# Numerical Optimal Control with DAEs Lecture 9: Indirect Optimal Control

Sébastien Gros

AWESCO PhD course

- Basics of the Pontryagin Maximum Principle (PMP)
- Numerical difficulties of PMP & how to tackle them
- Some properties of PMP
- Singular OCPs & their impact on Direct Optimal Control

# Outline

Introduction to the Pontryagin Maximum Principle (PMP)

Interpretation of  $H_{\mathbf{u}}$ 

3 Input bounds in Indirect Optimal 19

Singular Optimal Control problems

General constraints in Indirect Optimal Control

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Simple continuous problem:

$$\begin{split} \min_{\mathbf{x},\mathbf{u}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) \mathrm{d}t \\ \text{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x},\mathbf{u}\right), \\ & \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \end{split}$$

Define the Hamiltonian function

 $H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u})$ 

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Get the optimal control solution from  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ 

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Simple continuous problem:  $\min_{\mathbf{x},\mathbf{u}} \quad \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_0}^{t_{\mathrm{f}}} L(\mathbf{x}(t),\mathbf{u}(t)) \, \mathrm{d}t$ s.t.  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\mathbf{u}),$  $\mathbf{x}(t_0) = \mathbf{x}_0$ 

Define the Hamiltonian function

$$H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u})$$



Get the optimal control solution from  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$  with:

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Labelled a Two Points Boundary Value Problem (TPBVP)

	Continuous Equations	Discrete Equations
Global	Hamilton-Jacobi-Bellman (HJB)	Dynamic Programming (DP)
Local	Pontryagin (PMP)	Direct Optimal Control (DOC)

#### Continuous problem:

$$\begin{split} \min_{\mathbf{u} \in \mathcal{U}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \mathrm{d}t \\ \text{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \\ & \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \end{split}$$

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#### Continuous problem:

$$\begin{split} \min_{\mathbf{u}\in\mathcal{U}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} \mathcal{L}\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) \mathrm{d}t \\ \mathrm{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x},\mathbf{u}\right), \\ & \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \end{split}$$

**PMP**: define the Hamiltonian function

$$H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u})$$

Get the optimal input  $\mathbf{u}(\mathbf{x}, \lambda) = \arg \min_{\mathbf{u}} H(\mathbf{x}, \lambda, \mathbf{u})$ Use it in the state-costate integration:

$$\begin{array}{ll} \text{States}: & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), & \mathbf{x}(t_0) = \mathbf{x}_0 \\ \text{Costates}: & \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H\left(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}\right), & \boldsymbol{\lambda}\left(t_f\right) = \nabla_{\mathbf{x}} \phi\left(\mathbf{x}\left(t_f\right)\right) \end{array}$$

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#### Continuous problem:

- $$\begin{split} \min_{\mathbf{u} \in \mathcal{U}} \quad \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} \mathcal{L}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \mathrm{d}t \\ \text{s.t.} \quad \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \end{split}$$
  - $\mathbf{x}\left(t_{0}\right)=\mathbf{x}_{0}$

- PMP equations provide an " $\infty$ "-dimensional input profile u(.)
- State constraints hard to handle

PMP: define the Hamiltonian function

$$H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u})$$

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**D.O.C.** describes the solution as a finite set of variables  $\mathbf{w}$  transform the problem into a discrete one

Solve the resulting Nonlinear Program (NLP):

$$\begin{split} \min_{\mathbf{w}} & \Phi\left(\mathbf{w}\right) \\ \text{s.t.} & \mathbf{g}\left(\mathbf{w}\right) = \mathbf{0}, \\ & \mathbf{h}\left(\mathbf{w}\right) \leq \mathbf{0} \end{split}$$

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#### Continuous problem:

- $$\begin{split} \min_{\mathbf{u} \in \mathcal{U}} \quad \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \mathrm{d}t \\ \mathrm{s.t.} \quad \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \end{split}$$
  - $\mathbf{x}\left(t_{0}\right)=\mathbf{x}_{0}$

- Input profile restricted to a finite-dimensional space (e.g. piecewise-constant)
- Easy to treat all types of constraints

**D.O.C.** describes the solution as a finite set of variables  ${\bf w}$  transform the problem into a discrete one

Solve the resulting Nonlinear Program (NLP):

$$\begin{array}{ll} \min_{\mathbf{w}} & \Phi(\mathbf{w}) \\ \text{s.t.} & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{array}$$

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The big distinction in Optimal Control...

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The big distinction in Optimal Control...

"First optimise then discretize" (HJB & PMP)

- First write the continuous equations describing the solution to the problem
- ... then discretize the equations & solve

Note: the PMP "family" is referred to as Indirect optimal control here

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- First write the continuous equations describing the solution to the problem
- ... then discretize the equations & solve

Note: the PMP "family" is referred to as *Indirect* optimal control here **vs.** 

"First discretize then optimise " (DP & DOC)

- First discretize the continuous OCP into a discrete one...
- ... then write the **discrete equations** describing the solution & solve

Simple continuous problem:

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$\min_{\mathbf{x},\mathbf{u}}$	$\phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L(\mathbf{x}(t), \mathbf{u}(t))  \mathrm{d}t$
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#### **Two Points Boundary Value Problem**

• Integrate forward ? We have  $\mathbf{x}(t_0) = \mathbf{x}_0$ , but we don't have  $\lambda(t_0)...$ 

Simple continuous problem:

$\min_{\mathbf{x},\mathbf{u}}$	$\phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L(\mathbf{x}(t), \mathbf{u}(t))  \mathrm{d}t$
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#### **Two Points Boundary Value Problem**

- Integrate forward ? We have  $\mathbf{x}(t_0) = \mathbf{x}_0$ , but we don't have  $\lambda(t_0)$ ...
- Integrate forward-backward ? Integrate state forward from  $\mathbf{x}(t_0) = \mathbf{x}_0$  then backward from  $\lambda(t_f) = \nabla_{\mathbf{x}} \phi(\mathbf{x}(t_f))...$  but we don't know u...

Simple continuous problem:

$\min_{\mathbf{x},\mathbf{u}}$	$\phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L(\mathbf{x}(t), \mathbf{u}(t))  \mathrm{d}t$
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#### **Two Points Boundary Value Problem**

Note that the **entire** solution is "described by"  $\lambda(t_0)$ 



**Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > \operatorname{tol} \, do$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ TPBVP Compute:  $\frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_0}$  $\mathbf{r} = oldsymbol{\lambda}\left(t_{\mathrm{f}}
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abla_{\mathbf{x}}\phi\left(\mathbf{x}\left(t_{\mathrm{f}}
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ight)$  and 1.5 (<del>1)</del> × 0.5⊦  $\mathbf{x}(t_0)$  $\mathbf{x}(\mathbf{t}_{\mathbf{f}})$ 0.0 0.5 1.5 t 20  $\lambda(t_0)$ (10) (10) (10)  $\lambda(t_{\rm f})$ 0 0.5 í٥ 1.5 2 t 19<sup>th</sup> of February, 2016 S. Gros 7 / 22

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ight)$  and Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ (<del>1)</del> × 0.5⊦  $\mathbf{x}(t_0)$ 0.0 0.5 1.5 20  $\lambda(t_0)$ (10) (10) (10) 0 0.5 í٥ 1.5 19<sup>th</sup> of February, 2016 S. Gros

 $\mathbf{x}(\mathbf{t}_{\mathbf{f}})$ 

 $\lambda(t_{\rm f})$ 

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Input: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > \operatorname{tol} \operatorname{do}$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \lambda, \mathbf{u})$ :  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x}(t_0) = \mathbf{x}_0$   $\dot{\lambda} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \lambda, \mathbf{u}), \quad \lambda(t_0) = \lambda_0$ Compute:  $\mathbf{r} = \lambda(t_f) - \nabla_{\mathbf{x}} \phi(\mathbf{x}(t_f)) \text{ and } \frac{\partial \mathbf{r}}{\partial \lambda_0}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$  $\vdots$ 

Example:

$$\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2} \int_0^4 (x^2 + u^2) dt \\ \dot{x} = u - \sin(x), \quad x(0) = 1$$

TPBVP



**Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$ m Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ : x  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Compute:  $\mathbf{r} = \boldsymbol{\lambda} \left( t_{\mathrm{f}} 
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ight)$  and  $\frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ 1.5 Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ (<del>1</del>) × 0.5 ⊧ í٥  $H(x, u, \lambda) = \frac{1}{2} (x^{2} + u^{2}) + \lambda (u - \sin(x))$ 20 is minimised by  $u = -\lambda$ . Dynamics read as:

> $\dot{x} = -\lambda - \sin(x)$  $\dot{\lambda} = \lambda \cos(x) - x$

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

$$\lim_{x \to 0} \frac{1}{2} \int_0^4 (x^2 + u^2) dt \dot{x} = u - \sin(x), \quad x(0) = 1$$





 $19^{\mathrm{th}}$  of February, 2016

Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{x,y} \frac{1}{2} \int_{0}^{4} (x^{2} + u^{2}) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Guess  $\lambda_0 = 0.36$  / Current  $\lambda_0 = 0.36$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - \nabla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \text{ and } \frac{\partial \mathbf{r}}{\partial \lambda_{0}} \quad \underbrace{\underbrace{\vdots}}_{\mathbf{x}} \overset{10}{\mathbf{x}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 0 5  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$  $\widehat{t}$  $\lambda(4) = -4.3359$ -5 2 is minimised by  $u = -\lambda$ . Dynamics read as: u (t)  $\dot{x} = -\lambda - \sin(x)$ 0  $\dot{\lambda} = \lambda \cos(x) - x$ -5 2 3

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \ \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Guess  $\lambda_0 = 0.36$  / Current  $\lambda_0 = 0.45579$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - \nabla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \text{ and } \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ (t) ×0.5⊦ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 2  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$  $(\widehat{t})$  $\lambda(4) = -1.8854$ is minimised by  $u = -\lambda$ . Dynamics read as: u(t) $\dot{x} = -\lambda - \sin(x)$  $\dot{\lambda} = \lambda \cos(x) - x$ -2' 2

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

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**Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$ m Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ : x  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - 
abla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$  $^{(t)}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ -1 ÷0.5  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ 

is minimised by  $u = -\lambda$ . Dynamics read as:

$$\dot{x} = -\lambda - \sin(x)$$
  
 $\dot{\lambda} = \lambda \cos(x) - x$ 

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

$$\lim_{n \to \infty} \frac{1}{2} \int_0^4 (x^2 + u^2) dt \dot{x} = u - \sin(x), \quad x(0) = 1$$



Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$ m Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ : x  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - 
abla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ (t)Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 0 0.5 (t) $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ 0 -0.5 is minimised by  $u = -\lambda$ . Dynamics read as:

$$\lim_{\mathbf{u}} \quad \frac{1}{2} \int_{0}^{4} (x^{2} + u^{2}) dt \dot{x} = u - \sin(x), \quad x(0) = 1$$



Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

 $\dot{x} = -\lambda - \sin(x)$  $\dot{\lambda} = \lambda \cos(x) - x$ 

**Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - 
abla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 0

 $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ 

is minimised by  $u = -\lambda$ . Dynamics read as:

$$\dot{x} = -\lambda - \sin(x)$$
  
 $\dot{\lambda} = \lambda \cos(x) - x$ 

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

Example:

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$$\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2} \int_0^4 (x^2 + u^2) dt \\ \dot{x} = u - \sin(x), \quad x(0) = 1$$



Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \ \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Guess  $\lambda_0 = 0.41$  / Current  $\lambda_0 = 0.41$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - 
abla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ (t)Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 0 3 5  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$  $\widehat{f}$  $\lambda(4) = -3.1365$ -5<sup>1</sup> 0 2 3 is minimised by  $u = -\lambda$ . Dynamics read as: u(t) $\dot{x} = -\lambda - \sin(x)$  $\dot{\lambda} = \lambda \cos(x) - x$ -5<sup>L</sup> 2 3 Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$ m Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ : x  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Compute:  $(t) \times (t)$  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - 
abla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ -50<sup>l</sup> 40r  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ ÷20

is minimised by  $u = -\lambda$ . Dynamics read as:

$$\dot{x} = -\lambda - \sin(x)$$
  
 $\dot{\lambda} = \lambda \cos(x) - x$ 

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

$$\lim_{\mathbf{u}} \quad \frac{1}{2} \int_{0}^{4} (x^{2} + u^{2}) dt \dot{x} = u - \sin(x), \quad x(0) = 1$$



Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \ \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Compute: (t) $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - \nabla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \text{ and } \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ -20<sup>l</sup> 20  $H(x, u, \lambda) = \frac{1}{2} (x^{2} + u^{2}) + \lambda (u - \sin (x))$  $(\hat{t})_{10}$ 

is minimised by  $u = -\lambda$ . Dynamics read as:

$$\dot{x} = -\lambda - \sin(x)$$
  
 $\dot{\lambda} = \lambda \cos(x) - x$ 

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

(t) 10- n

-20

2

 $\dot{x} = u - \sin(x), \quad x(0) = 1$ 

Guess  $\lambda_0 = 0.41$  / Current  $\lambda_0 = 0.94265$ 

 $\lambda(4) = 17.1841$ 

3

3
Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{x,y} \frac{1}{2} \int_{0}^{4} (x^{2} + u^{2}) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Guess  $\lambda_0 = 0.41$  / Current  $\lambda_0 = 0.57211$  $10^{-10}$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - \nabla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \text{ and } \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ -10<sup>l</sup> 10r  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$  $\binom{t}{2}$  $\lambda(4) = 5.0793$ is minimised by  $u = -\lambda$ . Dynamics read as: *i* (*t*) -5  $\dot{x} = -\lambda - \sin(x)$  $\dot{\lambda} = \lambda \cos(x) - x$ -10<sup>L</sup> 2 3 Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ くぼう くほう くほう 19<sup>th</sup> of February, 2016

**Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \ \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Guess  $\lambda_0 = 0.41$  / Current  $\lambda_0 = 0.49677$ Compute:  $\mathbf{r} = \boldsymbol{\lambda} \left( t_{\mathrm{f}} \right) - 
abla_{\mathbf{x}} \phi \left( \mathbf{x} \left( t_{\mathrm{f}} \right) \right) \quad \mathsf{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ (t) Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ -5<u>`</u> 2  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$  $\widehat{\underline{t}}_{2}$ is minimised by  $u = -\lambda$ . Dynamics read as: 0 (t) $\dot{x} = -\lambda - \sin(x)$  $\dot{\lambda} = \lambda \cos(x) - x$ -40

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

2

3  $\lambda(4) = 3.186$ 

3

3

Example:

**Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - 
abla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$  $\lambda(t)$ 

is minimised by  $u = -\lambda$ . Dynamics read as:

$$\dot{x} = -\lambda - \sin(x)$$
  
 $\dot{\lambda} = \lambda \cos(x) - x$ 

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

$$\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2} \int_0^4 (x^2 + u^2) \, dt \\ \dot{x} = u - \sin(x) \,, \quad x(0) = 1$$



Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \ \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Guess  $\lambda_0 = 0.41$  / Current  $\lambda_0 = 0.38536$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - 
abla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ (t)Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 5  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$  $\widehat{f}$  $\lambda(4) = -3.4516$ -5<sup>1</sup> 0 2 is minimised by  $u = -\lambda$ . Dynamics read as: u(t) $\dot{x} = -\lambda - \sin(x)$  $\dot{\lambda} = \lambda \cos(x) - x$ -5` 0 2 Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

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**Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$ m Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ : x  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Compute:  $\mathbf{r} = \boldsymbol{\lambda} \left( t_{\mathrm{f}} \right) - 
abla_{\mathbf{x}} \phi \left( \mathbf{x} \left( t_{\mathrm{f}} \right) \right) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ (t) Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ -5<sub>0</sub>  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$  $\widehat{\underline{t}}_{2}$ 

is minimised by  $u = -\lambda$ . Dynamics read as:

$$\dot{x} = -\lambda - \sin(x)$$
  
 $\dot{\lambda} = \lambda \cos(x) - x$ 

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

$$\lim_{\mathbf{u}} \quad \frac{1}{2} \int_{0}^{4} (x^{2} + u^{2}) dt \dot{x} = u - \sin(x), \quad x(0) = 1$$



Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \ \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Guess  $\lambda_0 = 0.41$  / Current  $\lambda_0 = 0.33057$ Compute: (t) $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - \nabla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \text{ and } \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 10r  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$  $\lambda(t)$  $\lambda(4) = -5.9176$ 0 -10<sup>L</sup> n (t) 10, is minimised by  $u = -\lambda$ . Dynamics read as:  $\dot{x} = -\lambda - \sin(x)$  $\dot{\lambda} = \lambda \cos(x) - x$ -10<sup>L</sup> 2 Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ ・ 同 ト ・ ヨ ト ・ ヨ

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Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \ \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Guess  $\lambda_0 = 0.41$  / Current  $\lambda_0 = 0.43026$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - 
abla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ (t)Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 0 5  $H(x, u, \lambda) = \frac{1}{2} (x^{2} + u^{2}) + \lambda (u - \sin(x))$  $\widehat{f}$ -5<sup>1</sup> 0 2 is minimised by  $u = -\lambda$ . Dynamics read as: u(t) $\dot{x} = -\lambda - \sin(x)$  $\dot{\lambda} = \lambda \cos(x) - x$ -5` 0 2 Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

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 $\lambda(4) = -3.165$ 

3

3

Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$  $\frac{100}{100} \times (t)$ Compute:  $\mathbf{r} = \boldsymbol{\lambda}(t_{\mathrm{f}}) - 
abla_{\mathbf{x}}\phi(\mathbf{x}(t_{\mathrm{f}})) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ -500 400 ÷200  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ is minimised by  $u = -\lambda$ . Dynamics read as: (t) 200<sup>-</sup> n  $\dot{x} = -\lambda - \sin(x)$ 

 $\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2} \int_0^4 (x^2 + u^2) \, dt \\ \dot{x} = u - \sin(x) \,, \quad x(0) = 1$ 



Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

 $\dot{\lambda} = \lambda \cos(x) - x$ 

Input: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > \text{tol } \mathbf{do}$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \lambda, \mathbf{u})$ :  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$   $\dot{\lambda} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \lambda, \mathbf{u}), \quad \lambda(t_0) = \lambda_0$ Compute:  $\mathbf{r} = \lambda(t_f) - \nabla_{\mathbf{x}} \phi(\mathbf{x}(t_f)) \text{ and } \frac{\partial \mathbf{r}}{\partial \lambda_0}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 

$$H(x, u, \lambda) = \frac{1}{2} \left(x^2 + u^2\right) + \lambda \left(u - \sin(x)\right)$$

is minimised by  $u = -\lambda$ . Dynamics read as:

$$\dot{x} = -\lambda - \sin(x)$$
  
 $\dot{\lambda} = \lambda \cos(x) - x$ 

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0 !!$ 

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Example:

$$\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2} \int_0^4 (x^2 + u^2) dt \\ \dot{x} = u - \sin(x), \quad x(0) = 1$$

What is going on ?!?

Let's try to understand the relationship  $\lambda_{0}
ightarrow\lambda\left(t
ight)$ 







19<sup>th</sup> of February, 2016



Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ t = 2Compute: 10  $\mathbf{r} = \boldsymbol{\lambda} \left( t_{\mathrm{f}} 
ight) - 
abla_{\mathbf{x}} \phi \left( \mathbf{x} \left( t_{\mathrm{f}} 
ight) 
ight)$  and  $\frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 5  $\lambda(t)$ 0  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ -5 is minimised by  $u = -\lambda$ . Dynamics read as: -10 ∟ 0.2  $\dot{x} = -\lambda - \sin(x)$ 0.6 0.4 0.8  $\dot{\lambda} = \lambda \cos(x) - x$  $\lambda_0$ 

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0$  !!

Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ t = 2.4Compute: 10  $\mathbf{r} = \boldsymbol{\lambda} \left( t_{\mathrm{f}} 
ight) - 
abla_{\mathbf{x}} \phi \left( \mathbf{x} \left( t_{\mathrm{f}} 
ight) 
ight)$  and  $\frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 5  $\lambda(t)$ 0  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ -5 is minimised by  $u = -\lambda$ . Dynamics read as:  $\dot{x} = -\lambda - \sin(x)$ -10 L 0.6 0.4 0.8  $\dot{\lambda} = \lambda \cos(x) - x$  $\lambda_0$ 

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Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \ \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ t = 2.8Compute: 10  $\mathbf{r} = \boldsymbol{\lambda} \left( t_{\mathrm{f}} 
ight) - 
abla_{\mathbf{x}} \phi \left( \mathbf{x} \left( t_{\mathrm{f}} 
ight) 
ight)$  and  $\frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 5  $\lambda(t)$ 0  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ -5 is minimised by  $u = -\lambda$ . Dynamics read as:  $\dot{x} = -\lambda - \sin(x)$ -10 L 0.6 0.4 0.8  $\dot{\lambda} = \lambda \cos(x) - x$  $\lambda_0$ 

Find  $\lambda_0$  to enforce  $\lambda_f = \lambda(4) = 0$  !!

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Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \ \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{x} = u - \sin(x), \quad x(0) = 1$  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ t = 3.2Compute: 10  $\mathbf{r} = \boldsymbol{\lambda} \left( t_{\mathrm{f}} 
ight) - 
abla_{\mathbf{x}} \phi \left( \mathbf{x} \left( t_{\mathrm{f}} 
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ight)$  and  $\frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 5  $\lambda(t)$ 0  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ -5 is minimised by  $u = -\lambda$ . Dynamics read as:  $\dot{x} = -\lambda - \sin(x)$ -10 L 0.6 0.4 0.8  $\dot{\lambda} = \lambda \cos(x) - x$  $\lambda_0$ 

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Example: **Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$  $\min_{\mathbf{x},\mathbf{u}} \ \frac{1}{2} \int_{0}^{4} \left( x^{2} + u^{2} \right) dt$ Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ :  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ t = 3.6Compute: 10  $\mathbf{r} = \boldsymbol{\lambda} \left( t_{\mathrm{f}} 
ight) - 
abla_{\mathbf{x}} \phi \left( \mathbf{x} \left( t_{\mathrm{f}} 
ight) 
ight)$  and  $\frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 5  $\lambda(t)$ 0  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ -5 is minimised by  $u = -\lambda$ . Dynamics read as:  $\dot{x} = -\lambda - \sin(x)$ 

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**Input**: Initial conditions  $\mathbf{x}_0$ , guess  $\lambda_0$ while  $\|\mathbf{r}\| > tol \ do$ m Integrate with  $\mathbf{u} = \arg\min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$ : x  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$  $\dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}), \quad \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0$ Compute: 10  $\mathbf{r} = \boldsymbol{\lambda} \left( t_{\mathrm{f}} \right) - 
abla_{\mathbf{x}} \phi \left( \mathbf{x} \left( t_{\mathrm{f}} \right) \right) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial \boldsymbol{\lambda}_{0}}$ Newton step:  $\lambda_0 \leftarrow \lambda_0 - \frac{\partial \mathbf{r}}{\partial \lambda_0}^{-1} \mathbf{r}$ 5  $\lambda(t)$ 0  $H(x, u, \lambda) = \frac{1}{2} (x^2 + u^2) + \lambda (u - \sin(x))$ 

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$$\lim_{u} \frac{1}{2} \int_{0}^{4} (x^{2} + u^{2}) dt \dot{x} = u - \sin(x), \quad x(0) = 1$$



19<sup>th</sup> of February, 2016

Consider a compact domain D(t) in the  $\mathbf{x}(t)$ ,  $\lambda(t)$  space, with boundary  $\partial D(t)$ . The volume of D(t) say  $\rho(D(t)) \in \mathbb{R}$  is given by:

$$\rho\left(D\left(t\right)\right) = \int_{D(t)} \mathrm{d}D$$

and evolves with time as the integration of  $\mathbf{x}(t)$ ,  $\boldsymbol{\lambda}(t)$  proceeds.

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$$\frac{d}{dt}\rho\left(D\left(t\right)\right) = \int_{\partial D(t)} \left\langle \left[\begin{array}{c} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{array}\right], \mathbf{n}_{\partial D} \right\rangle \mathrm{d}\partial D$$

where is  $\mathbf{n}_{\partial D}$  the normal to the surface  $\partial D$ 

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where is  $\mathbf{n}_{\partial D}$  the normal to the surface  $\partial D$ , and div the divergence operator.

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$$\operatorname{div}\left(\left[\begin{array}{c}\dot{\mathbf{x}}\\\dot{\boldsymbol{\lambda}}\end{array}\right]\right) = \operatorname{Tr}\left(\frac{\partial\dot{\mathbf{x}}}{\partial\mathbf{x}} + \frac{\partial\dot{\boldsymbol{\lambda}}}{\partial\boldsymbol{\lambda}}\right) = \operatorname{Tr}\left(\frac{\partial\mathbf{F}}{\partial\mathbf{x}} - \frac{\partial}{\partial\boldsymbol{\lambda}}\frac{\partial H}{\partial\mathbf{x}}\right) = \mathbf{0}$$

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Consider a compact domain D(t) in the x(t),  $\lambda(t)$  space, with boundary  $\partial D(t)$ . The volume of D(t) say  $\rho(D(t)) \in \mathbb{R}$  is given by:

$$\rho\left(D\left(t\right)\right) = \int_{D(t)} \mathrm{d}D$$

and evolves with time as the integration of  $\mathbf{x}(t)$ ,  $\boldsymbol{\lambda}(t)$  proceeds. Its time-derivative reads as:

$$\frac{d}{dt}\rho\left(D\left(t\right)\right) = \int_{\partial D(t)} \left\langle \left[\begin{array}{c} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{array}\right], \mathbf{n}_{\partial D} \right\rangle \mathrm{d}\partial D \xrightarrow{Green'sth.} \int_{D(t)} \mathrm{div}\left(\left[\begin{array}{c} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{array}\right]\right) \mathrm{d}D$$

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because

$$\frac{\partial}{\partial \boldsymbol{\lambda}} \frac{\partial \boldsymbol{H}}{\partial \mathbf{x}} = \frac{\partial}{\partial \boldsymbol{\lambda}} \left( \frac{\partial \boldsymbol{L}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} + \frac{\partial \boldsymbol{\lambda}^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} \right) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}$$

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$$\frac{\partial}{\partial \lambda} \frac{\partial H}{\partial \mathbf{x}} = \frac{\partial}{\partial \lambda} \left( \frac{\partial L(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} + \frac{\partial \lambda^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} \right) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}$$

The volume of D(t) is constant throughout the integration of  $\mathbf{x}, \boldsymbol{\lambda}$  !!

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Consider a compact domain D(t) in the  $\mathbf{x}(t)$ ,  $\lambda(t)$  space, with boundary  $\partial D(t)$ . The volume of D(t) say  $\rho(D(t)) \in \mathbb{R}$  is given by:

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The volume of D(t) is constant<sup>\*</sup> throughout the integration of  $\mathbf{x}, \boldsymbol{\lambda}$  !!

\* See Liouville's Theorem on phase-space distribution functions in Hamiltonian mechanics

Problem:

$$\min_{\mathbf{x},\mathbf{u}} \quad \frac{1}{2} \int_0^{t_f} \left( x^2 + u^2 \right) dt \\ \dot{x} = u - \sin(x) \,, \quad x(0) = 1$$

Yields

$$H(x, u, \lambda) = \frac{1}{2} \left( x^2 + u^2 \right) + \lambda \left( u - \sin \left( x \right) \right)$$

Hamiltonian is minimised by  $u = -\lambda$ .

The dynamics become

$$\dot{x} = -\lambda - \sin(x)$$
  
 $\dot{\lambda} = \lambda \cos(x) - x$ 

Find  $\lambda_0$  to enforce  $\lambda_{
m f}=\lambda(t_{
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The punchline:

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Important consequence: the condition

$$oldsymbol{\lambda}\left(t_{\mathrm{f}}
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is often very sensitive to 
$$oldsymbol{\lambda}_0$$
 !!

Optimal Control with DAEs, lecture 9

**PMP equations**:  $\mathbf{u}^* = \arg \min_{\mathbf{u}} H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$  with:

$$\begin{split} & \text{States}: \quad \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \qquad \quad \mathbf{x}(t_0) = \bar{\mathbf{x}}_0 \\ & \text{Costates}: \quad \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} \mathcal{H}\left(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}\right), \quad \boldsymbol{\lambda}(t_f) = \nabla_{\mathbf{x}} \phi\left(\mathbf{x}\left(t_f\right)\right) \end{split}$$

Key idea: apply the multiple-shooting principle to the PMP equations

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Key idea: apply the multiple-shooting principle to the PMP equations State-costate Optimal control

$$\mathbf{u}\left(\mathbf{s}\right) = \arg\min_{\mathbf{u}} H\left(\mathbf{u}, \mathbf{s}\right)$$

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$$\mathbf{s} = \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} \qquad \qquad \mathbf{u}(\mathbf{s}) = \arg\min_{\mathbf{u}} H(\mathbf{u}, \mathbf{s})$$

Integrator  $\boldsymbol{\xi}(\mathbf{s}_k)$  over the time intervals  $t \in [t_k, t_{k+1}]$  with:

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**Root-finding** problem over the variables  $s_{0,...,N}$ :

$$\mathbf{r}(\mathbf{s}) = \begin{bmatrix} \mathbf{x}_0 - \bar{\mathbf{x}}_0 \\ \boldsymbol{\xi}(\mathbf{s}_0) - \mathbf{s}_1 \\ \vdots \\ \boldsymbol{\xi}(\mathbf{s}_{N-1}) - \mathbf{s}_N \\ \boldsymbol{\lambda}_N - \nabla_{\mathbf{x}} \phi(\mathbf{x}_N) \end{bmatrix} = \mathbf{0}$$

Problem:

PMP equations with  $u = -\lambda$ :

11 / 22

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# Outline

Introduction to the Pontryagin Maximum Principle (PMP)

- 2 Interpretation of  $H_{
  m u}$
- 3 Input bounds in Inducet Optimal 150
  - Singular Optimal Control problems

5 General constraints in Indirect Optimal Control

A (1) > A (2) > A

Consider the functional:

$$J[\mathbf{u}(.)] = \phi(\mathbf{x}(t_{f})) + \int_{t_{0}}^{t_{f}} L(\mathbf{x}, \mathbf{u}) dt$$
  
s.t.  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x}(t_{0}) = \mathbf{x}_{0}$ 

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that maps an input profile  $\mathbf{u}(.)$  into the corresponding cost.

**Gâteaux derivative** (see Calculus of Variations)  $\delta J[\mathbf{u}(.), \boldsymbol{\xi}(.)] = \lim_{\tau \to 0} \frac{J[\mathbf{u}(.) + \tau \boldsymbol{\xi}(.)] - J[\mathbf{u}(.)]}{\tau}$ 

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s.t.  $\dot{\mathbf{x}} = \mathbf{F} (\mathbf{x}, \mathbf{u}), \quad \mathbf{x} (t_0) = \mathbf{x}_0$ 

that maps an input profile  $\mathbf{u}(.)$  into the corresponding cost.

**Gâteaux derivative** (see Calculus of Variations)  $\delta J[\mathbf{u}(.), \boldsymbol{\xi}(.)] = \lim_{\tau \to 0} \frac{J[\mathbf{u}(.) + \tau \boldsymbol{\xi}(.)] - J[\mathbf{u}(.)]}{\tau}$ 

Note: Gâteaux derivatives are directional derivatives for functionals ("direction"  $\xi$ )

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Consider the functional:

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#### Optimality:

$$\delta J[\mathbf{u}^{\star}(.),\boldsymbol{\xi}(.)]=0, \quad \forall \boldsymbol{\xi}(.)$$

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**Interpretation of**  $H_{\mathbf{u}}$  (fundamental Lemma of Calculus of Variations):  $\delta J[\mathbf{u}(.), \boldsymbol{\xi}(.)] = \int_{0}^{t_{\mathrm{f}}} H_{\mathbf{u}}(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}(t)) \cdot \boldsymbol{\xi}(t) \, \mathrm{d}t$ 

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**Interpretation of**  $H_{u}$  (fundamental Lemma of Calculus of Variations):

$$\delta J\left[\mathbf{u}\left(.\right),\boldsymbol{\xi}\left(.\right)\right] = \int_{0}^{t_{\mathrm{f}}} H_{\mathbf{u}}\left(\mathbf{x}(t),\boldsymbol{\lambda}\left(t\right),\mathbf{u}\left(t\right)\right) \cdot \boldsymbol{\xi}\left(t\right) \mathrm{d}t$$

What if  $\mathbf{u}(.)$  is restricted to some (Banach) space ? E.g. piecewise-constant...

... then  $\xi(.)$  is restricted to the same space !
Interpretation of  $H_{\mathbf{u}}$  (cont')

Consider a piecewise-constant parametrization of  $\mathbf{u}(.)...$ 

... akin to a restriction of  $\mathbf{u}(.)$  to that Banach space !

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## Interpretation of $H_{\mathbf{u}}$ (cont')

Consider a piecewise-constant parametrization of  $\mathbf{u}(.)$ ... ... akin to a restriction of  $\mathbf{u}(.)$  to that Banach space !

Piecewise-constant parametrization

 $\begin{aligned} \mathbf{u}\left(t\right) &= \mathbf{u}_{k} \qquad \forall \, t \in [t_{k}, \, t_{k+1}] \\ \boldsymbol{\xi}\left(t\right) &= \boldsymbol{\xi}_{k} \qquad \forall \, t \in [t_{k}, \, t_{k+1}] \end{aligned}$ 

Interpretation of  $H_{\mathbf{u}}$  (cont')

Consider a piecewise-constant parametrization of  $\mathbf{u}(.)$ ... ... akin to a restriction of  $\mathbf{u}(.)$  to that Banach space !

Piecewise-constant parametrization

 $\mathsf{Functional} \to \mathsf{function}$ 

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 $J\left[\mathbf{u}\left(.\right)\right] \equiv \bar{J}\left(\mathbf{u}_{1},...,\mathbf{u}_{N-1}\right)$ 

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$$J\left[\mathbf{u}\left(.\right)\right]\equiv\bar{J}\left(\mathbf{u}_{1},...,\mathbf{u}_{N-1}\right)$$

Then optimality requires:

$$\delta J \left[ \mathbf{u}^{\star} \left( . \right), \boldsymbol{\xi} \left( . \right) \right] = \int_{0}^{t_{\mathrm{f}}} H_{\mathbf{u}} \left( \mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}^{\star}(t) \right) \cdot \boldsymbol{\xi}(t) \, \mathrm{d}t = \sum_{k=0}^{N-1} \int_{t_{k}}^{t_{k+1}} H_{\mathbf{u}} \left( \mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}_{k}^{\star} \right) \cdot \boldsymbol{\xi}_{k} \, \mathrm{d}t = 0, \quad \forall \boldsymbol{\xi}_{k}$$

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Functional  $\rightarrow$  function

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Hence **optimality condition** is  $\forall k$ :

$$\int_{t_{k}}^{t_{k+1}} H_{\mathbf{u}}\left(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}_{k}^{\star}\right) \, \mathrm{d}t = 0$$

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$$\int_{t_{k}}^{t_{k+1}} H_{\mathbf{u}}\left(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}_{k}^{\star}\right) \, \mathrm{d}t = 0$$

 $G\hat{a}teaux \rightarrow classic derivative$ 

$$\frac{\partial \bar{J}}{\partial \mathbf{u}_{k}} = \int_{t_{k}}^{t_{k+1}} H_{\mathbf{u}}\left(\mathbf{x}(t), \boldsymbol{\lambda}\left(t\right), \mathbf{u}_{k}\right) \, \mathrm{d}t$$

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 $J[\mathbf{u}(.)] \equiv \bar{J}(\mathbf{u}_1, ..., \mathbf{u}_{N-1})$ 

When solving an OCP using Direct Optimal Control, one can see the NLP solver as trying to get  $\int_{t_{\nu}}^{t_{k+1}} H_{\mathbf{u}}(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}_k) dt = 0 !!$ 

# Outline

Introduction to the Pontryagin Maximum Principle (PMP)

Interpretation of  $H_{\rm u}$ 

3 Input bounds in Indirect Optimal Control

Singular Optimal Control problems

General constraints in Indirect Optimal Control

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OCP with input bounds:

$$\begin{split} \min_{\mathbf{x},\mathbf{u}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) dt \\ \mathrm{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x},\mathbf{u}\right), \quad \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \end{split}$$

 $\mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max}$ 

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Define the Hamiltonian function

$$H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u}) + \boldsymbol{\mu}^{\top} \underbrace{\left[ egin{array}{c} \mathbf{u}_{\min} - \mathbf{u} \\ \mathbf{u} - \mathbf{u}_{\max} \end{array} 
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Get the input from:  $\mathbf{u} = \underset{\mathbf{u}}{\operatorname{argmin}} H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u})$ , with:

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 19<sup>th</sup> of February, 2016

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 $19^{\mathrm{th}}$  of February, 2016

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When  $\mathbf{u}$  hits the bounds,  $\boldsymbol{\mu}$  "creates a gradient" in  $H$  to enforce feasibility

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OCP with input bounds:

$$\begin{split} \min_{\mathbf{x},\mathbf{u}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) dt \\ \mathrm{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x},\mathbf{u}\right), \quad \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \\ & \mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max} \end{split}$$

An equivalent but simpler approach: define the Lagrange function

 $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u})$ 

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$$\label{eq:Getthe input from: } \begin{split} \text{Get the input from: } \mathbf{u} &= \mathop{\mathrm{argmin}}_{\mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max}} \mathcal{L}\left(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}\right) \text{, with: } \end{split}$$

OCP with input bounds:

$$\begin{split} \min_{\mathbf{x},\mathbf{u}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) dt \\ \mathrm{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x},\mathbf{u}\right), \quad \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \\ & \mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max} \end{split}$$

An equivalent but simpler approach: define the Lagrange function

$$\mathcal{L}\left(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}
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 $\begin{array}{ll} \mathsf{States}: & \dot{\mathbf{x}} = \mathbf{F}, & \mathbf{x}(t_0) = \mathbf{x}_0\\ \mathsf{Costates}: & \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}}\mathcal{L}, & \boldsymbol{\lambda}(t_\mathrm{f}) = \nabla_{\mathbf{x}}\phi\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) \end{array}$ 

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Note: optimality reads as...

$$\int_{0}^{t_{\mathrm{f}}} \mathcal{L}_{\mathbf{u}}\left(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}(t)\right) \cdot \boldsymbol{\xi}(t) \, \mathrm{d}t \geq 0 \quad \text{for any feasible direction } \boldsymbol{\xi}(.)$$

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Note:  $\mathbf{u}^*$  is now a **non-smooth** function of  $\mathbf{x}$ ,  $\boldsymbol{\lambda}$ . Must be handled carefully when solving TPBVP via Newton !!

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# Outline

Introduction to the Bontryagin Maximum Principle (PMP)

Interpretation of  $H_{\rm u}$ 

3 Input bounds in Indirect Optimal Co

# Singular Optimal Control problems

General constraints in Indirect Optimal Control

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 $\min_{\mathbf{u},\mathbf{x}} \quad \phi\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right)$ s.t.  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x}(\mathbf{0}) = \bar{\mathbf{x}}_0$  $\mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max}$ 

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What if  $\mathcal{L}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})$  is afine in  $\mathbf{u}$  ? E.g.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

yields  $\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda) = \lambda^{\top} \mathbf{f}(\mathbf{x}) + \lambda^{\top} \mathbf{g}(\mathbf{x}) \mathbf{u}$ then  $\mathbf{u} = \begin{cases} \mathbf{u}_{\max} & \text{if } \lambda^{\top} \mathbf{g}(\mathbf{x}) < 0 \end{cases}$ 

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u

Find **u** when  $\lambda^{\top} \mathbf{g}(\mathbf{x}) = 0$  ?

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$$\label{eq:max_max_matrix} \text{then } \mathbf{u} = \left\{ \begin{array}{ll} \mathbf{u}_{\max} & \text{if} \quad \boldsymbol{\lambda}^{\top} \mathbf{g}\left(\mathbf{x}\right) < \mathbf{0} \\ \mathbf{u}_{\min} & \text{if} \quad \boldsymbol{\lambda}^{\top} \mathbf{g}\left(\mathbf{x}\right) > \mathbf{0} \\ ? & \text{if} \quad \boldsymbol{\lambda}^{\top} \mathbf{g}\left(\mathbf{x}\right) = \mathbf{0} \end{array} \right.$$





Find  $\mathbf{u}$  when  $\mathbf{\lambda}^{ op} \mathbf{g}(\mathbf{x}) = 0$  ?

Use 
$$\frac{\mathrm{d}^{i}}{\mathrm{d}t^{i}}\mathcal{L}_{\mathbf{u}} = \mathbf{0}, \quad i > \mathbf{0}$$

for some *i*, the input **u** appears in  $\frac{d^{i}}{dt^{i}}\mathcal{L}_{\mathbf{u}}$ . Then solve  $\frac{d^{i}}{dt^{i}}\mathcal{L}_{\mathbf{u}} = 0$  for **u** !!

$$\min_{\mathbf{x}(.),\mathbf{u}(.)} \quad \frac{1}{2} \int_0^1 \mathbf{x}_1^2 \, \mathrm{d}t$$
s.t. 
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$-5 \le \mathbf{u} \le 5$$

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$$\begin{split} \min_{\mathbf{x}(.),\mathbf{u}(.)} & \frac{1}{2} \int_{0}^{1} \mathbf{x}_{1}^{2} dt \\ \text{s.t.} & \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & -5 \leq \mathbf{u} \leq 5 \\ \mathcal{L}_{\mathbf{u}} = \boldsymbol{\lambda}_{2} & \frac{d}{dt} \mathcal{L}_{\mathbf{u}} = -\boldsymbol{\lambda}_{1} & \frac{d^{2}}{dt^{2}} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_{1} & \frac{d^{3}}{dt^{3}} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_{2} & \frac{d^{4}}{dt^{4}} \mathcal{L}_{\mathbf{u}} = \mathbf{u} \end{split}$$

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$$\begin{split} \min_{\mathbf{x}(.),\mathbf{u}(.)} & \frac{1}{2} \int_{0}^{1} \mathbf{x}_{1}^{2} dt \\ \text{s.t.} & \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & -5 \leq \mathbf{u} \leq 5 \\ \mathcal{L}_{\mathbf{u}} = \lambda_{2} & \frac{d}{dt} \mathcal{L}_{\mathbf{u}} = -\lambda_{1} & \frac{d^{2}}{dt^{2}} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_{1} & \frac{d^{3}}{dt^{3}} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_{2} & \frac{d^{4}}{dt^{4}} \mathcal{L}_{\mathbf{u}} = \mathbf{u} \end{split}$$

#### **Optimal input**:

$$\mathbf{u}^{\star} = \left\{ \begin{array}{ll} \mathbf{u}_{\min} & \text{if} \quad \boldsymbol{\lambda}_2 > \mathbf{0} \\ \mathbf{u}_{\max} & \text{if} \quad \boldsymbol{\lambda}_2 < \mathbf{0} \\ \mathbf{0} & \text{if} \quad \frac{\mathrm{d}^i}{\mathrm{d}t'} \mathcal{L}_{\mathbf{u}} = \mathbf{0} \end{array} \right.$$

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$$\begin{split} \min_{\mathbf{x}(.),\mathbf{u}(.)} & \frac{1}{2} \int_0^1 \mathbf{x}_1^2 \, \mathrm{d}t \\ \text{s.t.} & \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & -5 \leq \mathbf{u} \leq 5 \\ \mathcal{L}_{\mathbf{u}} = \lambda_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}_{\mathbf{u}} = -\lambda_1 \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_1 \qquad \frac{\mathrm{d}^3}{\mathrm{d}t^3} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_2 \qquad \frac{\mathrm{d}^4}{\mathrm{d}t^4} \mathcal{L}_{\mathbf{u}} = \mathbf{u} \end{split}$$

#### **Optimal input:**

• Input is either in the bounds or zero !!

$$\mathbf{u}^{\star} = \left\{ egin{array}{ccc} \mathbf{u}_{\min} & \mbox{if} & \lambda_2 > 0 \ \mathbf{u}_{\max} & \mbox{if} & \lambda_2 < 0 \ 0 & \mbox{if} & rac{\mathrm{d}^{\prime}}{\mathrm{d}t^{\prime}}\mathcal{L}_{\mathbf{u}} = 0 \end{array} 
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$$\begin{split} \min_{\mathbf{x}(.),\mathbf{u}(.)} & \frac{1}{2} \int_0^1 \mathbf{x}_1^2 \, \mathrm{d}t \\ \text{s.t.} & \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & -5 \leq \mathbf{u} \leq 5 \\ \mathcal{L}_{\mathbf{u}} = \boldsymbol{\lambda}_2 & \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}_{\mathbf{u}} = -\boldsymbol{\lambda}_1 & \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_1 & \frac{\mathrm{d}^3}{\mathrm{d}t^3} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_2 & \frac{\mathrm{d}^4}{\mathrm{d}t^4} \mathcal{L}_{\mathbf{u}} = \mathbf{u} \end{split}$$

#### Optimal input:

- Input is either in the bounds or zero !!
- Bang-bang input until  $\mathbf{x}, \, \boldsymbol{\lambda} = \mathsf{0}$  (4 conditions)

$$\mathbf{u}^{\star} = \left\{ \begin{array}{ll} \mathbf{u}_{\min} & \text{if} \quad \boldsymbol{\lambda}_2 > 0 \\ \mathbf{u}_{\max} & \text{if} \quad \boldsymbol{\lambda}_2 < 0 \\ 0 & \text{if} \quad \frac{d^i}{dt^i} \mathcal{L}_{\mathbf{u}} = 0 \end{array} \right.$$
$$\begin{split} \min_{\mathbf{x}(.),\mathbf{u}(.)} & \frac{1}{2} \int_0^1 \mathbf{x}_1^2 \, \mathrm{d}t \\ \text{s.t.} & \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & -5 \leq \mathbf{u} \leq 5 \\ \mathcal{L}_{\mathbf{u}} = \boldsymbol{\lambda}_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}_{\mathbf{u}} = -\boldsymbol{\lambda}_1 \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_1 \qquad \frac{\mathrm{d}^3}{\mathrm{d}t^3} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_2 \qquad \frac{\mathrm{d}^4}{\mathrm{d}t^4} \mathcal{L}_{\mathbf{u}} = \mathbf{u} \end{split}$$

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- Input is either in the bounds or zero !!
- Bang-bang input until  $\mathbf{x}, \lambda = 0$  (4 conditions)
- $\bullet~\mathcal{L}_{\mathbf{u}}$  is "controlled" via its 4  $^{\mathrm{th}}\text{-order}$  derivative

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$$\begin{split} \min_{\mathbf{x}(.),\mathbf{u}(.)} & \frac{1}{2} \int_0^1 \mathbf{x}_1^2 \, \mathrm{d}t \\ \text{s.t.} & \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & -5 \leq \mathbf{u} \leq 5 \\ \mathcal{L}_{\mathbf{u}} = \boldsymbol{\lambda}_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}_{\mathbf{u}} = -\boldsymbol{\lambda}_1 \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_1 \qquad \frac{\mathrm{d}^3}{\mathrm{d}t^3} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_2 \qquad \frac{\mathrm{d}^4}{\mathrm{d}t^4} \mathcal{L}_{\mathbf{u}} = \mathbf{u} \end{split}$$

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- $\mathcal{L}_{\mathbf{u}}$  is "controlled" via its 4<sup>th</sup>-order derivative
- Problem has a *degree of singularity* of <sup>4</sup>/<sub>2</sub> = 2 (# of derivatives to get u is always even)

$$\begin{split} \min_{\mathbf{x}(.),\mathbf{u}(.)} & \frac{1}{2} \int_0^1 \mathbf{x}_1^2 \, \mathrm{d}t \\ \text{s.t.} & \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & -5 \leq \mathbf{u} \leq 5 \\ \mathcal{L}_{\mathbf{u}} = \lambda_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}_{\mathbf{u}} = -\lambda_1 \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_1 \qquad \frac{\mathrm{d}^3}{\mathrm{d}t^3} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_2 \qquad \frac{\mathrm{d}^4}{\mathrm{d}t^4} \mathcal{L}_{\mathbf{u}} = \mathbf{u} \end{split}$$

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- Input is either in the bounds or zero !!
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- Degrees of freedom: λ(0) ∈ ℝ<sup>2</sup> and switching times in the bang-bang...

$$\begin{split} \min_{\mathbf{x}(.),\mathbf{u}(.)} & \frac{1}{2} \int_0^1 \mathbf{x}_1^2 \, \mathrm{d}t \\ \text{s.t.} & \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & -5 \leq \mathbf{u} \leq 5 \\ \mathcal{L}_{\mathbf{u}} = \boldsymbol{\lambda}_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}_{\mathbf{u}} = -\boldsymbol{\lambda}_1 \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_1 \qquad \frac{\mathrm{d}^3}{\mathrm{d}t^3} \mathcal{L}_{\mathbf{u}} = \mathbf{x}_2 \qquad \frac{\mathrm{d}^4}{\mathrm{d}t^4} \mathcal{L}_{\mathbf{u}} = \mathbf{u} \end{split}$$

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- Problem has a *degree of singularity* of <sup>4</sup>/<sub>2</sub> = 2 (# of derivatives to get u is always even)
- Degrees of freedom: λ(0) ∈ ℝ<sup>2</sup> and switching times in the bang-bang...
- We will have 2 switching times, to have 2+2=4

$$\min_{\mathbf{x}(.),\mathbf{u}(.)} \quad \frac{1}{2} \int_0^1 \mathbf{x}_1^2 \, \mathrm{d}t$$
s.t. 
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$-5 \le \mathbf{u} \le 5$$

#### **Optimal solution**



 $19^{\mathrm{th}}$  of February, 2016

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$$\min_{\mathbf{x}(.),\mathbf{u}(.)} \quad \frac{1}{2} \int_0^1 \mathbf{x}_1^2 \, \mathrm{d}t$$
s.t. 
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$-5 \le \mathbf{u} \le 5$$

**Optimal solution** vs. solution from multiple-shooting  $(t_{k+1} - t_k = 0.01)$ 



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$$\min_{\mathbf{x}(.),\mathbf{u}(.)} \quad \frac{1}{2} \int_0^1 \mathbf{x}_1^2 \, \mathrm{d}t$$
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$$-5 \le \mathbf{u} \le 5$$

**Optimal solution** vs. solution from multiple-shooting  $(t_{k+1} - t_k = 0.01)$ 





Consider the problem:

 $\begin{aligned} J\left(\mathbf{u}\right) &= & \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} & \dot{\mathbf{x}}(t) = \mathbf{F}\left(\mathbf{x}(t),\mathbf{u}(t)\right) \\ & & \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \\ & & & \mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max} \end{aligned}$ 

with  $\mathbf{u}(t) = \mathbf{u}_k$  for  $t \in [t_k, t_{k+1}]$ .



Consider the problem:

$$\begin{split} J(\mathbf{u}) &= \phi\left(\mathbf{x}\left(.\right),\mathbf{u}\left(.\right)\right) \\ \text{s.t.} \quad \dot{\mathbf{x}}(t) &= \mathbf{F}\left(\mathbf{x}(t),\mathbf{u}(t)\right) \\ \mathbf{x}\left(t_{0}\right) &= \mathbf{x}_{0} \\ \mathbf{u}_{\min} &\leq \mathbf{u} \leq \mathbf{u}_{\max} \end{split}$$

with  $\mathbf{u}(t) = \mathbf{u}_k$  for  $t \in [t_k, t_{k+1}]$ . Then

$$\frac{\partial J}{\partial \mathbf{u}_{k}} = \int_{t_{k}}^{t_{k+1}} \mathcal{L}_{\mathbf{u}}\left(\mathbf{x}(t), \boldsymbol{\lambda}\left(t\right), \mathbf{u}_{k}\right) \, \mathrm{d}t$$

is zero when  $\mathbf{u}_k$  is off the bounds.







Consider the problem:

$$J(\mathbf{u}) = \phi(\mathbf{x}(.), \mathbf{u}(.))$$
  
s.t.  $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{u}(t))$   
 $\mathbf{x}(t_0) = \mathbf{x}_0$   
 $\mathbf{u}_{\min} \le \mathbf{u} \le \mathbf{u}_{\max}$   
with  $\mathbf{u}(t) = \mathbf{u}_k$  for  $t \in [t_k, t_{k+1}]$ . Then

$$\frac{\partial J}{\partial \mathbf{u}_{k}} = \int_{t_{k}}^{t_{k+1}} \mathcal{L}_{\mathbf{u}}\left(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}_{k}\right) \, \mathrm{d}t$$

is zero when  $\mathbf{u}_k$  is off the bounds.



# Outline

Introduction to the Bontryagin Maximum Principle (PMP)

Interpretation of  $H_{
m u}$ 

3 Input bounds in Indirect Optimal I

Singular Optimal Control poplems

5 General constraints in Indirect Optimal Control

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OCP with state (mixed) constraints:

$$\begin{split} \min_{\mathbf{x},\mathbf{u}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) dt \\ \mathrm{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x},\mathbf{u}\right), \quad \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \\ & \mathbf{h}\left(\mathbf{x},\mathbf{u}\right) \leq \mathbf{0} \end{split}$$

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Define the Hamiltonian function

$$H\left(\mathbf{x}, oldsymbol{\lambda}, oldsymbol{\mu}, \mathbf{u}
ight) = L\left(\mathbf{x}, \mathbf{u}
ight) + oldsymbol{\lambda}^{ op} \mathbf{F}\left(\mathbf{x}, \mathbf{u}
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ight)$$

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OCP with state (mixed) constraints:

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ight) + \boldsymbol{\mu}^{ op} \mathbf{h}\left(\mathbf{x}, \mathbf{u}
ight)$$

Get the optimal control solution from:  $\mathbf{u}^* = \operatorname{argmin} H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u})$ , with: u

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OCP with state (mixed) constraints:

$$\begin{split} \min_{\mathbf{x},\mathbf{u}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) dt \\ \mathrm{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x},\mathbf{u}\right), \quad \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \\ & \mathbf{h}\left(\mathbf{x},\mathbf{u}\right) \leq \mathbf{0} \end{split}$$

Define the Hamiltonian function

$$H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u}) + \boldsymbol{\mu}^{\top} \mathbf{h}(\mathbf{x}, \mathbf{u})$$

Get the optimal control solution from:  $\mathbf{u}^{\star} = \underset{\mathbf{u}}{\operatorname{argmin}} H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u})$ , with:

 $\begin{array}{ll} \text{States}: & \dot{\mathbf{x}} = \mathbf{F}, & \mathbf{x}(t_0) = \mathbf{x}_0\\ \text{Costates}: & \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} \boldsymbol{H}, & \boldsymbol{\lambda}(t_f) = \nabla_{\mathbf{x}} \phi\left(\mathbf{x}\left(t_f\right)\right) \end{array}$ 

OCP with state (mixed) constraints:

$$\begin{split} \min_{\mathbf{x},\mathbf{u}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) dt \\ \mathrm{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x},\mathbf{u}\right), \quad \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \\ & \mathbf{h}\left(\mathbf{x},\mathbf{u}\right) \leq \mathbf{0} \end{split}$$

Define the Hamiltonian function

$$H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u}) + \boldsymbol{\mu}^{\top} \mathbf{h}(\mathbf{x}, \mathbf{u})$$

Get the optimal control solution from:  $\mathbf{u}^{\star} = \underset{\mathbf{u}}{\operatorname{argmin}} H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u})$ , with:

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OCP with state (mixed) constraints:

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$$H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\top} \mathbf{F}(\mathbf{x}, \mathbf{u}) + \boldsymbol{\mu}^{\top} \mathbf{h}(\mathbf{x}, \mathbf{u})$$

Get the optimal control solution from:  $\mathbf{u}^{\star} = \underset{\mathbf{u}}{\operatorname{argmin}} H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u})$ , with:

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OCP with state (mixed) constraints:

$$\begin{split} \min_{\mathbf{x},\mathbf{u}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) dt \\ \mathrm{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x},\mathbf{u}\right), \quad \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \\ & \mathbf{h}\left(\mathbf{x},\mathbf{u}\right) \leq \mathbf{0} \end{split}$$

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Get the optimal control solution from:  $\mathbf{u}^{\star} = \underset{\mathbf{u}}{\operatorname{argmin}} H(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{u})$ , with:

 $\begin{array}{ccc} \text{States}: & \dot{\mathbf{x}} = \mathbf{F}, & \mathbf{x}(t_0) = \mathbf{x}_0 & & \mathsf{T} \\ \text{Costates}: & \dot{\boldsymbol{\lambda}} = -\nabla_{\mathbf{x}} \mathcal{H}, & \boldsymbol{\lambda}(t_{\mathrm{f}}) = \nabla_{\mathbf{x}} \phi\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right) & \\ \text{Feasibility}: & \mathbf{h}\left(\mathbf{x},\mathbf{u}\right) \leq 0, & \boldsymbol{\mu} \geq 0 & \\ \text{Complementary slack.:} & \boldsymbol{\mu}^{\top} \mathbf{h}\left(\mathbf{x},\mathbf{u}\right) = 0 & \\ \end{array}$ 

The PMP equations can be hard to solve in general. No good PMP-based general-purpose solver available.

OCP with state (mixed) constraints:

$$\begin{split} \min_{\mathbf{x},\mathbf{u}} & \phi(\mathbf{x}(t_{\mathrm{f}})) + \int_{t_{0}}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) dt \\ \mathrm{s.t.} & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x},\mathbf{u}\right), \quad \mathbf{x}\left(t_{0}\right) = \mathbf{x}_{0} \\ & \mathbf{h}\left(\mathbf{x},\mathbf{u}\right) \leq \mathbf{0} \end{split}$$

Tentative solutions based on IP method

$$\begin{array}{ll} \mbox{Stationarity}: & H_{\mathbf{u}}\left(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{u}\right)=0\\ & \mbox{States}: & \dot{\mathbf{x}}=\mathbf{F}\left(\mathbf{x},\mathbf{u}\right), & \mathbf{x}(t_{0})=\mathbf{x}_{0}\\ & \mbox{Costates}: & \dot{\boldsymbol{\lambda}}=-\nabla_{\mathbf{x}}H\left(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{u}\right), & \boldsymbol{\lambda}(t_{f})=\nabla_{\mathbf{x}}\phi\left(\mathbf{x}\left(t_{f}\right)\right)\\ & \mbox{Feasibility}: & \mathbf{h}\left(\mathbf{x},\mathbf{u}\right)\leq 0, & \boldsymbol{\mu}\geq 0\\ & \mbox{Complementary slack}: & \boldsymbol{\mu}^{\top}\mathbf{h}\left(\mathbf{x},\mathbf{u}\right)=\tau \end{array}$$

• Handle dynamics + constraints  $H_{\mathbf{u}} = 0$  and  $\boldsymbol{\mu}^{\top} \mathbf{h} = \tau$  as a DAE (c.f. next week)

- Handle  $\mathbf{h}(\mathbf{x},\mathbf{u}) \leq 0$  and  $\mu \geq 0$  via step length (c.f. IP lecture)
- Also done using Primal IP approach (move **h** in the cost using log barrier)