The same happens with direct collocation...

... pattern of the constraints Jacobian

Ordering $\mathbf{w} = \{\mathbf{x}_0, \mathbf{u}_0, \dots, \mathbf{x}_{N-1}, \mathbf{u}_{N-1}, \mathbf{x}_N\}$

$\nabla \mathbf{g}(\mathbf{w})^\top$



Periodic OCP & Multiple-shooting - Sparsity pattern

Equality constraints...

$$\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \mathbf{x}_0 - \mathbf{x}_N \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \ldots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{bmatrix}$$

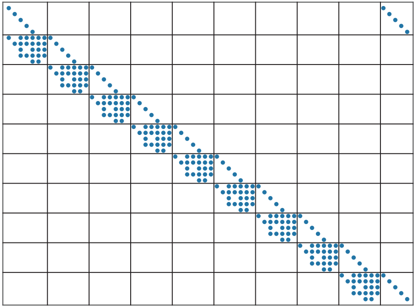
Observe that we have lost the "classical" banded structure of multiple-shooting !! The same happens with direct collocation...

... pattern of the constraints Jacobian

Ordering $\mathbf{w} = \{\mathbf{x}_0, \mathbf{u}_0, \ldots, \mathbf{x}_{N-1}, \mathbf{u}_{N-1}, \mathbf{x}_N\}$

$\nabla \mathbf{g}(\mathbf{w})^\top$

$\mathbf{g}(\mathbf{w})$



$\mathbf{w}$

Structure-exploiting solvers dedicated to direct optimal control are not all designed for taking in the "off-band" block !! Have that in mind when selecting tools for solving periodic optimal control problems using direct methods...

Outline

A simple but problematic Periodic OCP

Consider the π-OCP:

$$\begin{aligned}\min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x}, \\ & \mathbf{x}(1) - \mathbf{x}(0) = 0\end{aligned}$$

A simple but problematic Periodic OCP

Consider the π**-OCP**:

$$\begin{aligned} \min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}, \\ & \mathbf{x}(1) - \mathbf{x}(0) = 0 \end{aligned}$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A simple but problematic Periodic OCP

Consider the π-OCP:

$$\begin{aligned}\min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}, \\ & \mathbf{x}(1) - \mathbf{x}(0) = 0\end{aligned}$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

π-**constraint** becomes:

$$\boldsymbol{\pi} = [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and requires $\theta = 2k\pi$.

A simple but problematic Periodic OCP

Consider the π-OCP:

$$\begin{aligned} \min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}, \\ & \mathbf{x}(1) - \mathbf{x}(0) = 0 \end{aligned}$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

π-**constraint** becomes:

$$\boldsymbol{\pi} = [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and requires $\theta = 2k\pi$.

The π-OCP can be reformulated as:

$$\begin{aligned} \min_{\mathbf{x}_0,u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\theta}(t) = u(t) \\ & \boldsymbol{\pi} = \underbrace{[R(\theta) - I]}_{=0 \text{ at solution}}\mathbf{x}_0 = 0 \end{aligned}$$

A simple but problematic Periodic OCP

Consider the π-OCP:

$$\min_{\mathbf{x}(.),u(.)} \quad \frac{1}{2}\int_0^1 u(t)^2$$
$$\text{s.t.} \quad \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x},$$
$$\mathbf{x}(1) - \mathbf{x}(0) = 0$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

π-**constraint** becomes:

$$\boldsymbol{\pi} = [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and requires $\theta = 2k\pi$.

The π-OCP can be reformulated as:

$$\min_{\mathbf{x}_0,u(.)} \quad \frac{1}{2}\int_0^1 u(t)^2$$
$$\text{s.t.} \quad \dot{\theta}(t) = u(t)$$
$$\boldsymbol{\pi} = \underbrace{[R(\theta) - I]}_{=0 \text{ at solution}}\mathbf{x}_0 = 0$$

If $u(t)$ piecewise-constant, i.e. $\theta = \frac{1}{N}\sum_k \mathbf{u}_k$:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$

at the solution

A simple but problematic Periodic OCP

Consider the π-OCP:

$$\begin{aligned} \min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}, \\ & \mathbf{x}(1) - \mathbf{x}(0) = 0 \end{aligned}$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0)\,, \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

π-**constraint** becomes:

$$\boldsymbol{\pi} = [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and requires $\theta = 2k\pi$.

The π-OCP can be reformulated as:

$$\begin{aligned} \min_{\mathbf{x}_0,u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\theta}(t) = u(t) \\ & \boldsymbol{\pi} = \underbrace{[R(\theta) - I]}_{=0 \text{ at solution}}\mathbf{x}_0 = 0 \end{aligned}$$

If $u(t)$ piecewise-constant, i.e. $\theta = \frac{1}{N}\sum_k \mathbf{u}_k$:

$$\frac{\partial\boldsymbol{\pi}}{\partial\mathbf{x}_0} = 0, \qquad \frac{\partial\boldsymbol{\pi}}{\partial\mathbf{u}} = \frac{\partial R}{\partial\theta}\mathbf{x}_0\frac{\mathbf{1}}{N}^\top$$

at the solution, where $\frac{\partial R}{\partial\theta} = R\left(\frac{\pi}{2}\right)$.

A simple but problematic Periodic OCP

Consider the π-OCP:

$$\begin{aligned} \min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}, \\ & \mathbf{x}(1) - \mathbf{x}(0) = 0 \end{aligned}$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

π-**constraint** becomes:

$$\boldsymbol{\pi} = [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and requires $\theta = 2k\pi$.

The π-OCP can be reformulated as:

$$\begin{aligned} \min_{\mathbf{x}_0,u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\theta}(t) = u(t) \\ & \boldsymbol{\pi} = \underbrace{[R(\theta) - I]}_{=0 \text{ at solution}}\mathbf{x}_0 = 0 \end{aligned}$$

If $u(t)$ piecewise-constant, i.e. $\theta = \frac{1}{N}\sum_k \mathbf{u}_k$:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$

at the solution, where $\frac{\partial R}{\partial \theta} = R\left(\frac{\pi}{2}\right)$. Hence:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{w}} = \begin{bmatrix} 0 & R\left(\frac{\pi}{2}\right)\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top \end{bmatrix}$$

where $\mathbf{w} = \{\mathbf{x}_0, \mathbf{u}\}$

A simple but problematic Periodic OCP

Consider the π-OCP:

$$\begin{aligned} \min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}, \\ & \mathbf{x}(1) - \mathbf{x}(0) = 0 \end{aligned}$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

where

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

π-**constraint** becomes:

$$\boldsymbol{\pi} = [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and requires $\theta = 2k\pi$.

The π-OCP can be reformulated as:

$$\begin{aligned} \min_{\mathbf{x}_0,u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\theta}(t) = u(t) \\ & \boldsymbol{\pi} = \underbrace{[R(\theta) - I]}_{=0 \text{ at solution}}\mathbf{x}_0 = 0 \end{aligned}$$

If $u(t)$ piecewise-constant, i.e. $\theta = \frac{1}{N}\sum_k \mathbf{u}_k$:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$

at the solution, where $\frac{\partial R}{\partial \theta} = R\left(\frac{\pi}{2}\right)$. Hence:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{w}} = \begin{bmatrix} 0 & R\left(\frac{\pi}{2}\right)\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top \end{bmatrix}$$

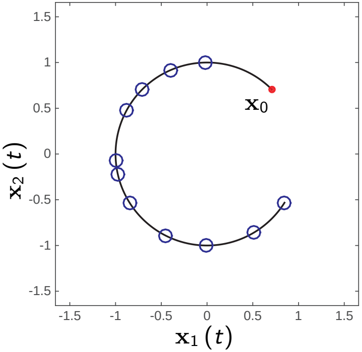
where $\mathbf{w} = \{\mathbf{x}_0, \mathbf{u}\}$, then $\mathbf{x}_0^\top R\left(\frac{\pi}{2}\right)\mathbf{x}_0 = 0$ and

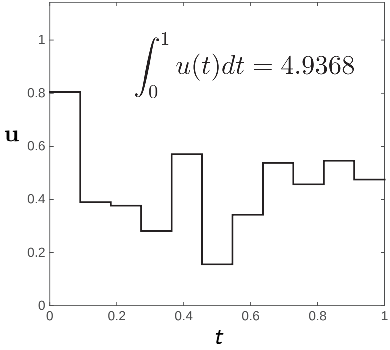
$$\mathbf{x}_0^\top \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{w}} = 0 \quad \textbf{s.t. LICQ fails !!}$$

A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \pi}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \pi}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}_{\mathbf{x}_0} \frac{\mathbf{1}}{N}^{\top}$$

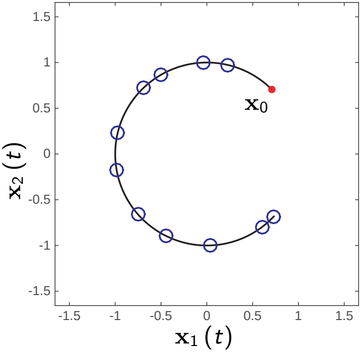


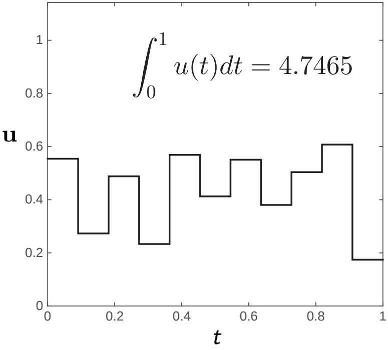


A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \pi}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \pi}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$

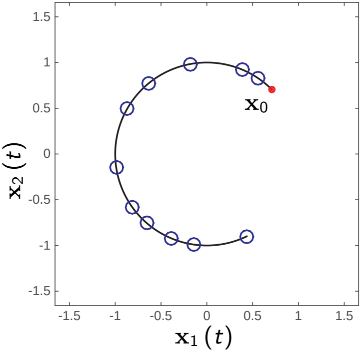


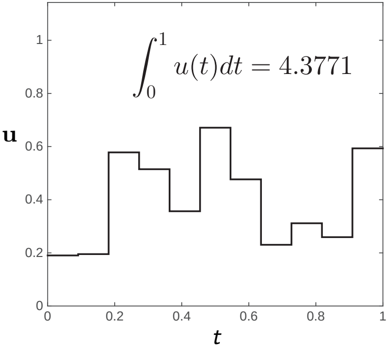


A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \pi}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \pi}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$

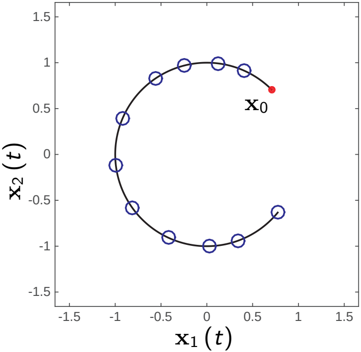


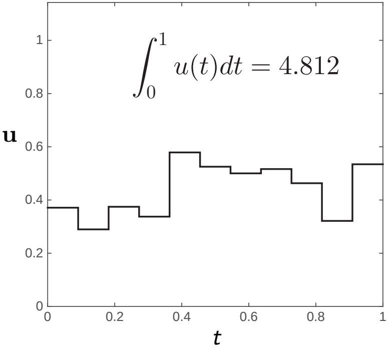


A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \pi}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \pi}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}_{\mathbf{x}_0} \frac{\mathbf{1}}{N}^\top$$

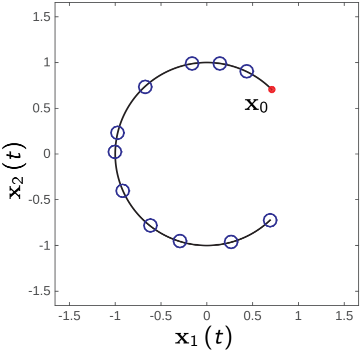


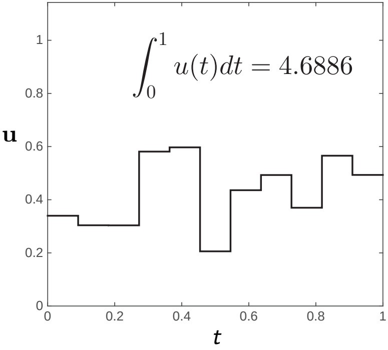


A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \pi}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \pi}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}_{\mathbf{x}_0} \frac{\mathbf{1}}{N}^{\top}$$

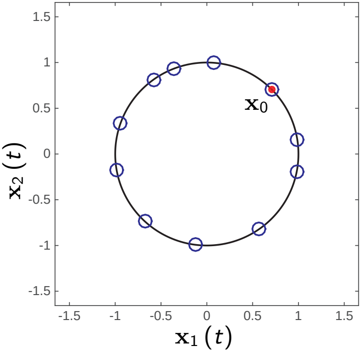


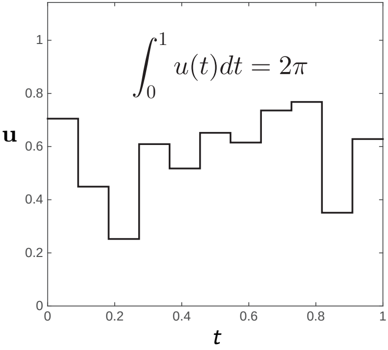


A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$

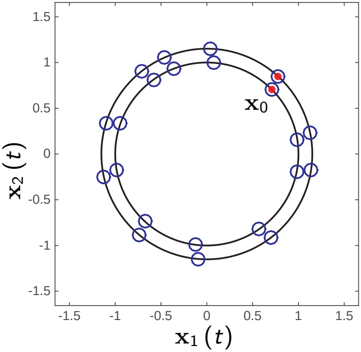


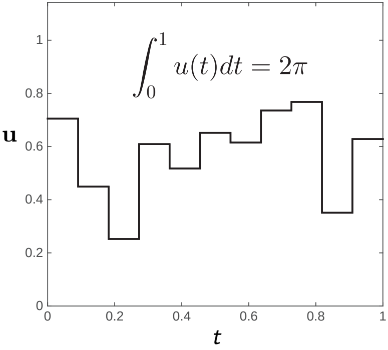


A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta} \mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$

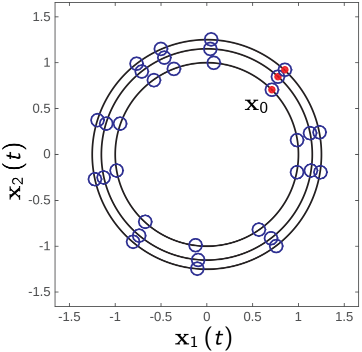


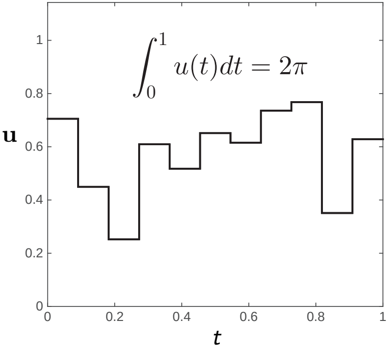


A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}_{\mathbf{x}_0} \frac{\mathbf{1}}{N}^\top$$

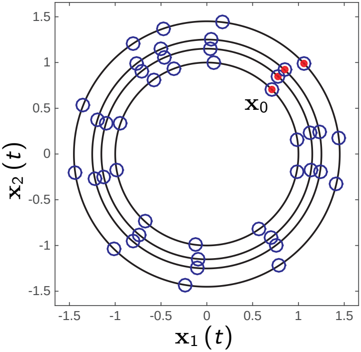


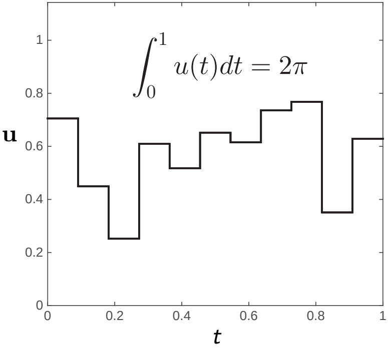


A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \pi}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \pi}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}_{\mathbf{x}_0} \frac{\mathbf{1}}{N}^\top$$

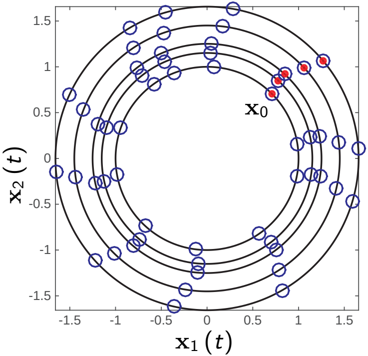


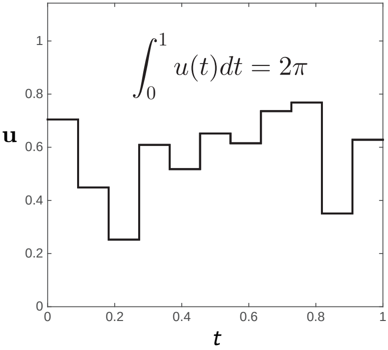


A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \pi}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \pi}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}_{\mathbf{x}_0} \frac{\mathbf{1}}{N}^\top$$

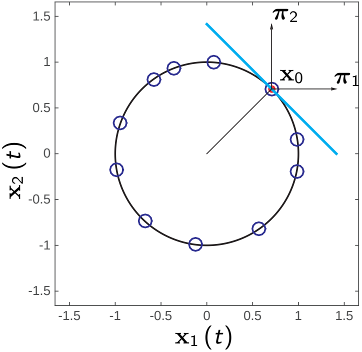


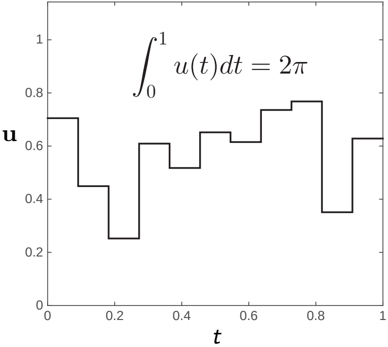


A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

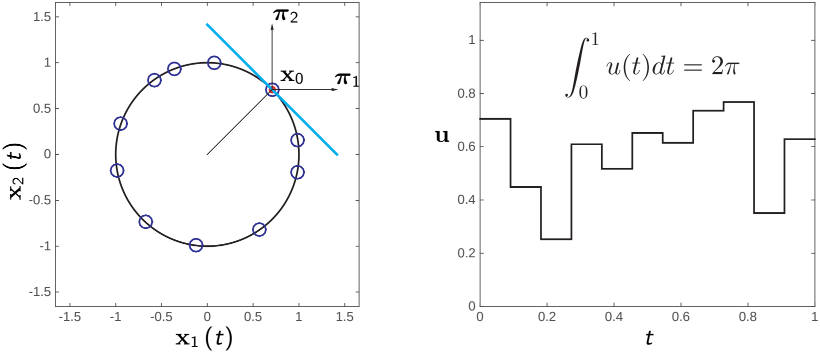
$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$





A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$

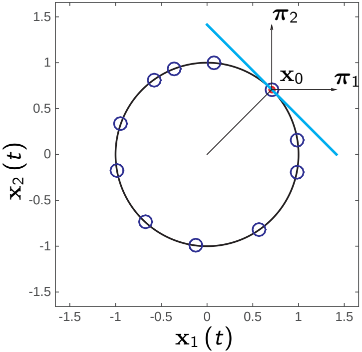


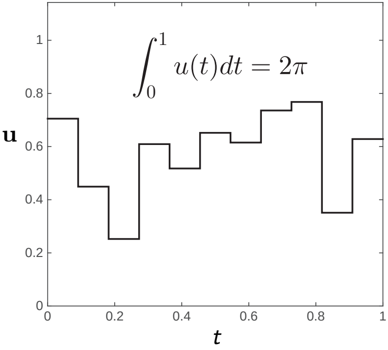
- $\mathbf{u}$ allows for satisfying $\boldsymbol{\pi} = 0$
- $\mathbf{x}_0$ has no impact on $\boldsymbol{\pi}$

A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta} \mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$



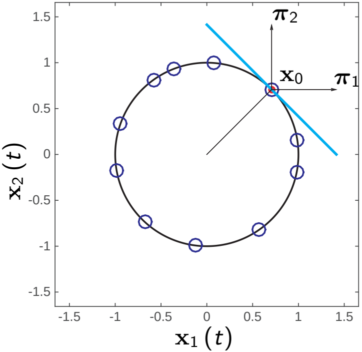


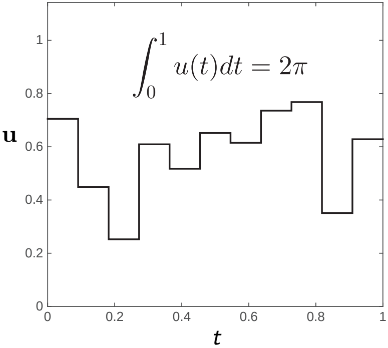
- $\mathbf{u}$ allows for satisfying $\boldsymbol{\pi} = 0$
- $\mathbf{x}_0$ has no impact on $\boldsymbol{\pi}$
- $\Delta\boldsymbol{\pi} = \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{w}} \Delta\mathbf{w}$ is restricted to the tangent

A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}_{\mathbf{x}_0} \frac{\mathbf{1}}{N}^\top$$

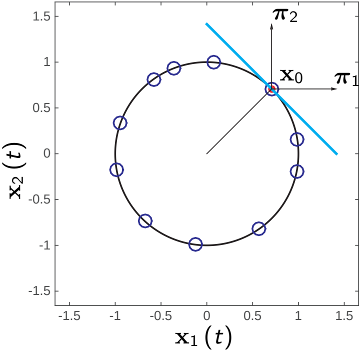


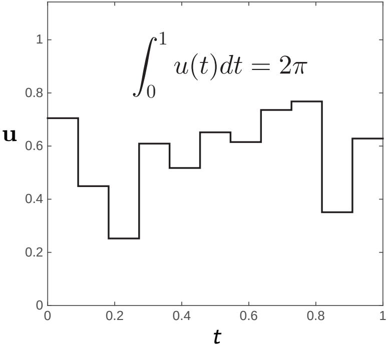


- $\mathbf{u}$ allows for satisfying $\boldsymbol{\pi} = 0$
- $\mathbf{x}_0$ has no impact on $\boldsymbol{\pi}$
- $\Delta\boldsymbol{\pi} = \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{w}} \Delta \mathbf{w}$ is restricted to the tangent
- $\boldsymbol{\pi} \in \mathbb{R}^2$ has (locally) only one degree of freedom

A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} \qquad\qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$

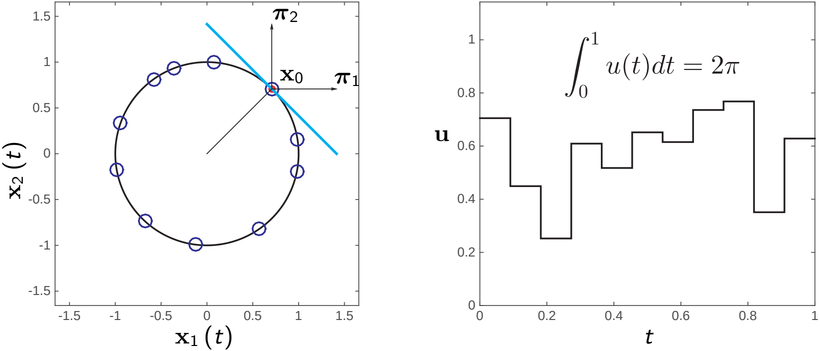




- $\mathbf{u}$ allows for satisfying $\boldsymbol{\pi} = 0$
- $\mathbf{x}_0$ has no impact on $\boldsymbol{\pi}$
- $\Delta\boldsymbol{\pi} = \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{w}}\Delta\mathbf{w}$ is restricted to the tangent
- $\boldsymbol{\pi} \in \mathbb{R}^2$ has (locally) only one degree of freedom
- i.e. $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are (locally) linearly dependent.

A simple but problematic Periodic OCP - What is going on ?!?

$$\dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0, \qquad \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$



Intuition: we impose 2 constraints via $\boldsymbol{\pi}$, but if $\mathbf{x}_1(0) = \mathbf{x}_1(1)$, then (locally) $\mathbf{x}_2(0) = \mathbf{x}_2(1)$ (or vice-versa), i.e. we have **redundant constraints**. What can we do ?!?

What about a simple constraint elimination ?

Consider the π-OCP:

$$\begin{aligned} \min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}, \\ & \mathbf{x}_1(1) - \mathbf{x}_1(0) = 0 \end{aligned}$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

What about a simple constraint elimination ?

Consider the π-OCP:

$$\begin{aligned}\min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x}, \\ & \mathbf{x}_1(1) - \mathbf{x}_1(0) = 0\end{aligned}$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

π-**constraint** reduced to 1^{st} component:

$$\boldsymbol{\pi} = \begin{bmatrix} 1 & 0 \end{bmatrix}\left[R(\theta) - I\right]\mathbf{x}_0 = 0$$

What about a simple constraint elimination ?

Consider the π-OCP:

$$\begin{aligned} \min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}, \\ & \mathbf{x}_1(1) - \mathbf{x}_1(0) = 0 \end{aligned}$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

π-**constraint** reduced to 1st component:

$$\boldsymbol{\pi} = \begin{bmatrix} 1 & 0 \end{bmatrix} [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and @ the solution:

$$\begin{aligned} \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} &= 0, \\ \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top \end{aligned}$$

What about a simple constraint elimination ?

Consider the π**-OCP**:

$$\min_{\mathbf{x}(.),u(.)} \quad \frac{1}{2}\int_0^1 u(t)^2$$
$$\text{s.t.} \quad \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x},$$
$$\mathbf{x}_1(1) - \mathbf{x}_1(0) = 0$$

Solution of the ODE reads as:

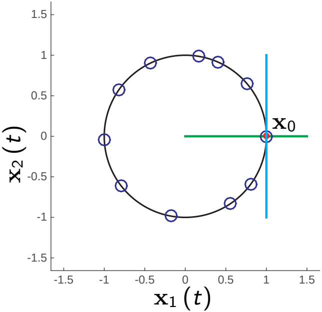
$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

π**-constraint** reduced to 1^{st} component:

$$\boldsymbol{\pi} = \begin{bmatrix} 1 & 0 \end{bmatrix} [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and @ the solution:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0,$$
$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^{\top}$$



What about a simple constraint elimination ?

Consider the π-OCP:

$$\min_{\mathbf{x}(.),u(.)} \quad \frac{1}{2}\int_0^1 u(t)^2$$
$$\text{s.t.} \quad \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x},$$
$$\mathbf{x}_1(1) - \mathbf{x}_1(0) = 0$$

Solution of the ODE reads as:

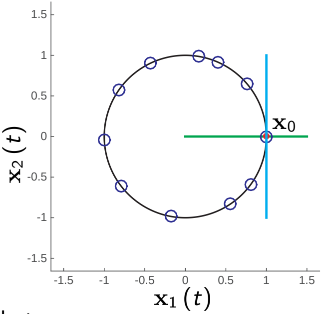
$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

π-**constraint** reduced to 1$^{\text{st}}$ component:

$$\boldsymbol{\pi} = \begin{bmatrix} 1 & 0 \end{bmatrix} [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and @ the solution:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0,$$
$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$



Note that:

$$\frac{\partial R}{\partial \theta} = R\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

What about a simple constraint elimination ?

Consider the π-OCP:

$$\begin{aligned} \min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}, \\ & \mathbf{x}_1(1) - \mathbf{x}_1(0) = 0 \end{aligned}$$

Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

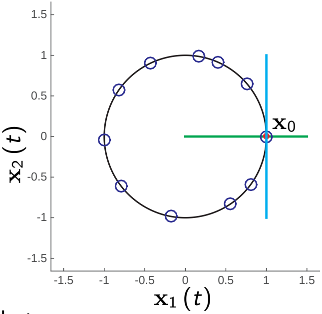
π-**constraint** reduced to 1^{st} component:

$$\boldsymbol{\pi} = \begin{bmatrix} 1 & 0 \end{bmatrix} [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and @ the solution:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0,$$

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$



Note that:

$$\frac{\partial R}{\partial \theta} = R\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

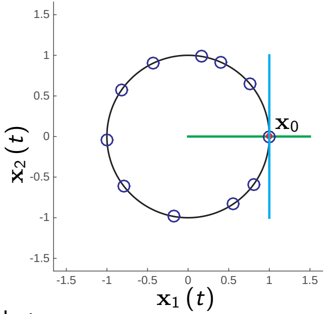
If $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\mathbf{x}_0^\top \frac{\partial R}{\partial \theta}\mathbf{x}_0 = 0$ and:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\mathbf{1}}{N}^\top = 0$$

What about a simple constraint elimination ?

Consider the π-OCP:

$$\min_{\mathbf{x}(.),u(.)} \quad \frac{1}{2}\int_0^1 u(t)^2$$
$$\text{s.t.} \quad \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x},$$
$$\mathbf{x}_1(1) - \mathbf{x}_1(0) = 0$$



Solution of the ODE reads as:

$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

π-**constraint** reduced to 1st component:

$$\boldsymbol{\pi} = \begin{bmatrix} 1 & 0 \end{bmatrix} [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and @ the solution:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0,$$
$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$

Note that:

$$\frac{\partial R}{\partial \theta} = R\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

If $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\mathbf{x}_0^\top \frac{\partial R}{\partial \theta}\mathbf{x}_0 = 0$ and:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{w}} = \begin{bmatrix} \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} & \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} \end{bmatrix} = 0 \quad \text{LICQ failure !}$$

What about a simple constraint elimination ?

Consider the π-OCP:

$$\min_{\mathbf{x}(.),u(.)} \quad \frac{1}{2}\int_0^1 u(t)^2$$
$$\text{s.t.} \quad \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x},$$
$$\mathbf{x}_1(1) - \mathbf{x}_1(0) = 0$$

Solution of the ODE reads as:

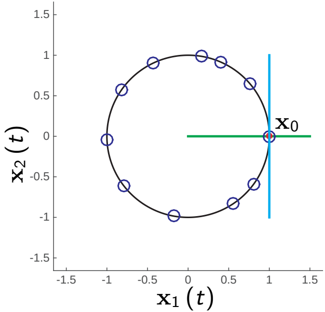
$$\mathbf{x}(1) = R(\theta)\,\mathbf{x}(0), \quad \theta = \int_0^1 u(t)\mathrm{d}t$$

π-**constraint** reduced to 1st component:

$$\boldsymbol{\pi} = \begin{bmatrix} 1 & 0 \end{bmatrix} [R(\theta) - I]\,\mathbf{x}_0 = 0$$

and @ the solution:

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}_0} = 0,$$
$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{u}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\partial R}{\partial \theta}\mathbf{x}_0 \frac{\mathbf{1}}{N}^\top$$



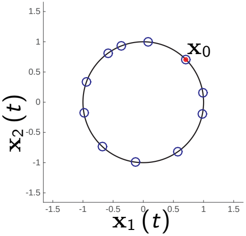
Arbitrary eliminations of redundant constraints in Periodic OCPs can yield "degenerate" situations, with $\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{w}} = 0$!!

Outline

What is the problem with this Periodic OCP ?

Consider the π-OCP:

$$\begin{aligned} \min_{\mathbf{x}(.),u(.)} \quad & \frac{1}{2}\int_0^1 u(t)^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}, \\ & \mathbf{x}(1) - \mathbf{x}(0) = 0 \end{aligned}$$



Invariant

The dynamics have a **1-dimensional invariant**. It reads as:

$$\mathcal{I}(\mathbf{x}(t)) = \frac{1}{2}\mathbf{x}(t)^\top \mathbf{x}(t) = \text{constant} \quad \forall\, \mathbf{x}(0),\, u(t)$$

$$\text{indeed } \dot{\mathcal{I}} = u\mathbf{x}^\top \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} = 0, \qquad \forall u,\, \mathbf{x}$$

such that $\mathbf{x}(t) \in \mathbb{R}^2$ is forced to evolve on a $2 - 1 = 1$-dimensional manifold.

- 2 periodic constraints are then redundant...
- Because the invariant is a manifold (not a linear space), a simple constraint elimination cannot guarantee a well-behaved OCP

Invariants in Periodic Optimal Control - Generalization

Periodic OCP

$$\begin{aligned}
\min_{\mathbf{u}(.),\mathbf{x}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\
& \mathbf{x}\left(t_{\mathrm{f}}\right) - \mathbf{x}\left(0\right) = 0
\end{aligned}$$

Invariants in Periodic Optimal Control - Generalization

Periodic OCP

$$\begin{aligned}
\min_{\mathbf{u}(.),\mathbf{x}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\
& \mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}\left(0\right) = 0
\end{aligned}$$

with a finite input parametrization $\mathbf{u}$, and using the integration function on $[0,\ t_\mathrm{f}]$ (single-shooting):

$$\mathbf{x}\left(t_\mathrm{f}\right) = \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right)$$

OCP can be written as NLP:

$$\begin{aligned}
\min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0, \mathbf{u}\right) \\
\text{s.t.} \quad & \boldsymbol{\pi} = \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right) - \mathbf{x}_0 = 0
\end{aligned}$$

Invariants in Periodic Optimal Control - Generalization

Periodic OCP

$$\begin{aligned} \min_{\mathbf{u}(.),\mathbf{x}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}\left(0\right) = 0 \end{aligned}$$

with a finite input parametrization $\mathbf{u}$, and using the integration function on $[0,\ t_\mathrm{f}]$ (single-shooting):

$$\mathbf{x}\left(t_\mathrm{f}\right) = \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)$$

OCP can be written as NLP:

$$\begin{aligned} \min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0,\mathbf{u}\right) \\ \text{s.t.} \quad & \boldsymbol{\pi} = \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right) - \mathbf{x}_0 = 0 \end{aligned}$$

Dynamics with invariants:

$$\mathcal{I}\left(\mathbf{x}\left(t\right)\right) = \mathcal{I}\left(\mathbf{x}\left(0\right)\right), \quad \forall t, \mathbf{u}, \mathbf{x}\left(0\right)$$

Invariants in Periodic Optimal Control - Generalization

Periodic OCP

$$\begin{aligned} \min_{\mathbf{u}(.),\mathbf{x}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{x}\left(t_\text{f}\right) - \mathbf{x}\left(0\right) = 0 \end{aligned}$$

with a finite input parametrization $\mathbf{u}$, and using the integration function on $[0,\ t_\text{f}]$ (single-shooting):

$$\mathbf{x}\left(t_\text{f}\right) = \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)$$

OCP can be written as NLP:

$$\begin{aligned} \min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0,\mathbf{u}\right) \\ \text{s.t.} \quad & \boldsymbol{\pi} = \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right) - \mathbf{x}_0 = 0 \end{aligned}$$

Dynamics with invariants:

$$\mathcal{I}\left(\mathbf{x}\left(t\right)\right) = \mathcal{I}\left(\mathbf{x}\left(0\right)\right), \quad \forall t, \mathbf{u}, \mathbf{x}\left(0\right)$$

yields the property:

$$\mathcal{I}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right) - \mathcal{I}\left(\mathbf{x}_0\right) = 0, \quad \forall\, \mathbf{w} = \left[\begin{array}{c} \mathbf{u} \\ \mathbf{x}_0 \end{array}\right]$$

Invariants in Periodic Optimal Control - Generalization

Periodic OCP

$$\begin{aligned} \min_{\mathbf{u}(.),\mathbf{x}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}\left(0\right) = 0 \end{aligned}$$

with a finite input parametrization $\mathbf{u}$, and using the integration function on $[0,\ t_\mathrm{f}]$ (single-shooting):

$$\mathbf{x}\left(t_\mathrm{f}\right) = \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)$$

OCP can be written as NLP:

$$\begin{aligned} \min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0,\mathbf{u}\right) \\ \text{s.t.} \quad & \boldsymbol{\pi} = \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right) - \mathbf{x}_0 = 0 \end{aligned}$$

Dynamics with invariants:

$$\mathcal{I}\left(\mathbf{x}\left(t\right)\right) = \mathcal{I}\left(\mathbf{x}\left(0\right)\right), \quad \forall t, \mathbf{u}, \mathbf{x}\left(0\right)$$

yields the property:

$$\mathcal{I}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right) - \mathcal{I}\left(\mathbf{x}_0\right) = 0, \quad \forall\, \mathbf{w} = \left[\begin{array}{c} \mathbf{u} \\ \mathbf{x}_0 \end{array}\right]$$

hence:

$$\begin{aligned} & \nabla_{\mathbf{w}}\left[\mathcal{I}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right) - \mathcal{I}\left(\mathbf{x}_0\right)\right] = \\ & \quad \left[\nabla_{\mathbf{w}}\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right) - \nabla_{\mathbf{w}}\mathbf{x}_0\right]\nabla_{\mathbf{x}_0}\mathcal{I}\left(\mathbf{x}_0\right) = 0 \end{aligned}$$

Invariants in Periodic Optimal Control - Generalization

Periodic OCP

$$\begin{aligned} \min_{\mathbf{u}(.),\mathbf{x}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}\left(0\right) = 0 \end{aligned}$$

with a finite input parametrization $\mathbf{u}$, and using the integration function on $[0,\ t_\mathrm{f}]$ (single-shooting):

$$\mathbf{x}\left(t_\mathrm{f}\right) = \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right)$$

OCP can be written as NLP:

$$\begin{aligned} \min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0, \mathbf{u}\right) \\ \text{s.t.} \quad & \boldsymbol{\pi} = \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right) - \mathbf{x}_0 = 0 \end{aligned}$$

Dynamics with invariants:

$$\mathcal{I}\left(\mathbf{x}\left(t\right)\right) = \mathcal{I}\left(\mathbf{x}\left(0\right)\right), \quad \forall t, \mathbf{u}, \mathbf{x}\left(0\right)$$

yields the property:

$$\mathcal{I}\left(\mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right)\right) - \mathcal{I}\left(\mathbf{x}_0\right) = 0, \quad \forall\, \mathbf{w} = \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_0 \end{bmatrix}$$

hence:

$$\nabla_{\mathbf{w}}\left[\mathcal{I}\left(\mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right)\right) - \mathcal{I}\left(\mathbf{x}_0\right)\right] = \underbrace{\left(\nabla_{\mathbf{w}}\mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right) - \nabla_{\mathbf{w}}\mathbf{x}_0\right)}_{\nabla_{\mathbf{w}}\boldsymbol{\pi}} \nabla_{\mathbf{x}_0}\mathcal{I}\left(\mathbf{x}_0\right) = 0$$

yields:

$$\nabla_{\mathbf{w}}\boldsymbol{\pi}\, \underbrace{\nabla_{\mathbf{x}_0}\mathcal{I}\left(\mathbf{x}_0\right)}_{\text{null space}} = 0$$

I.e. **LICQ deficiency !!**

Invariants in Periodic Optimal Control - Generalization

Periodic OCP

$$\begin{aligned}\min_{\mathbf{u}(.),\mathbf{x}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right)=0 \\ & \mathbf{x}\left(t_\mathrm{f}\right)-\mathbf{x}\left(0\right)=0\end{aligned}$$

with a finite input parametrization $\mathbf{u}$, and using the integration function on $[0,\ t_\mathrm{f}]$ (single-shooting):

$$\mathbf{x}\left(t_\mathrm{f}\right)=\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)$$

OCP can be written as NLP:

$$\begin{aligned}\min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0,\mathbf{u}\right) \\ \text{s.t.} \quad & \boldsymbol{\pi}=\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)-\mathbf{x}_0=0\end{aligned}$$

Invariants in the dynamics yield an **LICQ deficiency** in the periodicity constraints (also for ODEs) !!

Dynamics with invariants:

$$\mathcal{I}\left(\mathbf{x}\left(t\right)\right)=\mathcal{I}\left(\mathbf{x}\left(0\right)\right), \quad \forall t,\mathbf{u},\mathbf{x}\left(0\right)$$

yields the property:

$$\mathcal{I}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right)-\mathcal{I}\left(\mathbf{x}_0\right)=0, \quad \forall\, \mathbf{w}=\begin{bmatrix}\mathbf{u}\\ \mathbf{x}_0\end{bmatrix}$$

hence:

$$\nabla_{\mathbf{w}}\left[\mathcal{I}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right)-\mathcal{I}\left(\mathbf{x}_0\right)\right]= \underbrace{\left(\nabla_{\mathbf{w}}\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)-\nabla_{\mathbf{w}}\mathbf{x}_0\right)}_{\nabla_{\mathbf{w}}\boldsymbol{\pi}}\nabla_{\mathbf{x}_0}\mathcal{I}\left(\mathbf{x}_0\right)=0$$

yields:

$$\nabla_{\mathbf{w}}\boldsymbol{\pi}\underbrace{\nabla_{\mathbf{x}_0}\mathcal{I}\left(\mathbf{x}_0\right)}_{\text{null space}}=0$$

I.e. **LICQ deficiency !!**

Projection Method for Invariants

Periodic OCP:

$$
\begin{aligned}
\min_{\mathbf{x},\mathbf{z},\mathbf{u}} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}},\mathbf{x},\mathbf{z},\mathbf{u}\right) = 0 \\
& \mathbf{x}\left(t_{\mathrm{f}}\right) - \mathbf{x}\left(0\right) = 0
\end{aligned}
$$

where dynamics $\mathbf{F}$ have invariant $\mathcal{I}$.

Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u}} \quad & \phi\left(\mathbf{x},\mathbf{u}\right) \\ \text{s.t.} \quad & \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0, \mathbf{u}} \quad & \phi(\mathbf{x}, \mathbf{u}) \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

Build basis of the null space of $\nabla\mathcal{I}(\mathbf{x}_0)$:

$$Z^\top \nabla\mathcal{I}(\mathbf{x}_0) = 0$$

Note: Z is $n \times n - m$ with $\mathcal{I} \in \mathbb{R}^m$

Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u}} \quad & \phi(\mathbf{x},\mathbf{u}) \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}_0,\mathbf{u}) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

Build basis of the null space of $\nabla\mathcal{I}(\mathbf{x}_0)$:

$$Z^\top \nabla\mathcal{I}(\mathbf{x}_0) = 0$$

Note: Z is $n \times n - m$ with $\mathcal{I} \in \mathbb{R}^m$

Rewrite OCP as (NOCP):

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u}} \quad & \Phi(\mathbf{x}_0,\mathbf{u}) \\ \text{s.t.} \quad & Z^\top\left(\mathbf{f}(\mathbf{x}_0,\mathbf{u}) - \mathbf{x}_0\right) = 0 \end{aligned}$$

Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u}} \quad & \phi\left(\mathbf{x},\mathbf{u}\right) \\ \text{s.t.} \quad & \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

Build basis of the null space of $\nabla\mathcal{I}\left(\mathbf{x}_0\right)$:

$$Z^\top \nabla\mathcal{I}\left(\mathbf{x}_0\right) = 0$$

Note: Z is $n \times n - m$ with $\mathcal{I} \in \mathbb{R}^m$

Rewrite OCP as (NOCP):

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u},Z} \quad & \Phi\left(\mathbf{x}_0,\mathbf{u}\right) \\ \text{s.t.} \quad & Z^\top\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right) - \mathbf{x}_0\right) = 0 \end{aligned}$$

- Null-space Z is now function of $\mathbf{x}_0$!!

Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0, \mathbf{u}} \quad & \phi(\mathbf{x}, \mathbf{u}) \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

Build basis of the null space of $\nabla\mathcal{I}(\mathbf{x}_0)$:

$$Z^\top \nabla\mathcal{I}(\mathbf{x}_0) = 0$$

Note: Z is $n \times n - m$ with $\mathcal{I} \in \mathbb{R}^m$

Rewrite OCP as (NOCP):

$$\begin{aligned} \min_{\mathbf{x}_0, \mathbf{u}, Z} \quad & \Phi(\mathbf{x}_0, \mathbf{u}) \\ \text{s.t.} \quad & Z^\top (\mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{x}_0) = 0 \end{aligned}$$

- Null-space Z is now function of $\mathbf{x}_0$!!
- Can be hard to compute explicitly

Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0, \mathbf{u}} \quad & \phi(\mathbf{x}, \mathbf{u}) \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

Build basis of the null space of $\nabla \mathcal{I}(\mathbf{x}_0)$:

$$Z^\top \nabla \mathcal{I}(\mathbf{x}_0) = 0$$

Note: Z is $n \times n - m$ with $\mathcal{I} \in \mathbb{R}^m$

Rewrite OCP as (NOCP):

$$\begin{aligned} \min_{\mathbf{x}_0, \mathbf{u}, Z} \quad & \Phi(\mathbf{x}_0, \mathbf{u}) \\ \text{s.t.} \quad & Z^\top (\mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{x}_0) = 0 \\ & \frac{\partial \mathcal{I}}{\partial \mathbf{x}_0} Z = 0 \end{aligned}$$

- Null-space Z is now function of $\mathbf{x}_0$!!
- Can be hard to compute explicitly
- Can be done implicitly in the NLP

Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0, \mathbf{u}} \quad & \phi(\mathbf{x}, \mathbf{u}) \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

Build basis of the null space of $\nabla \mathcal{I}(\mathbf{x}_0)$:

$$Z^\top \nabla \mathcal{I}(\mathbf{x}_0) = 0$$

Note: Z is $n \times n - m$ with $\mathcal{I} \in \mathbb{R}^m$

Rewrite OCP as (NOCP):

$$\begin{aligned} \min_{\mathbf{x}_0, \mathbf{u}, Z} \quad & \Phi(\mathbf{x}_0, \mathbf{u}) \\ \text{s.t.} \quad & Z^\top (\mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{x}_0) = 0 \\ & \frac{\partial \mathcal{I}}{\partial \mathbf{x}_0} Z = 0 \end{aligned}$$

- Null-space Z is now function of $\mathbf{x}_0$!!
- Can be hard to compute explicitly
- Can be done implicitly in the NLP
- Note: Z is not unique !!

Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u}} \quad & \phi(\mathbf{x},\mathbf{u}) \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}_0,\mathbf{u}) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

Build basis of the null space of $\nabla\mathcal{I}(\mathbf{x}_0)$:

$$Z^\top \nabla\mathcal{I}(\mathbf{x}_0) = 0$$

Note: Z is $n \times n - m$ with $\mathcal{I} \in \mathbb{R}^m$

Rewrite OCP as (NOCP):

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u},Z} \quad & \Phi(\mathbf{x}_0,\mathbf{u}) \\ \text{s.t.} \quad & Z^\top(\mathbf{f}(\mathbf{x}_0,\mathbf{u}) - \mathbf{x}_0) = 0 \\ & \frac{\partial\mathcal{I}}{\partial\mathbf{x}_0} Z = 0, \quad Z^\top Z = I \end{aligned}$$

- Null-space Z is now function of $\mathbf{x}_0$!!
- Can be hard to compute explicitly
- Can be done implicitly in the NLP
- Note: Z is not unique !!

Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u}} \quad & \phi(\mathbf{x},\mathbf{u}) \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}_0,\mathbf{u}) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

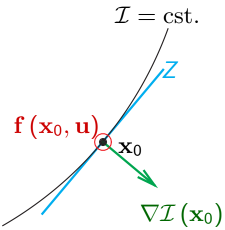
Build basis of the null space of $\nabla\mathcal{I}(\mathbf{x}_0)$:

$$Z^\top \nabla\mathcal{I}(\mathbf{x}_0) = 0$$

Note: Z is $n \times n - m$ with $\mathcal{I} \in \mathbb{R}^m$

Rewrite OCP as (NOCP):

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u},Z} \quad & \Phi(\mathbf{x}_0,\mathbf{u}) \\ \text{s.t.} \quad & Z^\top(\mathbf{f}(\mathbf{x}_0,\mathbf{u}) - \mathbf{x}_0) = 0 \\ & \frac{\partial\mathcal{I}}{\partial\mathbf{x}_0}Z = 0, \quad Z^\top Z = I \end{aligned}$$



Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u}} \quad & \phi(\mathbf{x},\mathbf{u}) \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}_0,\mathbf{u}) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

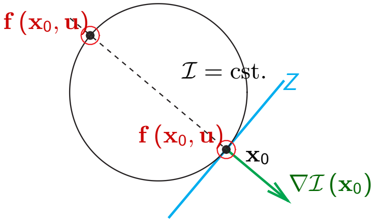
Build basis of the null space of $\nabla\mathcal{I}(\mathbf{x}_0)$:

$$Z^\top \nabla\mathcal{I}(\mathbf{x}_0) = 0$$

Note: Z is $n \times n - m$ with $\mathcal{I} \in \mathbb{R}^m$

Rewrite OCP as (NOCP):

$$\begin{aligned} \min_{\mathbf{x}_0,\mathbf{u},Z} \quad & \Phi(\mathbf{x}_0,\mathbf{u}) \\ \text{s.t.} \quad & Z^\top(\mathbf{f}(\mathbf{x}_0,\mathbf{u}) - \mathbf{x}_0) = 0 \\ & \frac{\partial \mathcal{I}}{\partial \mathbf{x}_0} Z = 0, \quad Z^\top Z = I \end{aligned}$$



Projection Method for Invariants

Periodic OCP with $\mathbf{x}_0 \in \mathbb{R}^n$

$$\begin{aligned} \min_{\mathbf{x}_0, \mathbf{u}} \quad & \phi(\mathbf{x}, \mathbf{u}) \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{x}_0 = 0 \end{aligned}$$

where integrator $\mathbf{f}$ over $[0,\ t_\mathrm{f}]$ preserves $\mathcal{I}$.

Build basis of the null space of $\nabla\mathcal{I}(\mathbf{x}_0)$:

$$Z^\top \nabla\mathcal{I}(\mathbf{x}_0) = 0$$

Note: Z is $n \times n - m$ with $\mathcal{I} \in \mathbb{R}^m$

NOCP has LICQ (under some assumptions). Projection creates "artificial" feasible solutions !

Rewrite OCP as (NOCP):

$$\begin{aligned} \min_{\mathbf{x}_0, \mathbf{u}, Z} \quad & \Phi(\mathbf{x}_0, \mathbf{u}) \\ \text{s.t.} \quad & Z^\top (\mathbf{f}(\mathbf{x}_0, \mathbf{u}) - \mathbf{x}_0) = 0 \\ & \frac{\partial \mathcal{I}}{\partial \mathbf{x}_0} Z = 0, \quad Z^\top Z = I \end{aligned}$$

Consistency in Periodic Optimal Control

Periodic OCP with consistency

$$\begin{aligned}\min_{\mathbf{u}(.),\mathbf{x}(.),\mathbf{z}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right)\\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{z}\left(t\right),\mathbf{u}\left(t\right)\right)=0\\ & \mathbf{x}\left(t_{\mathrm{f}}\right)-\mathbf{x}\left(0\right)=0\\ & \mathbf{C}\left(\mathbf{x}\left(0\right)\right)=0\end{aligned}$$

where $\mathbf{c}: \ \mathbb{R}^n \mapsto \mathbb{R}^m$.

Consistency in Periodic Optimal Control

Periodic OCP with consistency

$$\begin{aligned}\min_{\mathbf{u}(.),\mathbf{x}(.),\mathbf{z}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{z}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{x}\left(t_{\mathrm{f}}\right) - \mathbf{x}\left(0\right) = 0 \\ & \mathbf{C}\left(\mathbf{x}\left(0\right)\right) = 0\end{aligned}$$

where $\mathbf{c} : \mathbb{R}^n \mapsto \mathbb{R}^m$.

Consistency: for any $\mathbf{u}$,

$$\textbf{if} \quad \mathbf{C}\left(\mathbf{x}\left(0\right)\right) = 0 \quad \text{then} \quad \mathbf{C}\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right) = 0$$

Consistency in Periodic Optimal Control

Periodic OCP with consistency

$$\begin{aligned}\min_{\mathbf{u}(.),\mathbf{x}(.),\mathbf{z}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{z}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}\left(0\right) = 0 \\ & \mathbf{C}\left(\mathbf{x}\left(0\right)\right) = 0\end{aligned}$$

where $\mathbf{c} : \mathbb{R}^n \mapsto \mathbb{R}^m$. With a finite input parametrization $\mathbf{u}$, can be writen as (single-shooting):

$$\begin{aligned}\min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0,\mathbf{u}\right) \\ \text{s.t.} \quad & \mathbf{g} = \left[\begin{array}{c} \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right) - \mathbf{x}_0 \\ \mathbf{C}\left(\mathbf{x}_0\right)\end{array}\right] = 0\end{aligned}$$

using the integration function on $[0,\ t_\mathrm{f}]$:

$$\mathbf{x}\left(t_\mathrm{f}\right) = \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)$$

Consistency: for any $\mathbf{u}$,

if $\quad \mathbf{C}\left(\mathbf{x}\left(0\right)\right) = 0 \quad$ then $\quad \mathbf{C}\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) = 0$

Consistency in Periodic Optimal Control

Periodic OCP with consistency

$$\begin{aligned}
\min_{\mathbf{u}(.),\mathbf{x}(.),\mathbf{z}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{z}\left(t\right),\mathbf{u}\left(t\right)\right)=0 \\
& \mathbf{x}\left(t_{\mathrm{f}}\right)-\mathbf{x}\left(0\right)=0 \\
& \mathbf{C}\left(\mathbf{x}\left(0\right)\right)=0
\end{aligned}$$

where $\mathbf{c}:\ \mathbb{R}^n \mapsto \mathbb{R}^m$. With a finite input parametrization $\mathbf{u}$, can be writen as (single-shooting):

$$\begin{aligned}
\min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0,\mathbf{u}\right) \\
\text{s.t.} \quad & \mathbf{g}=\left[\begin{array}{c}\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)-\mathbf{x}_0 \\ \mathbf{C}\left(\mathbf{x}_0\right)\end{array}\right]=0
\end{aligned}$$

using the integration function on $[0,\ t_{\mathrm{f}}]$:

$$\mathbf{x}\left(t_{\mathrm{f}}\right)=\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)$$

Consistency: for any $\mathbf{u}$,

$$\textbf{if} \quad \mathbf{C}\left(\mathbf{x}\left(0\right)\right)=0 \quad \text{then} \quad \mathbf{C}\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right)=0$$

Observe that $\forall\,\mathbf{u}$

$$\mathbf{C}\left(\mathbf{x}_0\right)=0 \;\Rightarrow\; \mathbf{C}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right)=0$$

Consistency in Periodic Optimal Control

Periodic OCP with consistency

$$\begin{aligned} \min_{\mathbf{u}(.),\mathbf{x}(.),\mathbf{z}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{z}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}\left(0\right) = 0 \\ & \mathbf{C}\left(\mathbf{x}\left(0\right)\right) = 0 \end{aligned}$$

where $\mathbf{c} : \mathbb{R}^n \mapsto \mathbb{R}^m$. With a finite input parametrization $\mathbf{u}$, can be writen as (single-shooting):

$$\begin{aligned} \min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0,\mathbf{u}\right) \\ \text{s.t.} \quad & \mathbf{g} = \begin{bmatrix} \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right) - \mathbf{x}_0 \\ \mathbf{C}\left(\mathbf{x}_0\right) \end{bmatrix} = 0 \end{aligned}$$

using the integration function on $[0,\ t_\mathrm{f}]$:

$$\mathbf{x}\left(t_\mathrm{f}\right) = \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)$$

Consistency: for any $\mathbf{u}$,

$$\textbf{if} \quad \mathbf{C}\left(\mathbf{x}\left(0\right)\right) = 0 \quad \text{then} \quad \mathbf{C}\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) = 0$$

Observe that $\forall\, \mathbf{u}$

$$\mathbf{C}\left(\mathbf{x}_0\right) = 0 \;\Rightarrow\; \mathbf{C}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right) = 0$$

hence $\nabla_{\mathbf{u}}\mathbf{f}\,\nabla\mathbf{C} = 0$.

Consistency in Periodic Optimal Control

Periodic OCP with consistency

$$\min_{\mathbf{u}(.),\mathbf{x}(.),\mathbf{z}(.)} \quad \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right)$$
$$\text{s.t.} \quad \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{z}\left(t\right),\mathbf{u}\left(t\right)\right)=0$$
$$\mathbf{x}\left(t_{\mathrm{f}}\right)-\mathbf{x}\left(0\right)=0$$
$$\mathbf{C}\left(\mathbf{x}\left(0\right)\right)=0$$

where $\mathbf{c}:\mathbb{R}^n\mapsto\mathbb{R}^m$. With a finite input parametrization $\mathbf{u}$, can be writen as (single-shooting):

$$\min_{\mathbf{u},\mathbf{x}_0} \quad \Phi\left(\mathbf{x}_0,\mathbf{u}\right)$$
$$\text{s.t.} \quad \mathbf{g}=\left[\begin{array}{c}\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)-\mathbf{x}_0\\ \mathbf{C}\left(\mathbf{x}_0\right)\end{array}\right]=0$$

using the integration function on $[0,\ t_{\mathrm{f}}]$:

$$\mathbf{x}\left(t_{\mathrm{f}}\right)=\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)$$

Consistency: for any $\mathbf{u}$,

if $\quad \mathbf{C}\left(\mathbf{x}\left(0\right)\right)=0 \quad$ then $\quad \mathbf{C}\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right)=0$

Observe that $\forall\,\mathbf{u}$

$$\mathbf{C}\left(\mathbf{x}_0\right)=0 \;\Rightarrow\; \mathbf{C}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right)=0$$

hence $\nabla_{\mathbf{u}}\mathbf{f}\,\nabla\mathbf{C}=0$.

Moreover

$$\nabla_{\mathbf{x}_0}\mathbf{C}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right)=\nabla_{\mathbf{x}_0}\mathbf{f}\,\nabla\mathbf{C}\in\mathrm{span}\left\{\nabla\mathbf{C}\right\}$$

(see Proposition this morning)

Consistency in Periodic Optimal Control

Periodic OCP with consistency

$$\begin{aligned} \min_{\mathbf{u}(.),\mathbf{x}(.),\mathbf{z}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{z}\left(t\right),\mathbf{u}\left(t\right)\right) = 0 \\ & \mathbf{x}\left(t_{\mathrm{f}}\right) - \mathbf{x}\left(0\right) = 0 \\ & \mathbf{C}\left(\mathbf{x}\left(0\right)\right) = 0 \end{aligned}$$

where $\mathbf{c} : \mathbb{R}^n \mapsto \mathbb{R}^m$. With a finite input parametrization $\mathbf{u}$, can be writen as (single-shooting):

$$\begin{aligned} \min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0,\mathbf{u}\right) \\ \text{s.t.} \quad & \mathbf{g} = \left[\begin{array}{c} \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right) - \mathbf{x}_0 \\ \mathbf{C}\left(\mathbf{x}_0\right) \end{array}\right] = 0 \end{aligned}$$

using the integration function on $[0,\ t_{\mathrm{f}}]$:

$$\mathbf{x}\left(t_{\mathrm{f}}\right) = \mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)$$

Consistency: for any $\mathbf{u}$,

$$\textbf{if} \quad \mathbf{C}\left(\mathbf{x}\left(0\right)\right) = 0 \quad \text{then} \quad \mathbf{C}\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right) = 0$$

Observe that $\forall\, \mathbf{u}$

$$\mathbf{C}\left(\mathbf{x}_0\right) = 0 \;\Rightarrow\; \mathbf{C}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right) = 0$$

hence $\nabla_{\mathbf{u}}\mathbf{f}\,\nabla\mathbf{C} = 0$.

Moreover

$$\nabla_{\mathbf{x}_0}\mathbf{C}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right) = \nabla_{\mathbf{x}_0}\mathbf{f}\,\nabla\mathbf{C} \in \operatorname{span}\left\{\nabla\mathbf{C}\right\}$$

(see Proposition this morning) i.e.

$$\nabla_{\mathbf{x}_0}\mathbf{f}\,\nabla\mathbf{C} = \nabla\mathbf{C}M$$

for some $M \in \mathbb{R}^{m\times m}$.

Consistency in Periodic Optimal Control

Periodic OCP with consistency

$$\begin{aligned}\min_{\mathbf{u}(.),\mathbf{x}(.),\mathbf{z}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right)\\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{x}\left(t\right),\mathbf{z}\left(t\right),\mathbf{u}\left(t\right)\right)=0\\ & \mathbf{x}\left(t_\mathrm{f}\right)-\mathbf{x}\left(0\right)=0\\ & \mathbf{C}\left(\mathbf{x}\left(0\right)\right)=0\end{aligned}$$

where $\mathbf{c} : \mathbb{R}^n \mapsto \mathbb{R}^m$. With a finite input parametrization $\mathbf{u}$, can be writen as (single-shooting):

$$\begin{aligned}\min_{\mathbf{u},\mathbf{x}_0} \quad & \Phi\left(\mathbf{x}_0,\mathbf{u}\right)\\ \text{s.t.} \quad & \mathbf{g}=\left[\begin{array}{c}\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)-\mathbf{x}_0\\ \mathbf{C}\left(\mathbf{x}_0\right)\end{array}\right]=0\end{aligned}$$

using the integration function on $[0,\ t_\mathrm{f}]$:

$$\mathbf{x}\left(t_\mathrm{f}\right)=\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)$$

Consistency: for any $\mathbf{u}$,

$$\text{if} \quad \mathbf{C}\left(\mathbf{x}\left(0\right)\right)=0 \quad \text{then} \quad \mathbf{C}\left(\mathbf{x}\left(t_\mathrm{f}\right)\right)=0$$

Observe that $\forall\, \mathbf{u}$

$$\mathbf{C}\left(\mathbf{x}_0\right)=0 \;\Rightarrow\; \mathbf{C}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right)=0$$

hence $\nabla_\mathbf{u}\mathbf{f}\,\nabla\mathbf{C}=0$.

Moreover

$$\nabla_{\mathbf{x}_0}\mathbf{C}\left(\mathbf{f}\left(\mathbf{x}_0,\mathbf{u}\right)\right)=\nabla_{\mathbf{x}_0}\mathbf{f}\,\nabla\mathbf{C}\in\mathrm{span}\left\{\nabla\mathbf{C}\right\}$$

(see Proposition this morning) i.e.

$$\nabla_{\mathbf{x}_0}\mathbf{f}\,\nabla\mathbf{C}=\nabla\mathbf{C}M$$

for some $M\in\mathbb{R}^{m\times m}$. Then

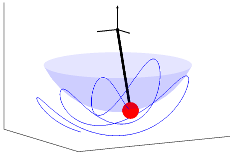
$$\underbrace{\left[\begin{array}{cc}\nabla_{\mathbf{x}_0}\mathbf{f}-I & \nabla\mathbf{C}\\ \nabla_\mathbf{u}\mathbf{f} & 0\end{array}\right]}_{=\nabla\mathbf{g}}\left[\begin{array}{c}\nabla\mathbf{C}\\ I-M\end{array}\right]=0$$

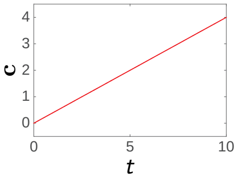
Consequence for Periodic OCPs with index-reduced DAEs

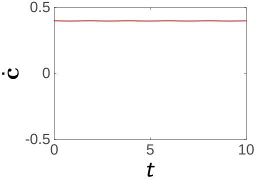
Index-1 DAE

$$\begin{bmatrix} mI & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix}$$

... is built to impose $\ddot{\mathbf{c}} = 0$ at all time.







Consequence for Periodic OCPs with index-reduced DAEs

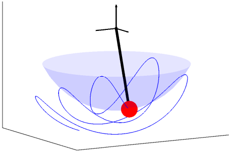
Index-1 DAE

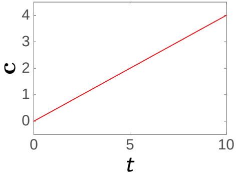
$$\begin{bmatrix} mI & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix}$$

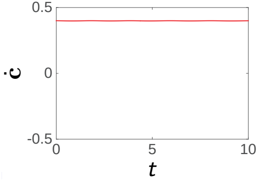
... is built to impose $\ddot{\mathbf{c}} = 0$ at all time. The state is:

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}$$

with invariant $\mathcal{I}(\mathbf{x}) = \mathbf{p}^\top \dot{\mathbf{p}} \quad (\equiv \dot{\mathbf{c}})$



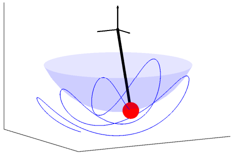




Consequence for Periodic OCPs with index-reduced DAEs

Index-1 DAE

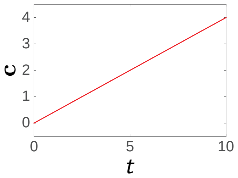
$$\begin{bmatrix} ml & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix}$$



... is built to impose $\ddot{\mathbf{c}} = 0$ at all time. The state is:

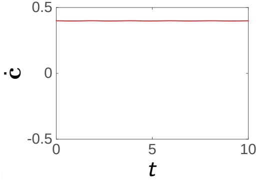
$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}$$

with invariant $\mathcal{I}(\mathbf{x}) = \mathbf{p}^\top \dot{\mathbf{p}} \quad (\equiv \dot{\mathbf{c}})$



Periodic OCP:

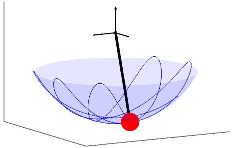
$$\begin{aligned} \min \quad & \phi(\mathbf{x}(.), \mathbf{u}(.)) \\ \text{s.t.} \quad & \begin{bmatrix} ml & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix} \\ & \mathbf{x}(t_\mathrm{f}) - \mathbf{x}(0) = 0 \end{aligned}$$



Consequence for Periodic OCPs with index-reduced DAEs

Index-1 DAE

$$\begin{bmatrix} ml & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix}$$



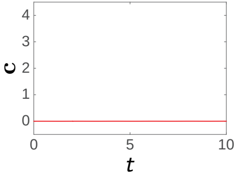
... is built to impose $\ddot{\mathbf{c}} = 0$ at all time. The state is:

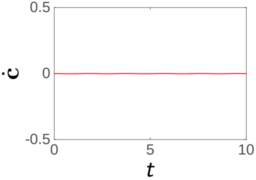
$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}$$

with invariant $\mathcal{I}(\mathbf{x}) = \mathbf{p}^\top \dot{\mathbf{p}} \quad (\equiv \dot{\mathbf{c}})$

Periodic OCP:

$$\begin{aligned} \min \quad & \phi(\mathbf{x}(.), \mathbf{u}(.)) \\ \text{s.t.} \quad & \begin{bmatrix} ml & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix} \\ & \mathbf{x}(t_\mathrm{f}) - \mathbf{x}(0) = 0 \\ & \mathbf{C} = \begin{bmatrix} \mathbf{c}(\mathbf{x}_0) \\ \dot{\mathbf{c}}(\mathbf{x}_0) \end{bmatrix} = 0 \end{aligned}$$

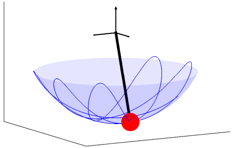




Consequence for Periodic OCPs with index-reduced DAEs

Index-1 DAE

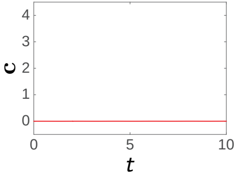
$$\begin{bmatrix} ml & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix}$$



... is built to impose $\ddot{\mathbf{c}} = 0$ at all time. The state is:

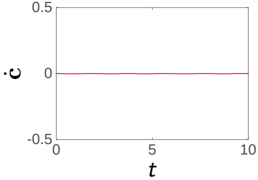
$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}$$

with invariant $\quad \mathcal{I}(\mathbf{x}) = \mathbf{p}^\top \dot{\mathbf{p}} \quad (\equiv \dot{\mathbf{c}})$



Periodic OCP:

$$\begin{aligned} \min \quad & \phi(\mathbf{x}(.), \mathbf{u}(.)) \\ \text{s.t.} \quad & \begin{bmatrix} ml & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix} \\ & \mathbf{x}(t_\mathrm{f}) - \mathbf{x}(0) = 0 \\ & \mathbf{C} = \begin{bmatrix} \mathbf{c}(\mathbf{x}_0) \\ \dot{\mathbf{c}}(\mathbf{x}_0) \end{bmatrix} = 0 \end{aligned}$$

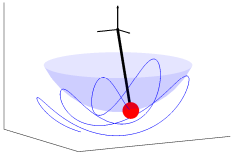


This Periodic OCP will have an LICQ deficiency !!

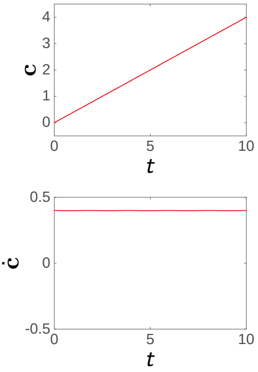
Consistency of index-reduced DAEs in Periodic Optimal Control

Periodic OCP:

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}(.),\mathbf{u}(.)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}},\mathbf{x},\mathbf{z},\mathbf{u}\right)=0 \\ & \mathbf{x}\left(t_{\mathrm{f}}\right)-\mathbf{x}\left(0\right)=0 \\ & \mathbf{c}\left(\mathbf{x}_0\right)=0, \qquad \dot{\mathbf{c}}\left(\mathbf{x}_0\right)=0 \end{aligned}$$



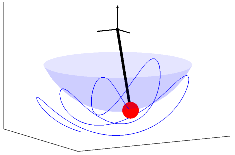
Proposition: if $\mathbf{x}\left(t_{\mathrm{f}}\right)-\mathbf{x}\left(0\right)=0$ and $\mathbf{c}\left(\mathbf{x}_0\right)=0$ hold, then $\dot{\mathbf{c}}\left(\mathbf{x}_0\right)=0$ holds.



Consistency of index-reduced DAEs in Periodic Optimal Control

Periodic OCP:

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}(.),\mathbf{u}(.)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}},\mathbf{x},\mathbf{z},\mathbf{u}\right) = 0 \\ & \mathbf{x}\left(t_{\mathrm{f}}\right) - \mathbf{x}\left(0\right) = 0 \\ & \mathbf{c}\left(\mathbf{x}_0\right) = 0, \qquad \dot{\mathbf{c}}\left(\mathbf{x}_0\right) = 0 \end{aligned}$$

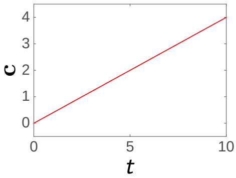


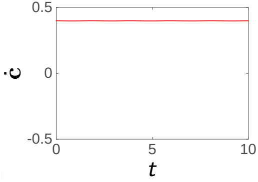
Proposition: if $\mathbf{x}\left(t_{\mathrm{f}}\right) - \mathbf{x}\left(0\right) = 0$ and $\mathbf{c}\left(\mathbf{x}_0\right) = 0$ hold, then $\dot{\mathbf{c}}\left(\mathbf{x}_0\right) = 0$ holds.

Proof: since $\ddot{\mathbf{c}} = 0$ is imposed by the dynamics, then

$$\begin{aligned} \dot{\mathbf{c}}\left(\mathbf{x}\left(t\right)\right) &= \dot{\mathbf{c}}\left(\mathbf{x}\left(0\right)\right) \quad \text{and} \\ \mathbf{c}\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right) &= \mathbf{c}\left(\mathbf{x}\left(0\right)\right) + \dot{\mathbf{c}}\left(\mathbf{x}\left(0\right)\right) t_{\mathrm{f}} \end{aligned}$$

Periodicity imposes that $\mathbf{c}\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right) = \mathbf{c}\left(\mathbf{x}\left(0\right)\right)$, hence $\dot{\mathbf{c}}\left(\mathbf{x}\left(0\right)\right) = 0$ must hold.

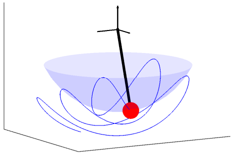




Consistency of index-reduced DAEs in Periodic Optimal Control

Periodic OCP:

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0 \\ & \mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}(0) = 0 \\ & \mathbf{c}\left(\mathbf{x}_0\right) = 0, \qquad \dot{\mathbf{c}}\left(\mathbf{x}_0\right) = 0 \end{aligned}$$



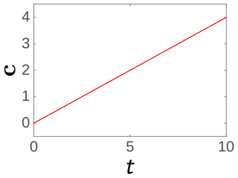
Proposition: if $\mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}(0) = 0$ and $\mathbf{c}\left(\mathbf{x}_0\right) = 0$ hold, then $\dot{\mathbf{c}}\left(\mathbf{x}_0\right) = 0$ holds.

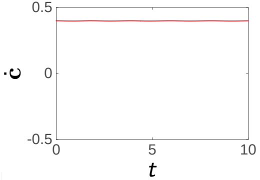
Proof: since $\ddot{\mathbf{c}} = 0$ is imposed by the dynamics, then

$$\begin{aligned} \dot{\mathbf{c}}\left(\mathbf{x}(t)\right) &= \dot{\mathbf{c}}\left(\mathbf{x}(0)\right) \quad \text{and} \\ \mathbf{c}\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) &= \mathbf{c}\left(\mathbf{x}(0)\right) + \dot{\mathbf{c}}\left(\mathbf{x}(0)\right) t_\mathrm{f} \end{aligned}$$

Periodicity imposes that $\mathbf{c}\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) = \mathbf{c}\left(\mathbf{x}(0)\right)$, hence $\dot{\mathbf{c}}\left(\mathbf{x}(0)\right) = 0$ must hold.

It is not necessary to impose $\dot{\mathbf{c}}\left(\mathbf{x}_0\right) = 0$ in a periodic OCP based on index-reduced, index-3 DAEs !

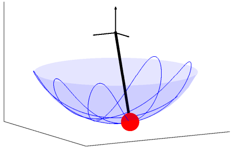




Consistency of index-reduced DAEs in Periodic Optimal Control

Periodic OCP:

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0 \\ & \mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}(0) = 0 \\ & \mathbf{c}\left(\mathbf{x}_0\right) = 0, \qquad \dot{\mathbf{c}}\left(\mathbf{x}_0\right) = 0 \end{aligned}$$

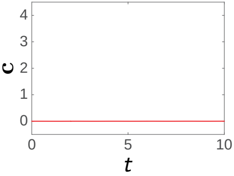


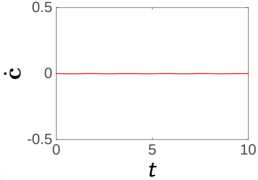
Proposition: if $\mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}(0) = 0$ and $\mathbf{c}\left(\mathbf{x}_0\right) = 0$ hold, then $\dot{\mathbf{c}}\left(\mathbf{x}_0\right) = 0$ holds.

Proof: since $\ddot{\mathbf{c}} = 0$ is imposed by the dynamics, then

$$\begin{aligned} \dot{\mathbf{c}}\left(\mathbf{x}(t)\right) &= \dot{\mathbf{c}}\left(\mathbf{x}(0)\right) \quad \text{and} \\ \mathbf{c}\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) &= \mathbf{c}\left(\mathbf{x}(0)\right) + \dot{\mathbf{c}}\left(\mathbf{x}(0)\right) t_\mathrm{f} \end{aligned}$$

Periodicity imposes that $\mathbf{c}\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) = \mathbf{c}\left(\mathbf{x}(0)\right)$, hence $\dot{\mathbf{c}}\left(\mathbf{x}(0)\right) = 0$ must hold.



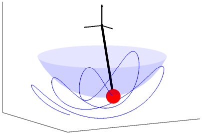


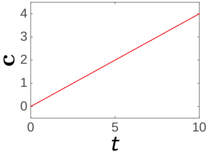
It is not necessary to impose $\dot{\mathbf{c}}\left(\mathbf{x}_0\right) = 0$ in a periodic OCP based on index-reduced, index-3 DAEs !

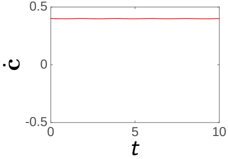
Have we solved our LICQ problem ?

Periodic OCP: with $\mathbf{c} \in \mathbb{R}^m$

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}\right) = 0 \\ & \mathbf{x}\left(t_\text{f}\right) - \mathbf{x}(0) = 0, \quad \mathbf{c}\left(\mathbf{x}_0\right) = 0 \end{aligned}$$



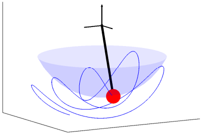


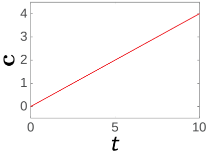


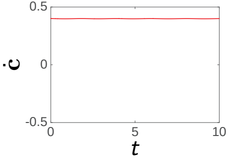
Consistency of index-reduced DAEs in Periodic Optimal Control

Periodic OCP: with $\mathbf{c} \in \mathbb{R}^m$

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}_0, \mathbf{u}\right) \\ \text{s.t.} \quad & \mathbf{g} = \left[\begin{array}{c} \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right) - \mathbf{x}_0 \\ \mathbf{c}\left(\mathbf{x}_0\right) \end{array} \right] = 0 \end{aligned}$$





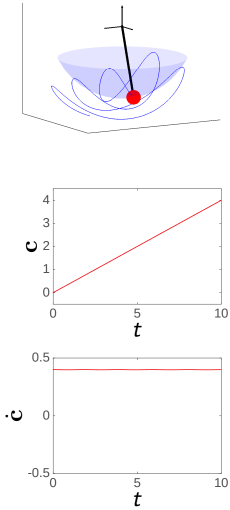


Consistency of index-reduced DAEs in Periodic Optimal Control

Periodic OCP: with $\mathbf{c} \in \mathbb{R}^m$

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}_0, \mathbf{u}\right) \\ \text{s.t.} \quad & \mathbf{g} = \left[\begin{array}{c} \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right) - \mathbf{x}_0 \\ \mathbf{c}\left(\mathbf{x}_0\right) \end{array}\right] = 0 \end{aligned}$$

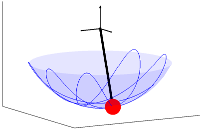
Note that: $\mathbf{c}\left(\mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right)\right) = \underbrace{\mathbf{c}\left(\mathbf{x}_0\right)}_{=0} + \dot{\mathbf{c}}\left(\mathbf{x}_0\right) \cdot t_{\mathrm{f}}$



Consistency of index-reduced DAEs in Periodic Optimal Control

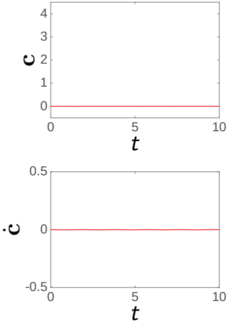
Periodic OCP: with $\mathbf{c} \in \mathbb{R}^m$

$$\begin{aligned} \text{min} \quad & \phi\left(\mathbf{x}_0, \mathbf{u}\right) \\ \text{s.t.} \quad & \mathbf{g} = \left[\begin{array}{c} \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right) - \mathbf{x}_0 \\ \mathbf{c}\left(\mathbf{x}_0\right) \end{array}\right] = 0 \end{aligned}$$



Note that: $\mathbf{c}\left(\mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right)\right) = \underbrace{\mathbf{c}\left(\mathbf{x}_0\right)}_{=0} + \dot{\mathbf{c}}\left(\mathbf{x}_0\right) \cdot t_\mathrm{f}$ hence at the solution:

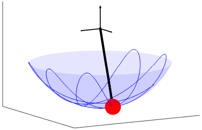
$$\nabla_{\mathbf{x}_0}\mathbf{f}\nabla\mathbf{c} = \nabla\dot{\mathbf{c}} \cdot t_\mathrm{f} \quad \text{and} \quad \nabla_{\mathbf{u}}\mathbf{f}\nabla\mathbf{c} = 0$$



Consistency of index-reduced DAEs in Periodic Optimal Control

Periodic OCP: with $\mathbf{c} \in \mathbb{R}^m$

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}_0, \mathbf{u}\right) \\ \text{s.t.} \quad & \mathbf{g} = \begin{bmatrix} \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right) - \mathbf{x}_0 \\ \mathbf{c}\left(\mathbf{x}_0\right) \end{bmatrix} = 0 \end{aligned}$$

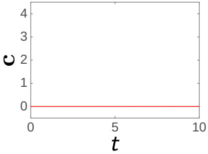


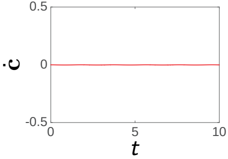
Note that: $\mathbf{c}\left(\mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right)\right) = \underbrace{\mathbf{c}\left(\mathbf{x}_0\right)}_{=0} + \dot{\mathbf{c}}\left(\mathbf{x}_0\right) \cdot t_{\mathrm{f}}$ hence at the solution:

$$\nabla_{\mathbf{x}_0}\mathbf{f}\nabla\mathbf{c} = \nabla\dot{\mathbf{c}} \cdot t_{\mathrm{f}} \quad \text{and} \quad \nabla_{\mathbf{u}}\mathbf{f}\nabla\mathbf{c} = 0$$

Then

$$\overbrace{\begin{bmatrix} \nabla_{\mathbf{x}_0}\mathbf{f} - I & \nabla\mathbf{c} \\ \nabla_{\mathbf{u}}\mathbf{f} & 0 \end{bmatrix}}^{=\nabla\mathbf{g}} \underbrace{\begin{bmatrix} \nabla\mathbf{c} \\ M \end{bmatrix}}_{\text{null-space?}} = \begin{bmatrix} t_{\mathrm{f}}\nabla\dot{\mathbf{c}} + \nabla\mathbf{c}\left(M - I\right) \\ 0 \end{bmatrix}$$

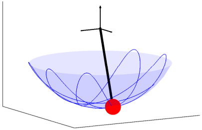




Consistency of index-reduced DAEs in Periodic Optimal Control

Periodic OCP: with $\mathbf{c} \in \mathbb{R}^m$

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}_0, \mathbf{u}\right) \\ \text{s.t.} \quad & \mathbf{g} = \left[\begin{array}{c} \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right) - \mathbf{x}_0 \\ \mathbf{c}\left(\mathbf{x}_0\right) \end{array}\right] = 0 \end{aligned}$$

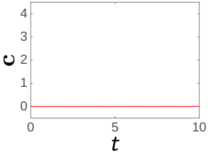


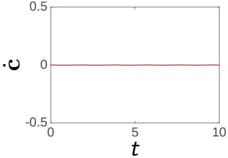
Note that: $\mathbf{c}\left(\mathbf{f}\left(\mathbf{x}_0, \mathbf{u}\right)\right) = \underbrace{\mathbf{c}\left(\mathbf{x}_0\right)}_{=0} + \dot{\mathbf{c}}\left(\mathbf{x}_0\right) \cdot t_\mathrm{f}$ hence at the solution:

$$\nabla_{\mathbf{x}_0}\mathbf{f}\nabla\mathbf{c} = \nabla\dot{\mathbf{c}} \cdot t_\mathrm{f} \quad \text{and} \quad \nabla_{\mathbf{u}}\mathbf{f}\nabla\mathbf{c} = 0$$

Then

$$\overbrace{\left[\begin{array}{cc} \nabla_{\mathbf{x}_0}\mathbf{f} - I & \nabla\mathbf{c} \\ \nabla_{\mathbf{u}}\mathbf{f} & 0 \end{array}\right]}^{=\nabla\mathbf{g}} \underbrace{\left[\begin{array}{c} \nabla\mathbf{c} \\ M \end{array}\right]}_{\text{null-space?}} = \left[\begin{array}{c} t_\mathrm{f}\nabla\dot{\mathbf{c}} + \nabla\mathbf{c}\left(M - I\right) \\ 0 \end{array}\right]$$





LICQ problem if $\nabla\dot{\mathbf{c}} \in \operatorname{span}\left\{\nabla\mathbf{c}\right\}$. This does not happen in index-reduced DAEs from Lagrange mechanics !!
(cannot say much in general though)

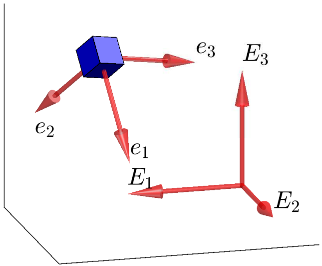
Outline

Representing Orientations (more on that tomorrow !!)

Orientations are represented via rotations...

- Let E be a fixed (e.g. inertial) reference frame
- Let e be a reference frame attached to the object

What's a rotation ?

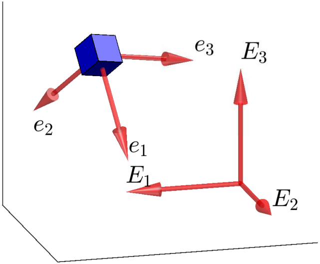


Representing Orientations (more on that tomorrow !!)

Orientations are represented via rotations...

- Let E be a fixed (e.g. inertial) reference frame
- Let e be a reference frame attached to the object
- Let $T = (\tau, \rho)$ be the transformation (translation + rotation) that brings E to e

What's a rotation ?

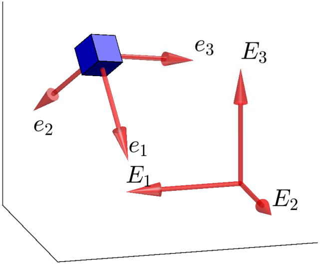


Representing Orientations (more on that tomorrow !!)

Orientations are represented via rotations...

- Let E be a fixed (e.g. inertial) reference frame
- Let e be a reference frame attached to the object
- Let $T = (\tau, \rho)$ be the transformation (translation + rotation) that brings E to e
- The orientation of the object is represented by ρ

What's a rotation ?

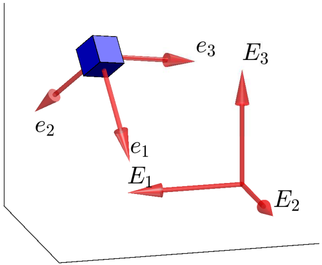


Direct Cosine Matrix (DCM)

Representation of the orientation as:

$$R = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the vectors of frame e given frame E. The orientation is then represented by the **9 numbers** making R.



Direct Cosine Matrix (DCM)

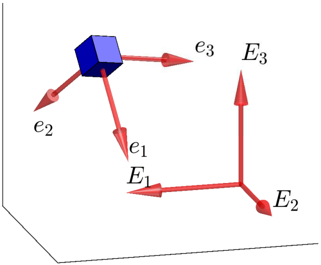
Representation of the orientation as:

$$R = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the vectors of frame e given frame E. The orientation is then represented by the **9 numbers** making R.

Orthonormality must hold, i.e.:

$$R^\top R - I = 0$$

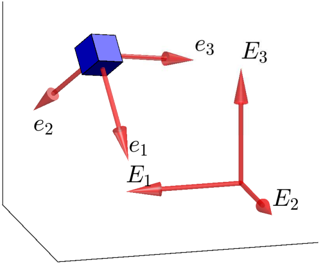


Direct Cosine Matrix (DCM)

Representation of the orientation as:

$$R = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the vectors of frame e given frame E. The orientation is then represented by the **9 numbers** making R.



Orthonormality must hold, i.e.:

$$R^\top R - I = 0$$

The time evolution of R is given by:

$$\dot{R} = R\Omega$$

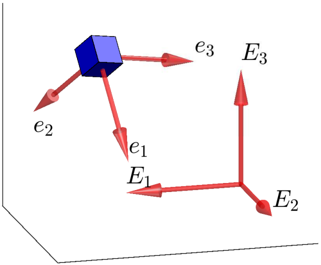
where Ω is a skew symmetric ($\Omega^\top = -\Omega$) matrix.

Direct Cosine Matrix (DCM)

Representation of the orientation as:

$$R = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the vectors of frame e given frame E. The orientation is then represented by the **9 numbers** making R.



Orthonormality must hold, i.e.:

$$R^\top R - I = 0$$

The time evolution of R is given by:

$$\dot{R} = R\Omega$$

where Ω is a skew symmetric ($\Omega^\top = -\Omega$) matrix.

The dynamics of R **preserve orthonormality**, i.e.:

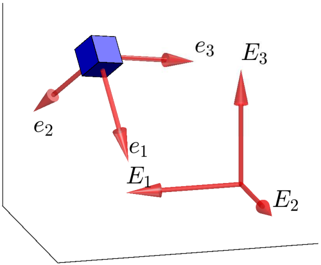
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R^\top R - I\right) = \dot{R}^\top R + R^\top \dot{R} = -\Omega R^\top R + R^\top R\Omega = 0$$

if $R^\top R - I = 0$.

Direct Cosine Matrix (DCM)

Representation of the orientation as:

$$R = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$



where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the vectors of frame e given frame E. The orientation is then represented by the **9 numbers** making R.

Orthonormality must hold, i.e.:

$$R^\top R - I = 0$$

The time evolution of R is given by:

$$\dot{R} = R\Omega$$

where Ω is a skew symmetric ($\Omega^\top = -\Omega$) matrix.

The dynamics of R **preserve orthonormality**, i.e.:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R^\top R - I\right) = \dot{R}^\top R + R^\top \dot{R} = -\Omega R^\top R + R^\top R\Omega = 0$$

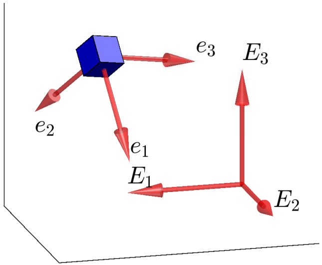
if $R^\top R - I = 0$. Hence, **imposing** $R(0)^\top R(0) - I = 0$ **yields** $R(t)^\top R(t) - I = 0$ **for all** t

Direct Cosine Matrix (DCM)

Representation of the orientation as:

$$R = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the vectors of frame e given frame E. The orientation is then represented by the **9 numbers** making R.



Orthonormality must hold, i.e.:

$$R^\top R - I = 0$$

The time evolution of R is given by:

$$\dot{R} = R\Omega$$

$\dot{R} = R\Omega$ is an example of ODE with consistency conditions !!

where Ω is a skew symmetric ($\Omega^\top = -\Omega$) matrix.

The dynamics of R **preserve orthonormality**, i.e.:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R^\top R - I\right) = \dot{R}^\top R + R^\top \dot{R} = -\Omega R^\top R + R^\top R\Omega = 0$$

if $R^\top R - I = 0$. Hence, **imposing** $R(0)^\top R(0) - I = 0$ **yields** $R(t)^\top R(t) - I = 0$ **for all** t

DCM in periodic optimal control

Periodic OCP with rotations

$$
\begin{aligned}
\min \quad & \phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}, R\right) = 0 \\
& \dot{R} = R\Omega\left(\mathbf{x}, \mathbf{z}, \mathbf{u}, R\right) \\
& \underbrace{\mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}\left(0\right) = 0}_{\textbf{Periodicity of } \mathbf{x}} \qquad \underbrace{\color{red}{R\left(t_\mathrm{f}\right) - R\left(0\right)} = 0}_{\textbf{Periodicity of } R} \\
& \underbrace{\mathbf{c}\left(\mathbf{x}_0\right) = 0}_{\textbf{Consistency of DAE}} \qquad \underbrace{R\left(0\right)^\top R(0) - I = 0}_{\textbf{Orthonormality of } R}
\end{aligned}
$$

DCM in periodic optimal control

Periodic OCP with rotations

$$\begin{aligned}
\min \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}},\mathbf{x},\mathbf{z},\mathbf{u},R\right)=0 \\
& \dot{R} = R\Omega\left(\mathbf{x},\mathbf{z},\mathbf{u},R\right) \\
& \underbrace{\mathbf{x}\left(t_{\mathrm{f}}\right)-\mathbf{x}\left(0\right)=0}_{\textbf{Periodicity of } \mathbf{x}} \qquad \underbrace{R\left(t_{\mathrm{f}}\right)-R\left(0\right)=0}_{\textbf{Periodicity of } R} \\
& \underbrace{\mathbf{c}\left(\mathbf{x}_0\right)=0}_{\textbf{Consistency of DAE}} \qquad \underbrace{R\left(0\right)^\top R(0)-I=0}_{\textbf{Orthonormality of } R}
\end{aligned}$$

Problem: periodicity & orthonormality constraints

$$R(0)^\top R(0) - I = 0, \qquad R(t_{\mathrm{f}}) - R(0) = 0$$

$+$ dynamics preserving orthonormality yield **LICQ deficiency**

DCM in periodic optimal control

Periodic OCP with rotations

$$
\begin{aligned}
\min \quad & \phi\left(\mathbf{x}\left(.\right), \mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}, R\right) = 0 \\
& \dot{R} = R\Omega\left(\mathbf{x}, \mathbf{z}, \mathbf{u}, R\right) \\
& \underbrace{\mathbf{x}\left(t_{\mathrm{f}}\right) - \mathbf{x}\left(0\right) = 0}_{\text{Periodicity of } \mathbf{x}} \qquad \underbrace{R\left(t_{\mathrm{f}}\right) - R\left(0\right) = 0}_{\text{Periodicity of } R} \\
& \underbrace{\mathbf{c}\left(\mathbf{x}_0\right) = 0}_{\text{Consistency of DAE}} \qquad \underbrace{R\left(0\right)^\top R(0) - I = 0}_{\text{Orthonormality of } R}
\end{aligned}
$$

Problem: periodicity & orthonormality constraints

$$
R(0)^\top R(0) - I = 0, \qquad R(t_{\mathrm{f}}) - R(0) = 0
$$

$+$ dynamics preserving orthonormality yield **LICQ deficiency**

- $\mathrm{SO}(3)$ has **3 dimensions**, $R(0) \in \mathbb{R}^{3\times 3}$ has **9 dimensions**

DCM in periodic optimal control

Periodic OCP with rotations

$$\begin{aligned}
\min \quad & \phi\left(\mathbf{x}\left(.\right), \mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}, R\right) = 0 \\
& \dot{R} = R\Omega\left(\mathbf{x}, \mathbf{z}, \mathbf{u}, R\right) \\
& \underbrace{\mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}\left(0\right) = 0}_{\text{Periodicity of } \mathbf{x}} \qquad \underbrace{R\left(t_\mathrm{f}\right) - R\left(0\right) = 0}_{\text{Periodicity of } R} \\
& \underbrace{\mathbf{c}\left(\mathbf{x}_0\right) = 0}_{\text{Consistency of DAE}} \qquad \underbrace{R\left(0\right)^\top R(0) - I = 0}_{\text{Orthonormality of } R}
\end{aligned}$$

Problem: periodicity & orthonormality constraints

$$R(0)^\top R(0) - I = 0, \qquad R(t_\mathrm{f}) - R(0) = 0$$

$+$ dynamics preserving orthonormality yield **LICQ deficiency**

- $\mathrm{SO}(3)$ has **3 dimensions**, $R(0) \in \mathbb{R}^{3\times 3}$ has **9 dimensions**
- **Orthonormality** condition $R(0)^\top R(0) - I = 0$ must **block** $9 - 3 = 6$ **dimensions**

DCM in periodic optimal control

Periodic OCP with rotations

$$\begin{aligned}
\min \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}},\mathbf{x},\mathbf{z},\mathbf{u},R\right) = 0 \\
& \dot{R} = R\Omega\left(\mathbf{x},\mathbf{z},\mathbf{u},R\right) \\
& \underbrace{\mathbf{x}\left(t_\mathrm{f}\right) - \mathbf{x}\left(0\right) = 0}_{\text{Periodicity of } \mathbf{x}} \qquad \underbrace{R\left(t_\mathrm{f}\right) - R\left(0\right) = 0}_{\text{Periodicity of } R} \\
& \underbrace{\mathbf{c}\left(\mathbf{x}_0\right) = 0}_{\text{Consistency of DAE}} \qquad \underbrace{R\left(0\right)^\top R(0) - I = 0}_{\text{Orthonormality of } R}
\end{aligned}$$

Problem: periodicity & orthonormality constraints

$$R(0)^\top R(0) - I = 0, \qquad R(t_\mathrm{f}) - R(0) = 0$$

$+$ dynamics preserving orthonormality yield **LICQ deficiency**

- $\mathrm{SO}(3)$ has **3 dimensions**, $R(0) \in \mathbb{R}^{3\times 3}$ has **9 dimensions**
- **Orthonormality** condition $R(0)^\top R(0) - I = 0$ must **block** $9 - 3 = 6$ **dimensions**
- **Periodicity** condition $R(t_\mathrm{f}) - R(0) = 0$ must **block 3 dimensions**

DCM in periodic optimal control

Periodic OCP with rotations

$$\begin{aligned}
\min \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}},\mathbf{x},\mathbf{z},\mathbf{u},R\right)=0 \\
& \dot{R} = R\Omega\left(\mathbf{x},\mathbf{z},\mathbf{u},R\right) \\
& \underbrace{\mathbf{x}\left(t_\mathrm{f}\right)-\mathbf{x}\left(0\right)=0}_{\text{Periodicity of } \mathbf{x}} \qquad \underbrace{R\left(t_\mathrm{f}\right)-R\left(0\right)=0}_{\text{Periodicity of } R} \\
& \underbrace{\mathbf{c}\left(\mathbf{x}_0\right)=0}_{\text{Consistency of DAE}} \qquad \underbrace{R\left(0\right)^\top R(0)-I=0}_{\text{Orthonormality of } R}
\end{aligned}$$

Problem: periodicity & orthonormality constraints

$$R(0)^\top R(0)-I=0, \qquad R(t_\mathrm{f})-R(0)=0$$

$+$ dynamics preserving orthonormality yield **LICQ deficiency**

- $\mathrm{SO}(3)$ has **3 dimensions**, $R(0)\in\mathbb{R}^{3\times3}$ has **9 dimensions**
- **Orthonormality** condition $R(0)^\top R(0)-I=0$ must **block** $9-3=6$ **dimensions**
- **Periodicity** condition $R(t_\mathrm{f})-R(0)=0$ must **block 3 dimensions**

We need to **pick** $3+6=9$ **constraints** among **Orthonormality** & **Periodicity**

DCM in periodic optimal control

Periodic OCP with rotations

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad & \mathbf{F}\left(\dot{\mathbf{x}},\mathbf{x},\mathbf{z},\mathbf{u},R\right)=0 \\ & \dot{R}=R\Omega\left(\mathbf{x},\mathbf{z},\mathbf{u},R\right) \\ & \underbrace{\mathbf{x}\left(t_{\mathrm{f}}\right)-\mathbf{x}\left(0\right)=0}_{\textbf{Periodicity of } \mathbf{x}} \qquad \underbrace{R\left(t_{\mathrm{f}}\right)-R\left(0\right)=0}_{\textbf{Periodicity of } R} \\ & \underbrace{\mathbf{c}\left(\mathbf{x}_0\right)=0}_{\textbf{Consistency of DAE}} \qquad \underbrace{R\left(0\right)^\top R(0)-I=0}_{\textbf{Orthonormality of } R} \end{aligned}$$

Problem: periodicity & orthonormality constraints

$$R(0)^\top R(0)-I=0, \qquad R(t_{\mathrm{f}})-R(0)=0$$

$+$ dynamics preserving orthonormality yield **LICQ deficiency**

How to pick the 9 constraints to eliminate ?

There is no good answer: regardless of your choice, there will be solutions for which LICQ fails for the rotations. This is related to the "simple elimination" slide !!

Handling rotations in periodic optimal control

Trick inspired from the projection method: $(R_0 = R(0), \quad R_N = R(t_\mathrm{f}))$

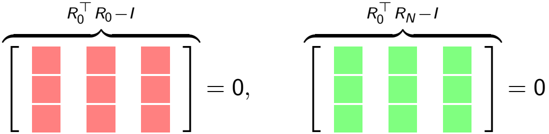
$$R_0^\top R_0 - I = 0, \qquad R_N - R_0 = 0$$

Rewrite as:

$$R_0^\top R_0 - I = 0, \qquad R_0^\top R_N - I = 0$$

Then select 9 constraints...

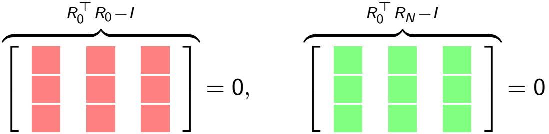
Trick inspired from the projection method: $(R_0 = R(0), \quad R_N = R(t_\mathrm{f}))$



Then select 9 constraints...

Handling rotations in periodic optimal control

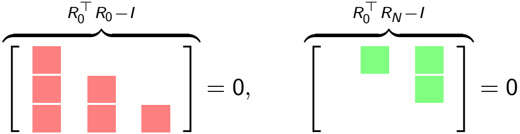
Trick inspired from the projection method: $(R_0 = R(0), \quad R_N = R(t_\mathrm{f}))$



Then select 9 constraints...

Proposition: select the *periodicity* constraints such that $-$ $^\top$ spans $\mathrm{so}(3)$ and the *orthogonality* constraints such that they do not "collide" with (total of 8 equally valid choices !!)

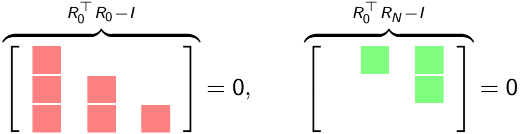
Trick inspired from the projection method: $(R_0 = R(0), \quad R_N = R(t_f))$



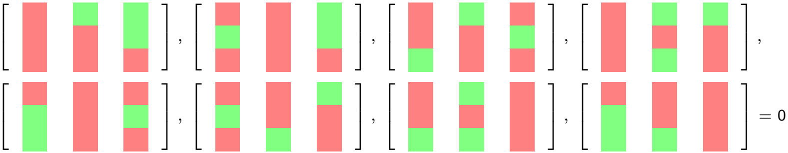
Proposition: select the *periodicity* constraints such that $\square - \square^\top$ spans $\mathrm{so}(3)$ and the *orthogonality* constraints such that they do not "collide" with $\square$ (total of 8 equally valid choices !!)

Handling rotations in periodic optimal control

Trick inspired from the projection method: $(R_0 = R(0), \quad R_N = R(t_f))$

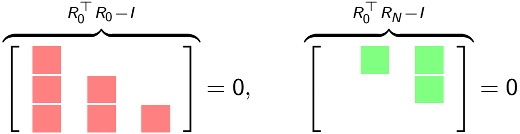


Proposition: select the *periodicity* constraints such that $-$ $^\top$ spans $\mathrm{so}(3)$ and the *orthogonality* constraints such that they do not "collide" with (total of 8 equally valid choices !!)



Handling rotations in periodic optimal control

Trick inspired from the projection method: $(R_0 = R(0), \quad R_N = R(t_\mathrm{f}))$



Proposition: select the *periodicity* constraints such that $- ^\top$ spans $\mathrm{so}(3)$ and the *orthogonality* constraints such that they do not "collide" with (total of 8 equally valid choices !!)

Intuition:

- *"blocks" the directions orthogonal to the* $\mathrm{SO}(3)$ *manifold*
- *"blocks" the directions tangent to the* $\mathrm{SO}(3)$ *manifold*

Proof: requires operating on the vector space $\mathbb{R}^{3\times 3}$, with associated scalar products and differential forms... *it is elaborate...*