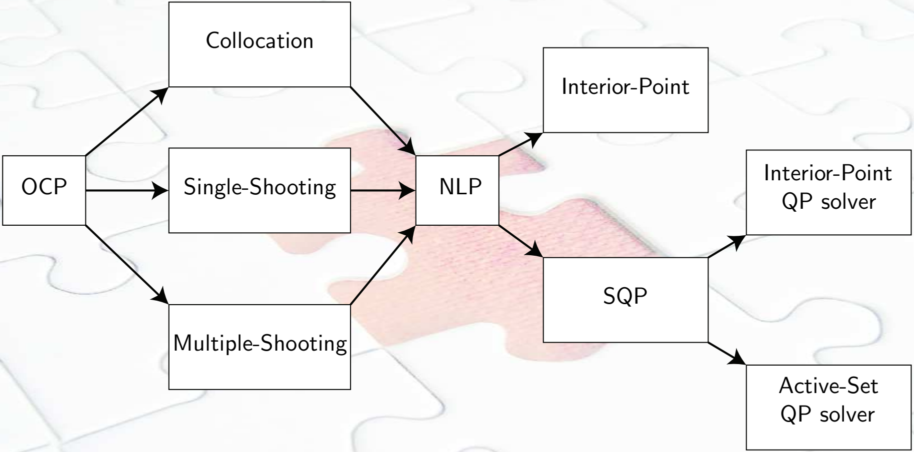
Numerical Optimal Control with DAEs
Lecture 8: Direct Collocation

Sébastien Gros

AWESCO PhD course

Survival map of Direct Optimal Control



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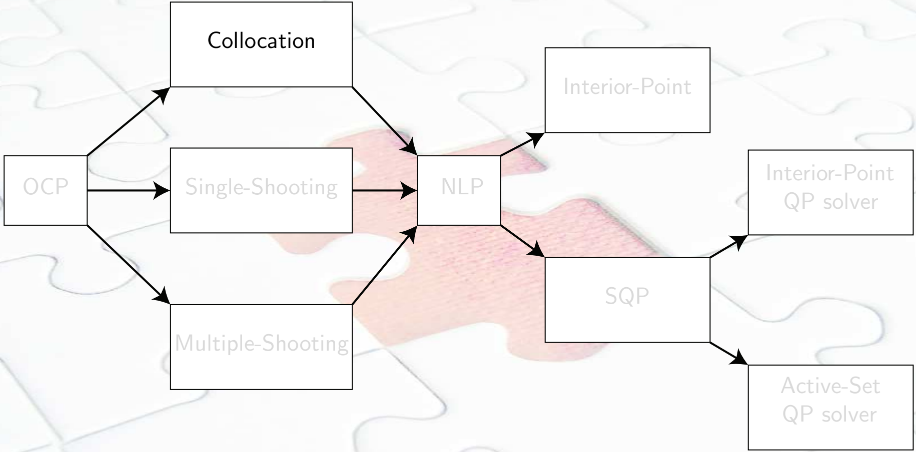

Another way for going from OCP to NLP

Outline

1. Polynomial interpolation
2. Collocation-based integration
3. Collocation in multiple-shooting
4. Direct Collocation
5. NLP from direct collocation

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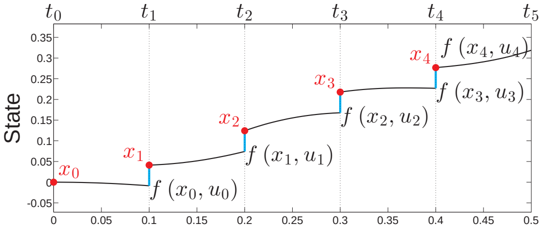
Polynomial interpolation

Consider a time grid:

$$\{t_{k,0}, \dots, t_{k,K}\} \in [t_k,\ t_{k+1}]$$



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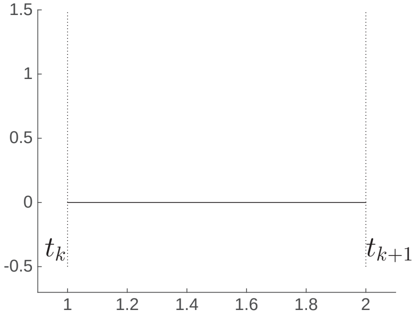

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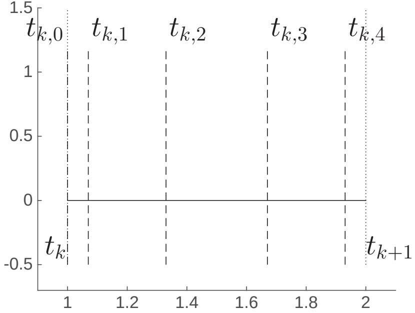

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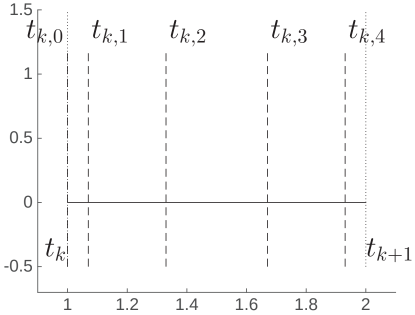

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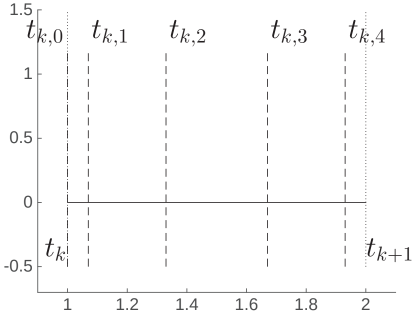

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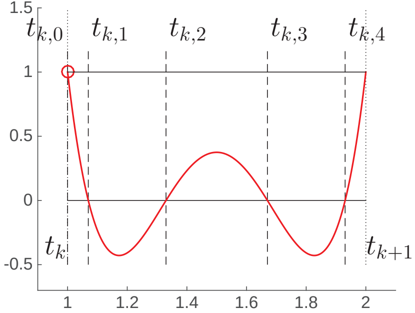

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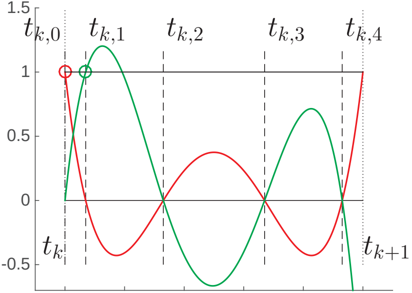

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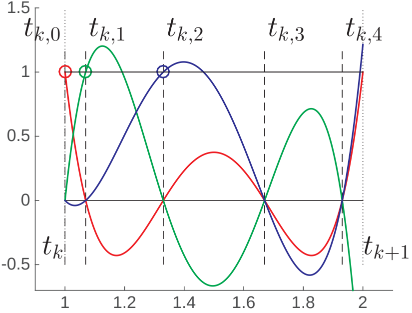

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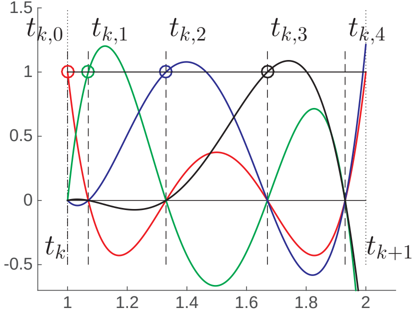

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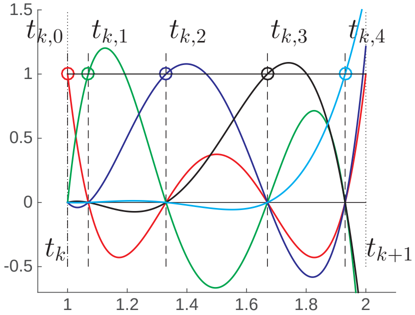

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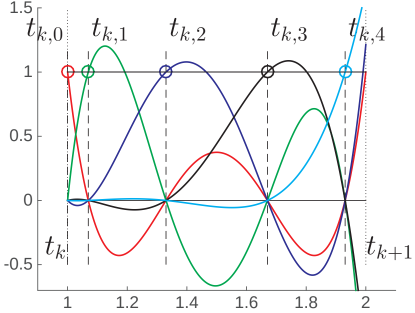

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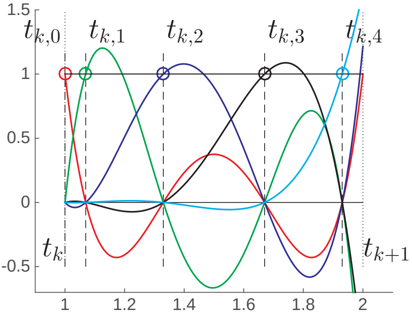

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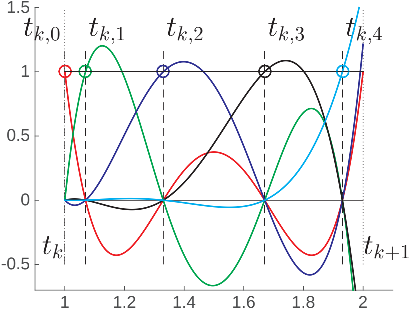

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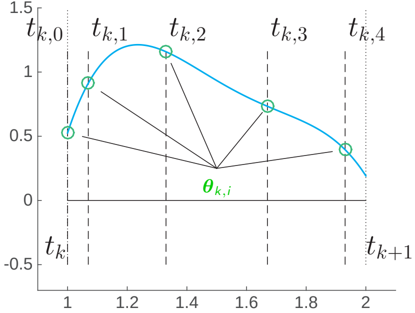

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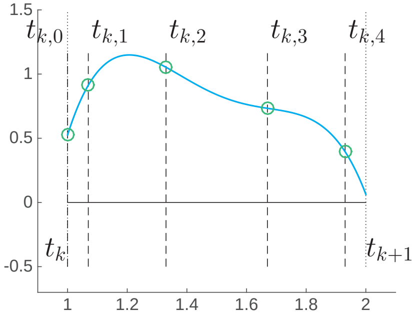

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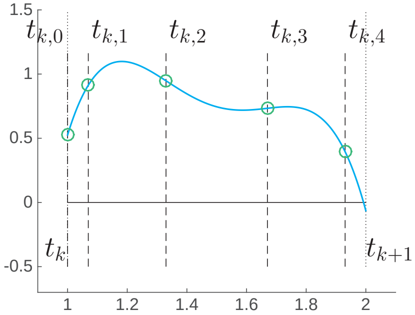

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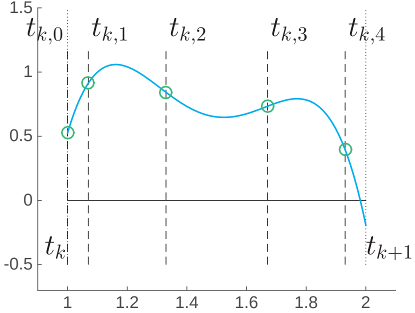

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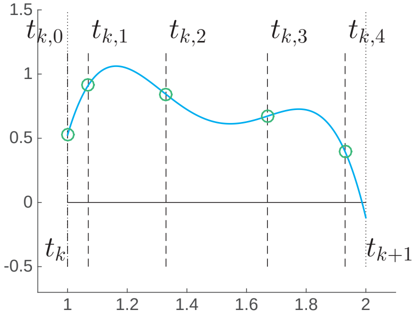

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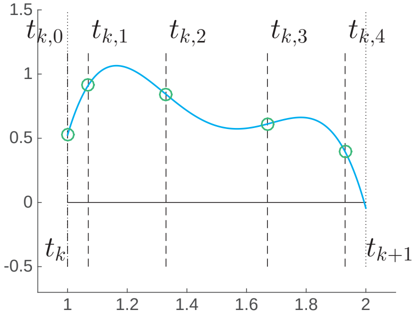

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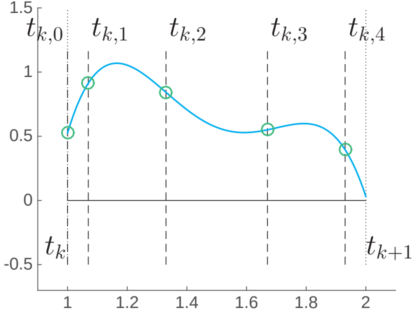

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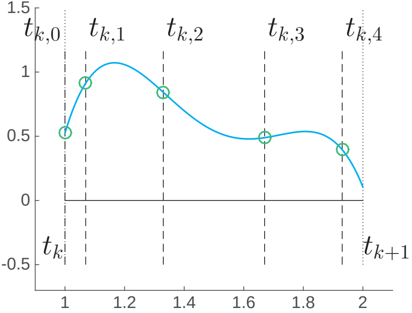

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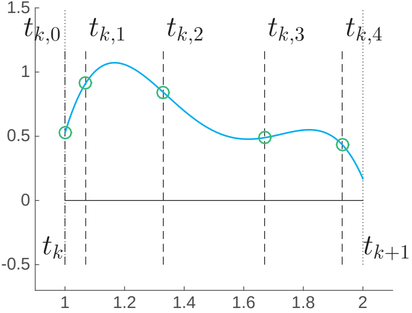

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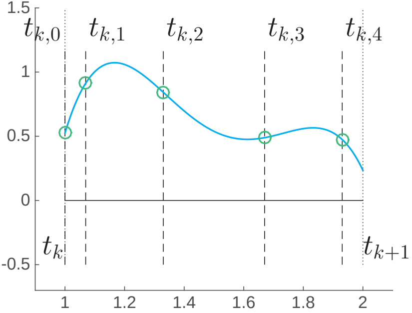

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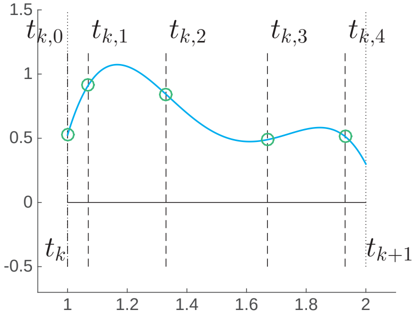

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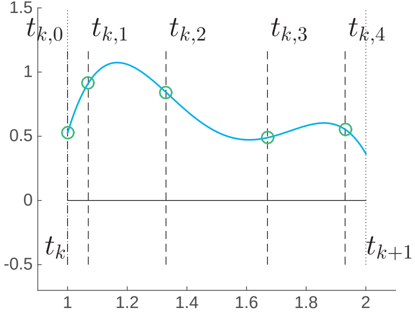

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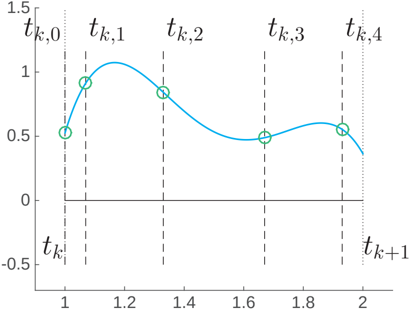


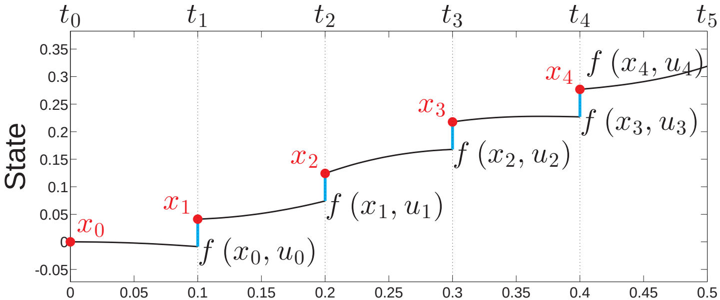
Note: the Lagrange polynomials are orthogonal, i.e.

$$\int_{t_k}^{t_{k+1}}P_{k,i}(t)P_{k,j}(t)\,\mathrm{d}t=0,\quad\forall i\neq j$$

Outline

Collocation methods - key idea

Approximate state trajectory $\mathbf{x}(t)$ via polynomials (order K)



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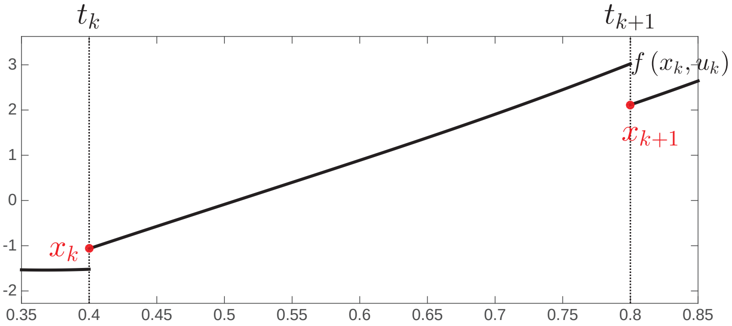

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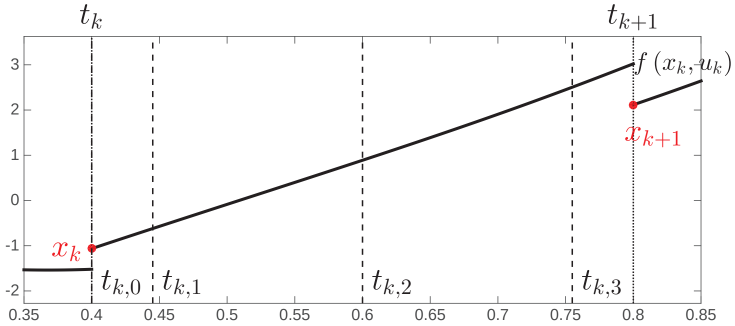

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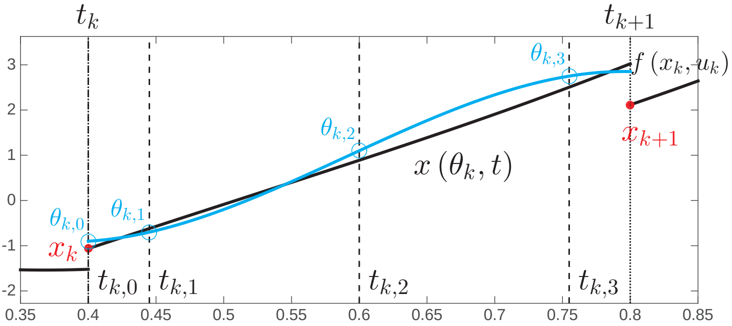

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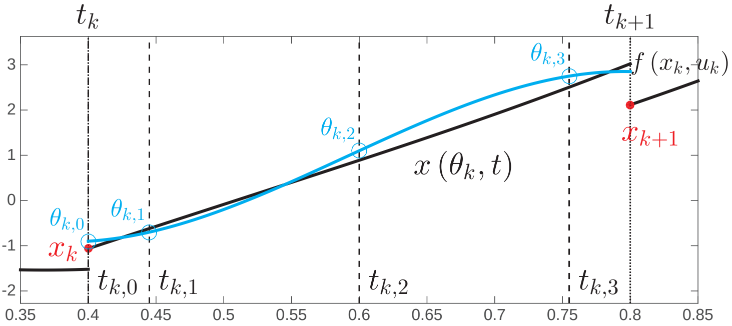

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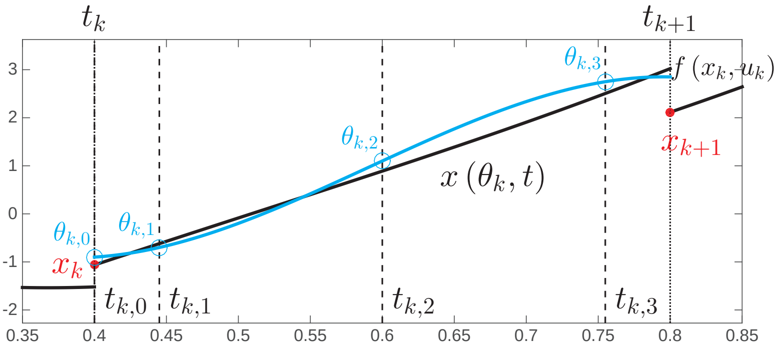

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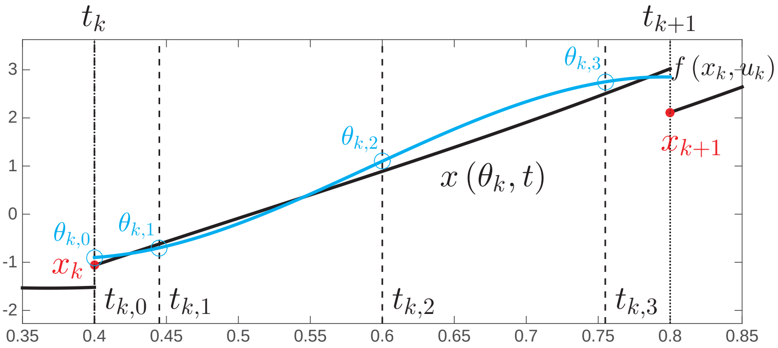

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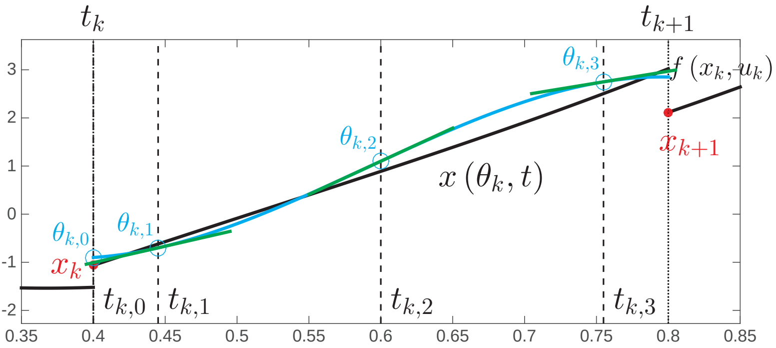

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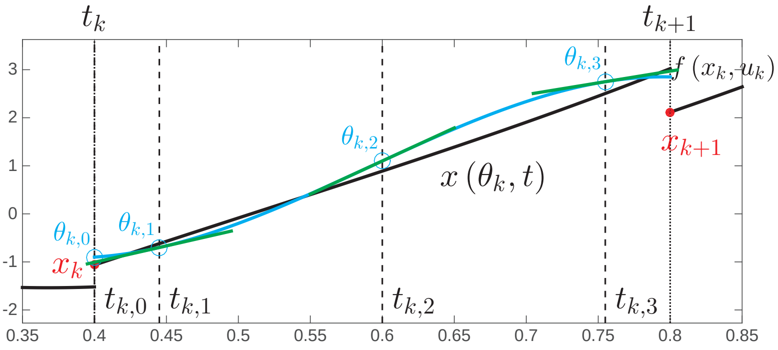

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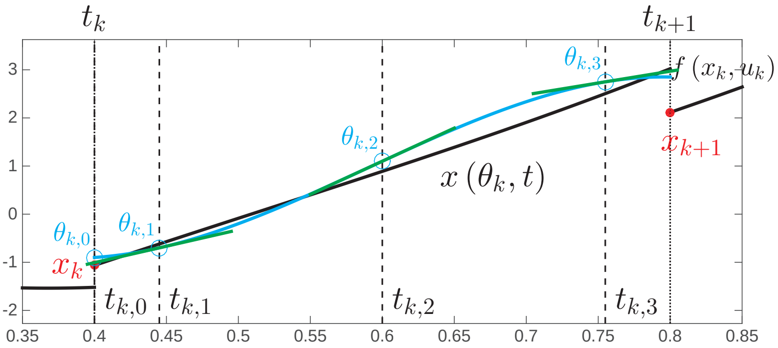

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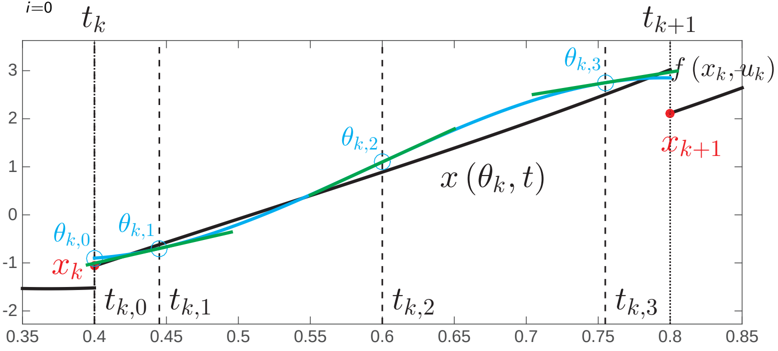

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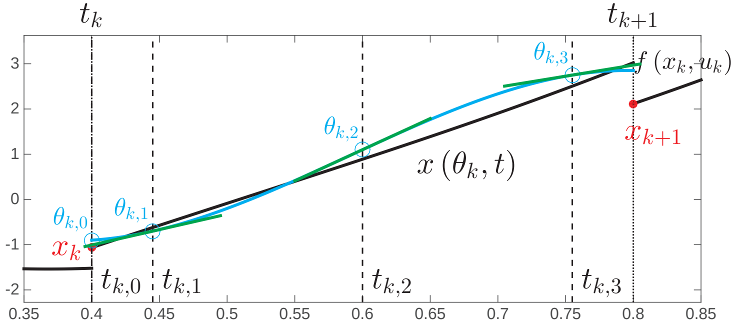

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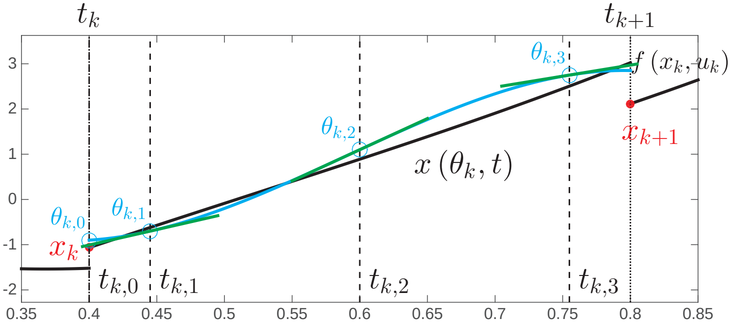

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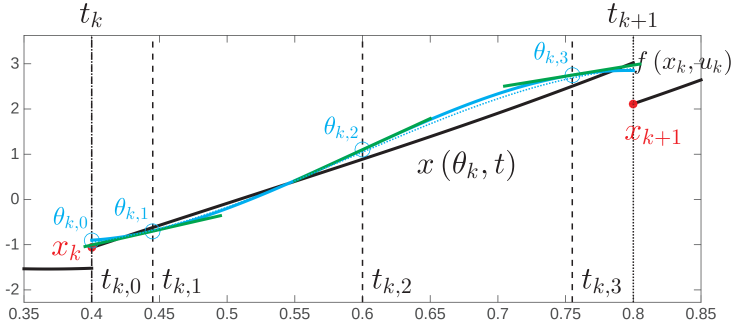

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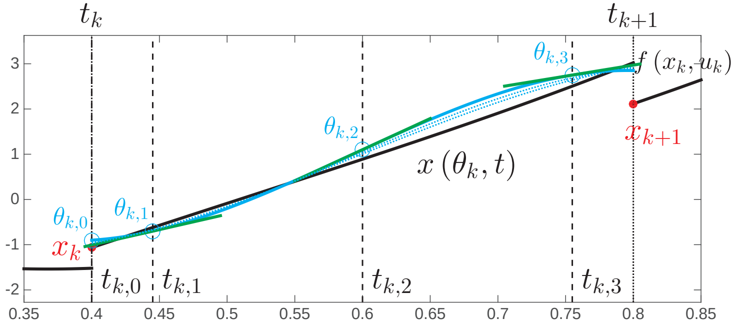

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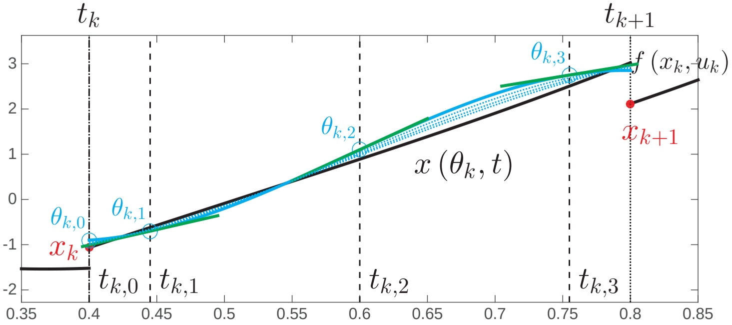

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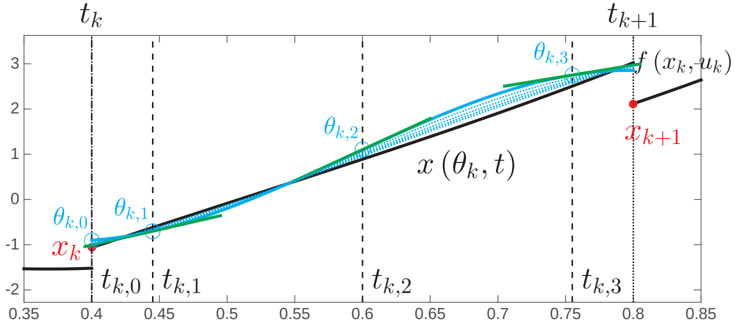

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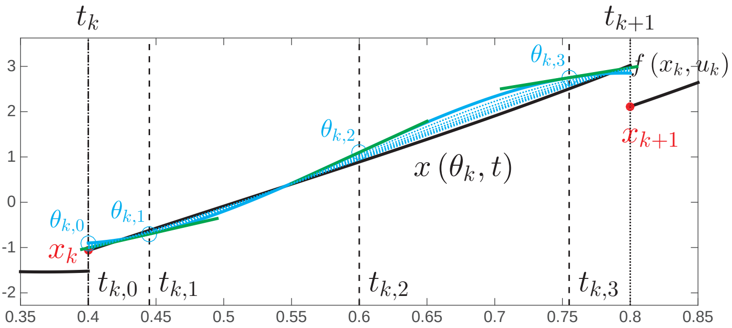
for $j = 1, ..., K$.

Solve for $\boldsymbol{\theta}_{k,i}$ using Newton

$$\begin{bmatrix} \boldsymbol{\theta}_{k,0} - \mathbf{x}_k \\ \sum_{i=0}^{K} \boldsymbol{\theta}_{k,i} \dot{P}_{k,i}(t_{k,1}) - \mathbf{F}\left(\boldsymbol{\theta}_{k,1}, \mathbf{u}_k\right) \\ \vdots \\ \sum_{i=0}^{K} \boldsymbol{\theta}_{k,i} \dot{P}_{k,i}(t_{k,K}) - \mathbf{F}\left(\boldsymbol{\theta}_{k,K}, \mathbf{u}_k\right) \end{bmatrix} = 0$$



Collocation methods - Implementation

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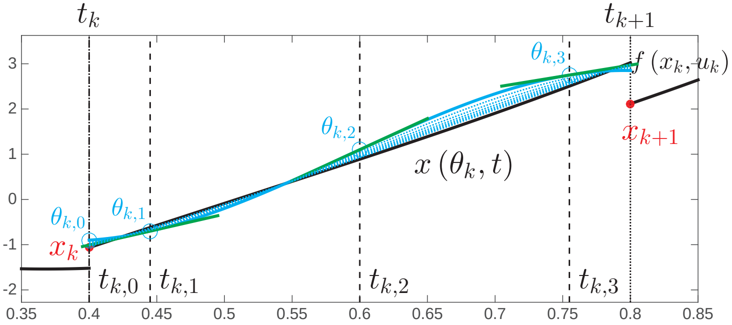

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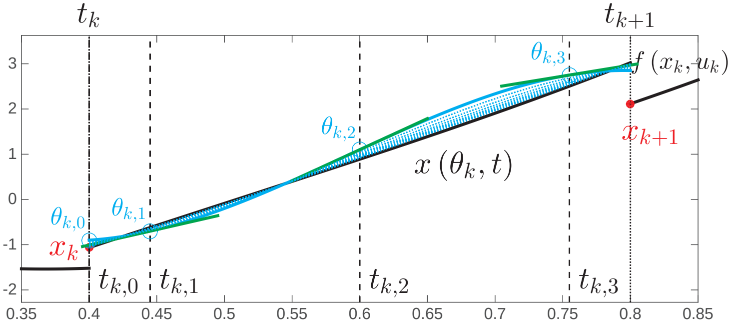

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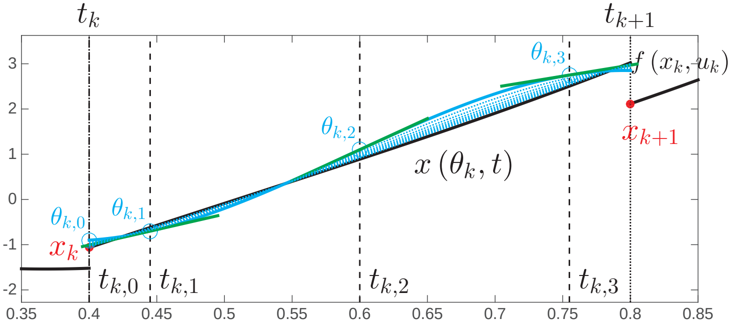

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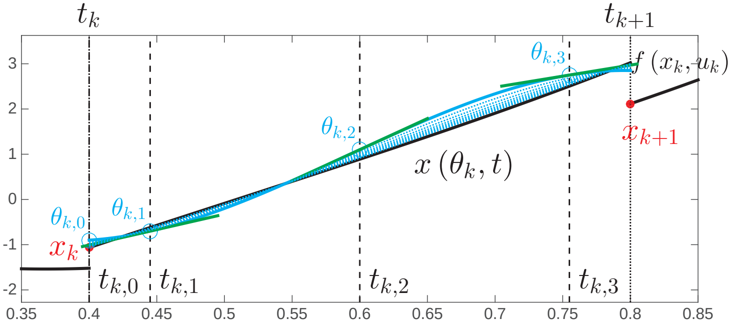
Shooting constraints

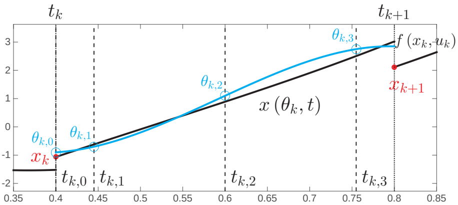
$$\underbrace{\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)}_{=\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right)} - \mathbf{x}_{k+1} = 0$$

becomes:

$$\sum_{i=0}^{K} \boldsymbol{\theta}_{k,i} P_{k,i}(t_{k+1}) - \mathbf{x}_{k+1} = 0$$



Selection of the time grid $t_{k,i}$



Selection of the time grid $t_{k,i}$

Collocation points on $[0,1]$:

K	Legendre	Radau
1	0.5	1.0
2	0.211325	0.333333
	0.788675	1.000000
3	0.112702	0.155051
	0.500000	0.644949
	0.887298	1.000000
4	0.069432	0.088588
	0.330009	0.409467
	0.669991	0.787659
	0.930568	1.000000
5	0.046910	0.057104
	0.230765	0.276843
	0.500000	0.583590
	0.769235	0.860240
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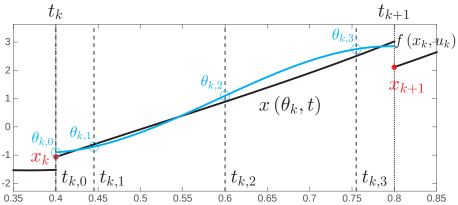

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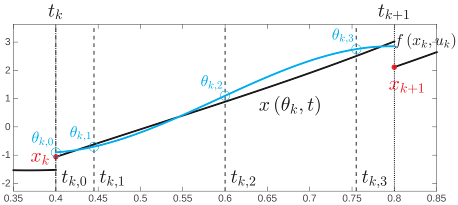

+one point at t_k to enforce continuity !

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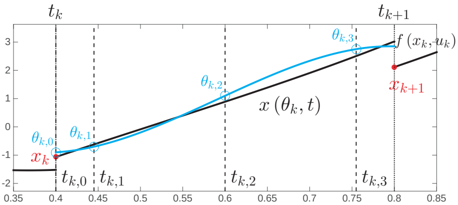

Why these points ?!? They deliver an exact integration for any polynomial $\mathbf{P}$ of order $< 2K$ (Legendre) and $< 2K - 1$ (Radau). I.e. for

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}) = \mathbf{P}(t)$$

the collocation equations deliver an exact solution, namely:

$$\mathbf{x}(t_{k+1}, \boldsymbol{\theta}_k) = \mathbf{x}_k + \int_{t_k}^{t_{k+1}} \mathbf{P}(\tau)\, \mathrm{d}\tau$$

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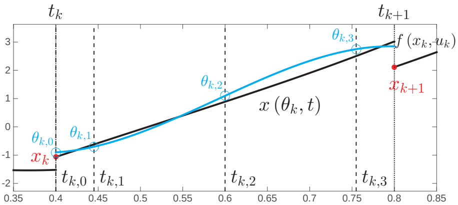

Interval $[t_k,\ t_{k+1}]$??

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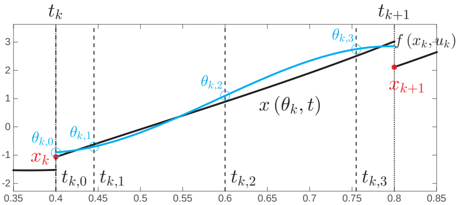

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- **Rescale & translate** the collocation points to $[t_k,\ t_{k+1}]$, or...
- Modification of the collocation equations with $h_k = t_{k+1} - t_k$:

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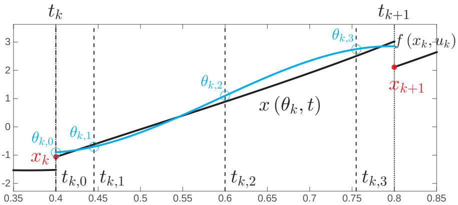

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Careful if $\mathbf{F}$ is time-dependent !

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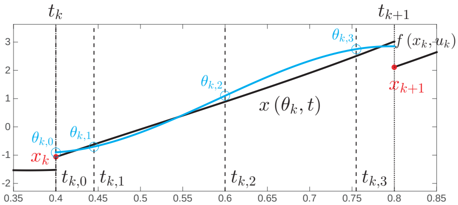

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Careful if $\mathbf{F}$ is time-dependent !

Note that Radau has a collocation point at the end of the interval, i.e. $\boldsymbol{\theta}_{k,K}$ provides the end-state of the integration !

Stability & Order

- Collocation methods are **A-stable** (i.e. can handle stiff equations). They have no stability limitation on the time intervals $h = t_{k+1} - t_k$ for stiff problems. I.e. even large time steps $h = t_{k+1} - t_k$ allow for capturing steady state and slow dynamics.

- Radau collocation is additionally **L-stable**. I.e. it can handle eigenvalues at $-\infty$.

- On an interval $h_k = t_{k+1} - t_k$, the **integration error** is $O(h_k^{2K})$ for Legendre and $O(h_k^{2K-1})$ for Radau. Losing one order is the "price" for having a collocation point at t_{k+1}.

- The integration error applies to the **end-state** of the integrator, but not to the intermediate points !

- Collocation-based integration is an **Implicit Runge-Kutta** scheme. Implicit Euler is an order-1 scheme !

Collocation - Sensitivity

Collocation constraints...

$$\boldsymbol{\theta}_{k,0} = \mathbf{x}_k$$

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right), \quad i = 1, ..., K$$

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$$\Delta\boldsymbol{\theta}_k = -\frac{\partial \mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right)}{\partial \boldsymbol{\theta}_k}^{-1} \mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right)$$

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Integrator function given by:

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Get sensitivities using:

$$\frac{\partial \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)}{\partial \mathbf{x}_k} = \frac{\partial \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right)}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{x}_k}, \qquad \frac{\partial \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)}{\partial \mathbf{u}_k} = \frac{\partial \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right)}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{u}_k}$$

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Hence:

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$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right)$$

Get sensitivities using:

$$\frac{\partial \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)}{\partial \mathbf{x}_k} = \frac{\partial \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right)}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{x}_k}, \qquad \frac{\partial \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)}{\partial \mathbf{u}_k} = \frac{\partial \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right)}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{u}_k}$$

Implicit function theorem states that

$$\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{x}_k} + \frac{\partial \mathbf{c}}{\partial \mathbf{x}_k} = 0, \qquad \frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{u}_k} + \frac{\partial \mathbf{c}}{\partial \mathbf{u}_k} = 0$$

Hence:

$$\frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{x}_k} = -\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k}^{-1} \frac{\partial \mathbf{c}}{\partial \mathbf{x}_k}, \qquad \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{u}_k} = -\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k}^{-1} \frac{\partial \mathbf{c}}{\partial \mathbf{u}_k}$$

Note that $\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k}^{-1}$ is computed in the Newton iteration, i.e. it comes for free !!

Outline

Collocation-based integrators in Multiple-shooting

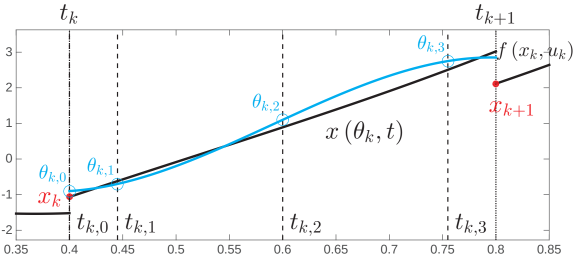
Collocation-based integrator solves:

$$\mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right) = 0$$

on each time interval $[t_k,\ t_{k+1}]$, provides:

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right)$$

with sensitivities.



Collocation-based integrators in Multiple-shooting

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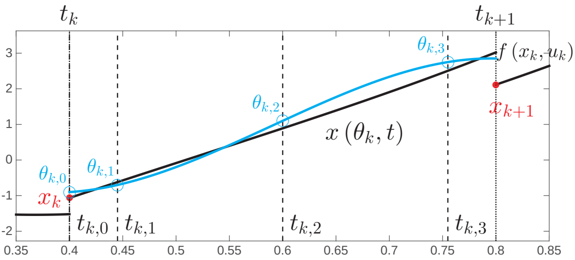
with sensitivities.

NLP with multiple-shooting

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \mathbf{x}_0 - \bar{\mathbf{x}}_0 \\ \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) - \mathbf{x}_N \end{bmatrix}$$

where $\mathbf{w} = \left\{\mathbf{x}_0,\ \mathbf{u}_0, \dots, \mathbf{x}_{N-1},\ \mathbf{u}_{N-1}, \mathbf{x}_N\right\}$



Collocation-based integrators in Multiple-shooting

Collocation-based integrator solves:

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on each time interval $[t_k,\ t_{k+1}]$, provides:

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right)$$

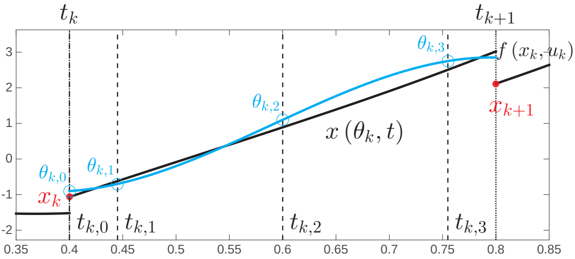
with sensitivities.

NLP with multiple-shooting

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \mathbf{x}_0 - \bar{\mathbf{x}}_0 \\ \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) - \mathbf{x}_N \end{bmatrix}$$

where $\mathbf{w} = \left\{\mathbf{x}_0,\ \mathbf{u}_0, \dots, \mathbf{x}_{N-1},\ \mathbf{u}_{N-1}, \mathbf{x}_N\right\}$



NLP solves:

$$\nabla_{\mathbf{w}} \mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}\right) = 0$$
$$\mathbf{g}\left(\mathbf{w}\right) = 0$$

Collocation-based integrators in Multiple-shooting

Collocation-based integrator solves:

$$\mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right) = 0$$

on each time interval $[t_k,\ t_{k+1}]$, provides:

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right)$$

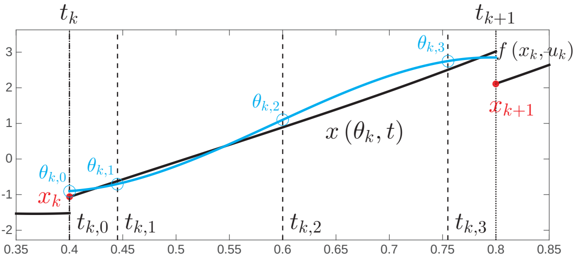
with sensitivities.

NLP with multiple-shooting

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \mathbf{x}_0 - \bar{\mathbf{x}}_0 \\ \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) - \mathbf{x}_N \end{bmatrix}$$

where $\mathbf{w} = \left\{\mathbf{x}_0,\ \mathbf{u}_0, \dots,\ \mathbf{x}_{N-1},\ \mathbf{u}_{N-1}, \mathbf{x}_N\right\}$



NLP solves:

$$\nabla_{\mathbf{w}} \mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}\right) = 0$$

$$\mathbf{g}\left(\mathbf{w}\right) = 0$$

Collocation-based integrator inside the NLP becomes a two-level Newton scheme !!

Collocation-based integrators in Multiple-shooting (cont')

NLP solver

$$\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w},\boldsymbol{\lambda}) = 0$$

$$\mathbf{g}(\mathbf{w}) = 0$$

$$\mathbf{w} = \{\mathbf{x}_0,\, \mathbf{u}_0, ..., \mathbf{x}_{N-1},\, \mathbf{u}_{N-1}, \mathbf{x}_N\}$$

Collocation-based integrators in Multiple-shooting (cont')

NLP solver

$$\nabla_{\mathbf{w}}\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda}\right)=0$$

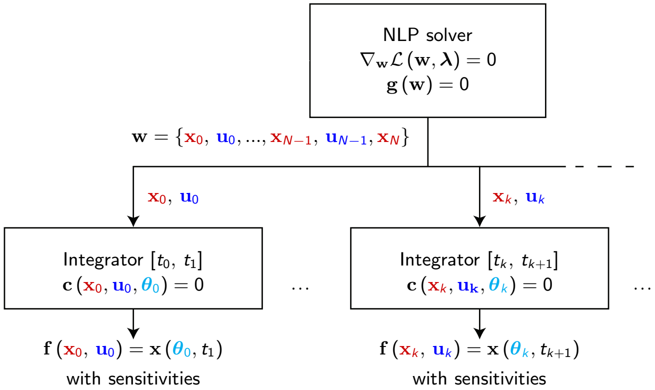
$$\mathbf{g}\left(\mathbf{w}\right)=0$$

$$\mathbf{w}=\{\mathbf{x}_0,\ \mathbf{u}_0,...,\mathbf{x}_{N-1},\ \mathbf{u}_{N-1},\mathbf{x}_N\}$$

NLP level

- Constraints $\mathbf{g}=0$
- Newton iterations (SQP/IP)

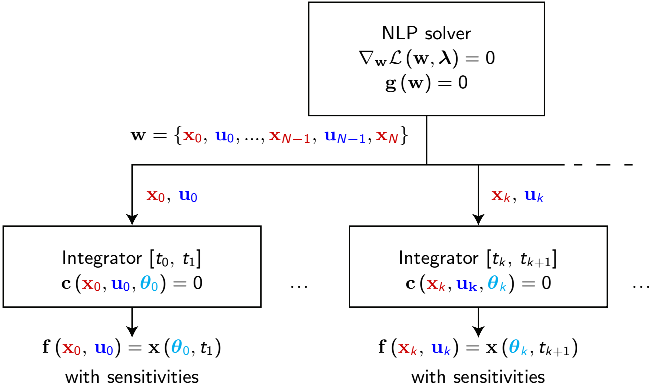
Collocation-based integrators in Multiple-shooting (cont')



NLP level

- Constraints $\mathbf{g} = 0$
- Newton iterations (SQP/IP)

Collocation-based integrators in Multiple-shooting (cont')



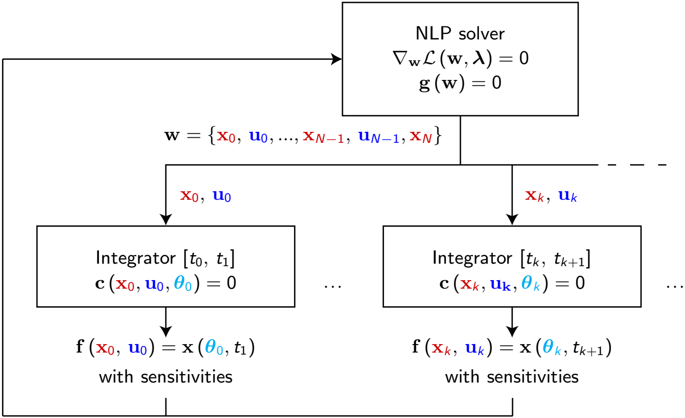
NLP level

- Constraints $\mathbf{g} = 0$
- Newton iterations (SQP/IP)

Integrator level

- Constraints $\mathbf{c} = 0$
- Newton iterations

Collocation-based integrators in Multiple-shooting (cont')



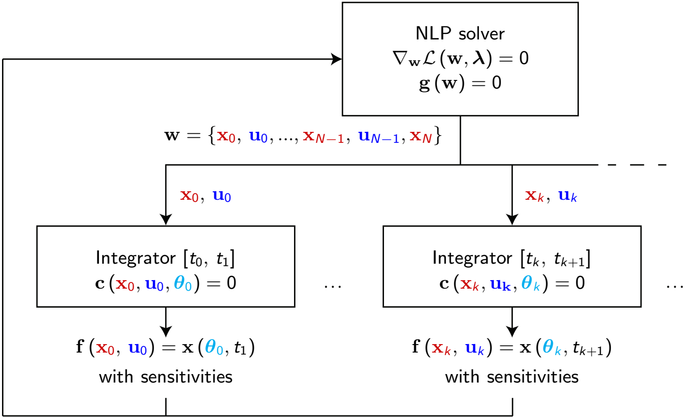
NLP level

- Constraints $\mathbf{g} = 0$
- Newton iterations (SQP/IP)

Integrator level

- Constraints $\mathbf{c} = 0$
- Newton iterations

Collocation-based integrators in Multiple-shooting (cont')



NLP level

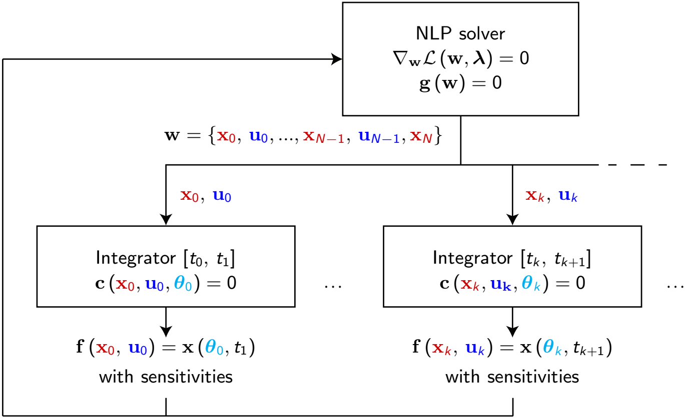
- Constraints $\mathbf{g} = 0$
- Newton iterations (SQP/IP)

Integrator level

- Constraints $\mathbf{c} = 0$
- Newton iterations

Constraints are solved at the NLP <u>and</u> at the integrator level separately !!

Collocation-based integrators in Multiple-shooting (cont')



NLP level

- Constraints $\mathbf{g} = 0$
- Newton iterations (SQP/IP)

Integrator level

- Constraints $\mathbf{c} = 0$
- Newton iterations

Constraints are solved at the NLP <u>and</u> at the integrator level separately !!

... what about handling them **altogether** in the NLP ?!?

Outline

1. Polynomial interpolation
2. Collocation-based integration
3. Collocation in multiple-shooting
4. **Direct Collocation**
5. NLP from direct collocation

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

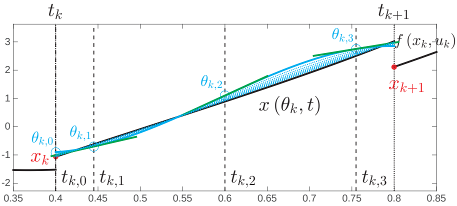
$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}_k\right)$$

is approximated using:

$$\mathbf{x}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{x}\left(\boldsymbol{\theta}_{k,i}, t_{k,i}\right) = \boldsymbol{\theta}_{k,i}$
- $K + 1$ degrees of freedom per state.



Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}_k\right)$$

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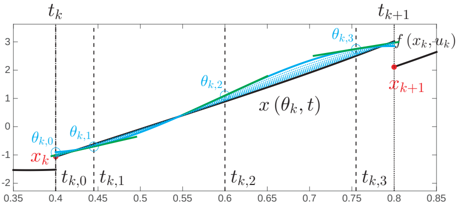

Integration constraints $(i = 1, ..., K)$

$$\frac{\partial}{\partial t}\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \mathbf{F}\left(\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right), \mathbf{u}_k\right)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}_k\right)$$

is approximated using:

$$\mathbf{x}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

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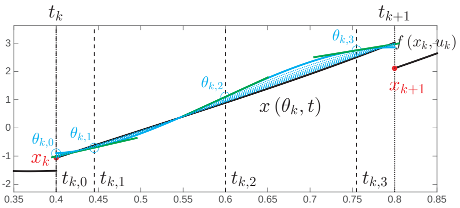


NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \left[\quad \right]$$

Integration constraints $(i = 1, ..., K)$

$$\frac{\partial}{\partial t}\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \mathbf{F}\left(\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right), \mathbf{u}_k\right)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

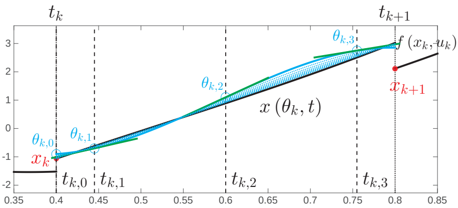
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}_k)$$

is approximated using:

$$\mathbf{x}(\boldsymbol{\theta}_k, t) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{x}(\boldsymbol{\theta}_{k,i}, t_{k,i}) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \\ \\ \\ \end{bmatrix}$$

Initial conditions $\bar{\mathbf{x}}_0$

Integration constraints $(i = 1, \ldots, K)$

$$\frac{\partial}{\partial t}\mathbf{x}(\boldsymbol{\theta}_k, t_{k,i}) = \mathbf{F}(\mathbf{x}(\boldsymbol{\theta}_k, t_{k,i}), \mathbf{u}_k)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

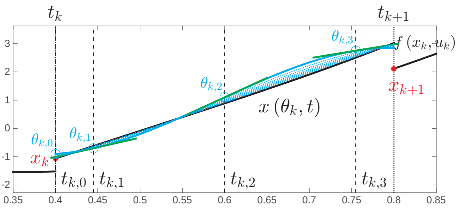
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}_k)$$

is approximated using:

$$\mathbf{x}(\boldsymbol{\theta}_k, t) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{x}(\boldsymbol{\theta}_{k,i}, t_{k,i}) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{x}(\boldsymbol{\theta}_0, t_1) - \boldsymbol{\theta}_{1,0} \\ \\ \\ \\ \end{bmatrix}$$

Continuity constraints ($\equiv$ shooting gaps)

Integration constraints ($i = 1, ..., K$)

$$\frac{\partial}{\partial t}\mathbf{x}(\boldsymbol{\theta}_k, t_{k,i}) = \mathbf{F}(\mathbf{x}(\boldsymbol{\theta}_k, t_{k,i}), \mathbf{u}_k)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}_k)$$

is approximated using:

$$\mathbf{x}(\boldsymbol{\theta}_k, t) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{x}(\boldsymbol{\theta}_{k,i}, t_{k,i}) = \boldsymbol{\theta}_{k,i}$
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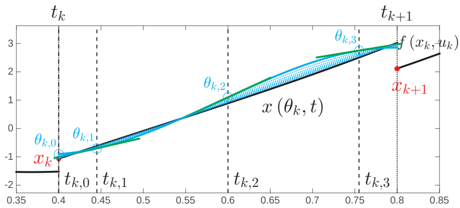


NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{x}(\boldsymbol{\theta}_0, t_1) - \boldsymbol{\theta}_{1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{0,i}, \mathbf{u}_0) - \sum_{j=0}^{K} \boldsymbol{\theta}_{0,j} \dot{P}_{0,j}(t_{0,i}) \\ \\ \\ \\ \end{bmatrix}$$

Integration constraints for $k = 0$

Integration constraints $(i = 1, ..., K)$

$$\frac{\partial}{\partial t}\mathbf{x}(\boldsymbol{\theta}_k, t_{k,i}) = \mathbf{F}(\mathbf{x}(\boldsymbol{\theta}_k, t_{k,i}), \mathbf{u}_k)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

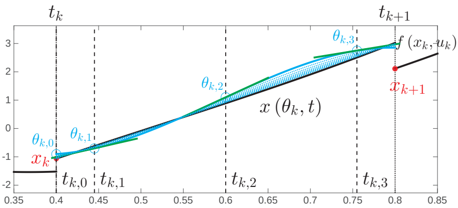
$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}_k\right)$$

is approximated using:

$$\mathbf{x}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{x}\left(\boldsymbol{\theta}_{k,i}, t_{k,i}\right) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{x}\left(\boldsymbol{\theta}_0, t_1\right) - \boldsymbol{\theta}_{1,0} \\ \mathbf{F}\left(\boldsymbol{\theta}_{0,i}, \mathbf{u}_0\right) - \sum_{j=0}^{K} \boldsymbol{\theta}_{0,j}\dot{P}_{0,j}(t_{0,i}) \\ \cdots \\ \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right) - \boldsymbol{\theta}_{k+1,0} \\ \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j}\dot{P}_{k,j}(t_{k,i}) \\ \cdots \end{bmatrix}$$

Remaining integration constraints $k = 1, \ldots, N-1$

Integration constraints $(i = 1, \ldots, K)$

$$\frac{\partial}{\partial t}\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \mathbf{F}\left(\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right), \mathbf{u}_k\right)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j}\dot{P}_{k,j}(t_{k,i}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

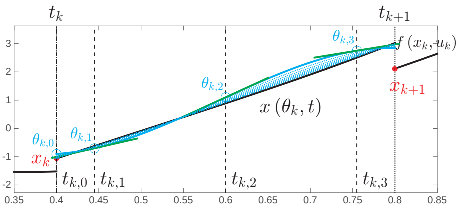
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}_k)$$

is approximated using:

$$\mathbf{x}(\boldsymbol{\theta}_k, t) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{x}(\boldsymbol{\theta}_{k,i}, t_{k,i}) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{x}(\boldsymbol{\theta}_0, t_1) - \boldsymbol{\theta}_{1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{0,i}, \mathbf{u}_0) - \sum_{j=0}^{K} \boldsymbol{\theta}_{0,j} \dot{P}_{0,j}(t_{0,i}) \\ \ldots \\ \mathbf{x}(\boldsymbol{\theta}_k, t_{k+1}) - \boldsymbol{\theta}_{k+1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) \\ \ldots \end{bmatrix}$$

Decision variables:

$$\mathbf{w} = \left\{\boldsymbol{\theta}_{0,0}, \ldots, \boldsymbol{\theta}_{0,K}, \mathbf{u}_0, \ldots, \boldsymbol{\theta}_{N-1,0}, \ldots, \boldsymbol{\theta}_{N-1,K}, \mathbf{u}_{N-1}\right\}$$

Integration constraints $(i = 1, \ldots, K)$

$$\frac{\partial}{\partial t}\mathbf{x}(\boldsymbol{\theta}_k, t_{k,i}) = \mathbf{F}(\mathbf{x}(\boldsymbol{\theta}_k, t_{k,i}), \mathbf{u}_k)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)$$

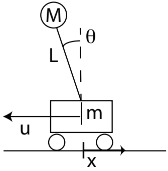
Direct Collocation - Example: swing-up of a pendulum

OCP

$$\min_{u_0,\ldots,u_{N-1}} \quad \sum_{k=0}^{N-1} u_k^2$$

$$\text{s.t.} \quad \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, u_k\right), \quad \forall t \in [t_k, t_{k+1}]$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 & \pi & 0 & 0 \end{bmatrix}, \quad \mathbf{x}\left(t_\mathrm{f}\right) = 0$$



$$\mathbf{x} = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^\top$$

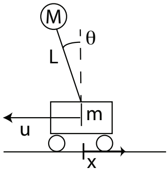
Direct Collocation - Example: swing-up of a pendulum

OCP

$$\min_{u_0,\ldots,u_{N-1}} \quad \sum_{k=0}^{N-1} u_k^2$$

$$\text{s.t.} \quad \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, u_k\right), \quad \forall t \in [t_k, t_{k+1}]$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 & \pi & 0 & 0 \end{bmatrix}, \quad \mathbf{x}\left(t_\mathrm{f}\right) = 0$$



$$\mathbf{x} = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^\top$$

$N = 20$
$K = 4$ with Legendre, order 8 !!
420 variables
404 constraints

Reminder:

$$\mathbf{x}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \boldsymbol{\theta}_{k,i} \cdot P_{k,i}(t)$$

$$\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \boldsymbol{\theta}_{k,i}$$

NLP with direct collocation

$$\min_{\mathbf{w}} \quad \sum_{k=0}^{N-1} u_k^2$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{x}\left(\boldsymbol{\theta}_0, t_1\right) - \boldsymbol{\theta}_{1,0} \\ \mathbf{F}\left(\boldsymbol{\theta}_{0,i}, \mathbf{u}_0\right) - \sum_{j=0}^{K} \boldsymbol{\theta}_{0,j} \dot{P}_{0,j}(t_{0,i}) \\ \ldots \\ \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right) - \boldsymbol{\theta}_{k+1,0} \\ \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) \\ \ldots \\ \mathbf{x}\left(\boldsymbol{\theta}_{N-1}, t_N\right) \end{bmatrix} = 0$$

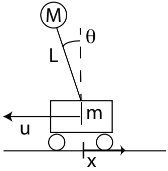
Direct Collocation - Example: swing-up of a pendulum

OCP

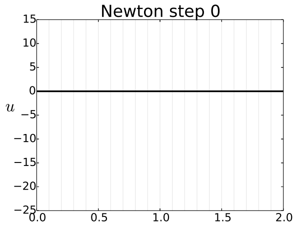
$$\min_{u_0,\ldots,u_{N-1}} \quad \sum_{k=0}^{N-1} u_k^2$$

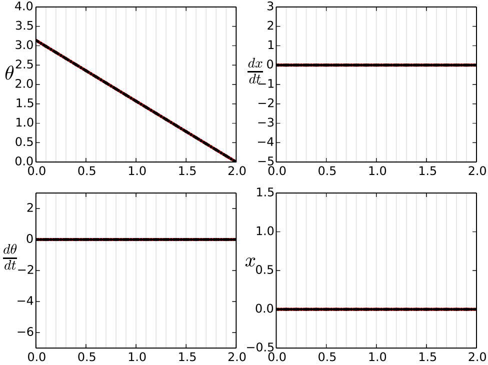
$$\text{s.t.} \quad \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, u_k\right), \quad \forall t \in [t_k, t_{k+1}]$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 & \pi & 0 & 0 \end{bmatrix}, \quad \mathbf{x}\left(t_\mathrm{f}\right) = 0$$



$$\mathbf{x} = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^\top$$





- $K + 1 = 5$
- all nodes are initialised

Direct Collocation - Example: swing-up of a pendulum

OCP

$$\min_{u_0,\ldots,u_{N-1}} \quad \sum_{k=0}^{N-1} u_k^2$$

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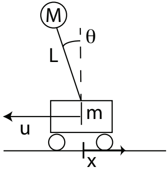

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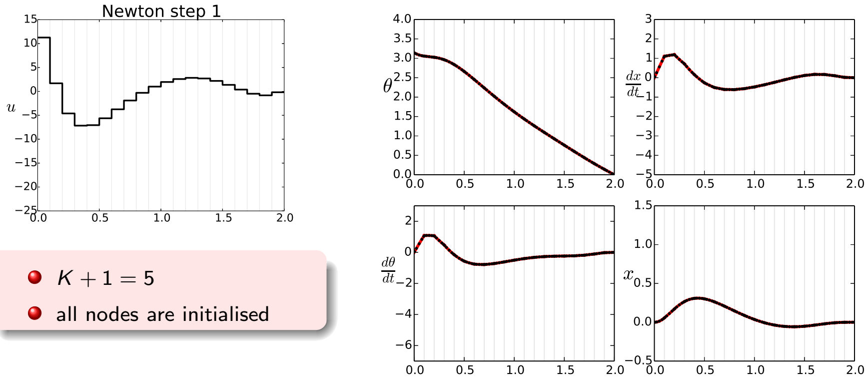

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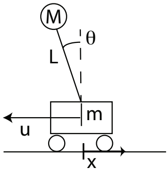

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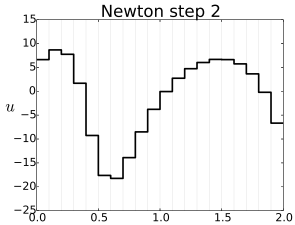

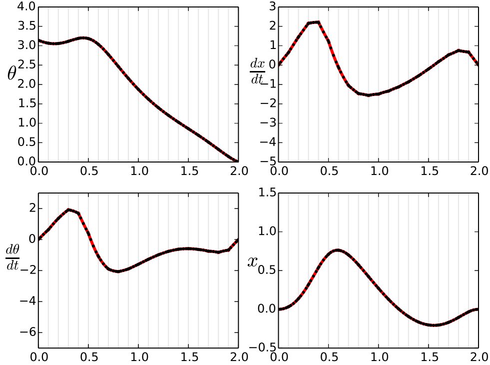

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Direct Collocation - Example: swing-up of a pendulum

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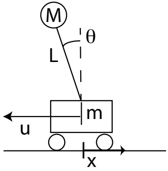

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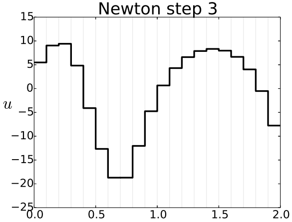

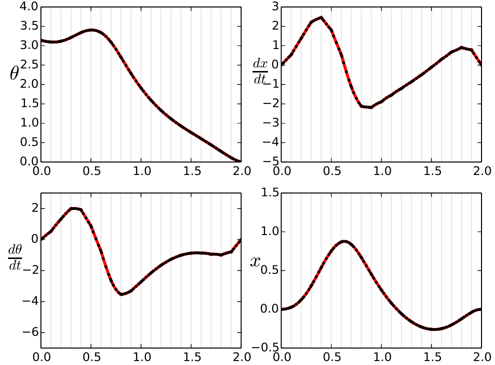

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Direct Collocation - Example: swing-up of a pendulum

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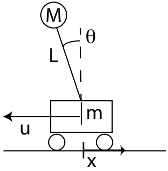

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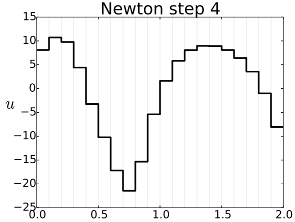

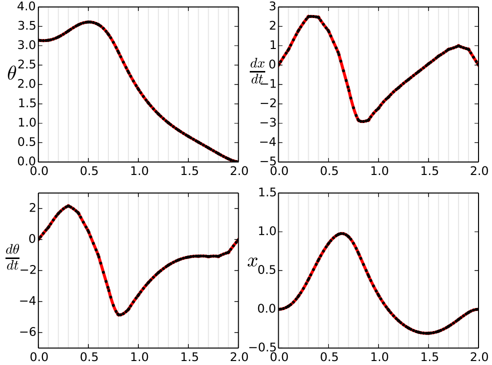

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Direct Collocation - Example: swing-up of a pendulum

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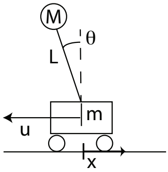

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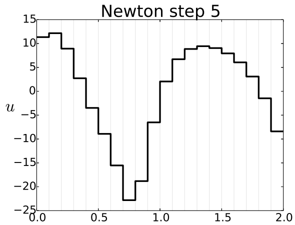

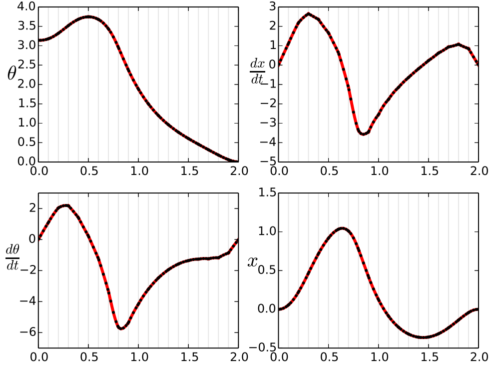

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Direct Collocation - Example: swing-up of a pendulum

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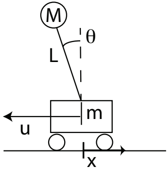

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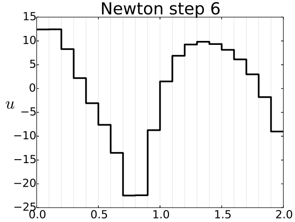

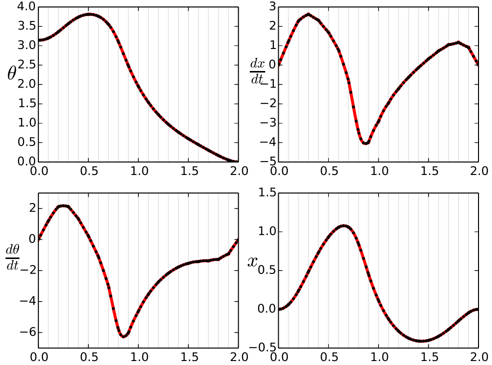

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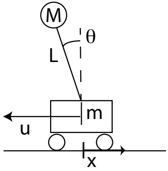

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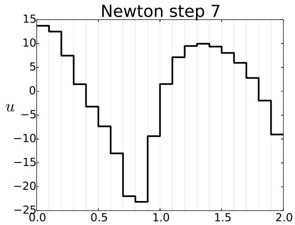

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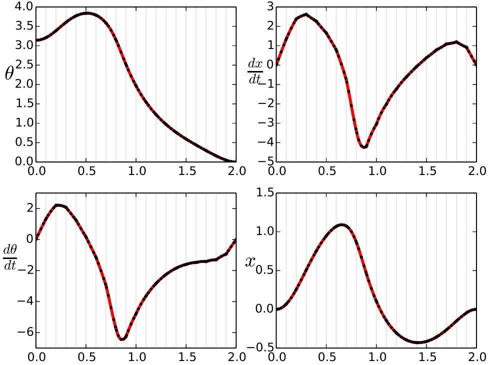

Direct Collocation - Example: swing-up of a pendulum

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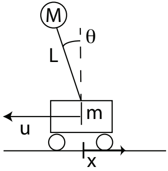

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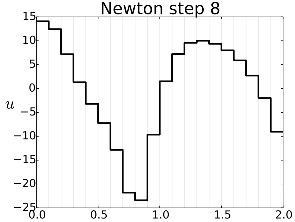

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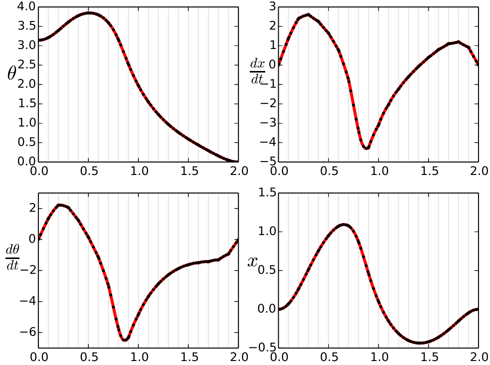


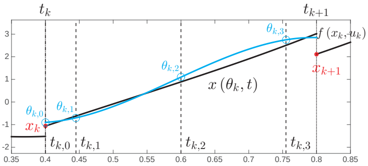
Cost and constraints discretisation in Direct Collocation

OCP:

$$\min \quad T\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right) + \int_0^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right)$$

$$\mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0$$



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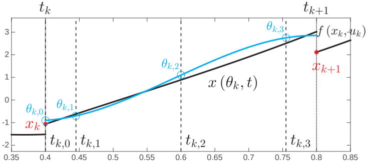


- Inequality constraints: $\mathbf{h}\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) \leq 0$ can be enforced on all collocation nodes:

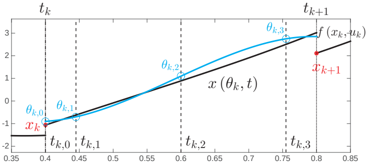
$$\mathbf{h}\left(\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right),\mathbf{u}_k\right) \leq 0, \quad \forall\, k = 0,\ldots,N-1, \quad i = 0,\ldots,K$$

but often only on the "shooting" nodes $t_{0,0}$, $t_{1,0}$, ..., $t_{N,0}$

Cost and constraints discretisation in Direct Collocation

OCP:

$$\begin{aligned} \min \quad & T\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right) + \int_0^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) \\ & \mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0 \end{aligned}$$



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  but often only on the "shooting" nodes $t_{0,0},\ t_{1,0},\ ...,\ t_{N,0}$

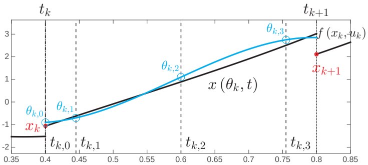
- Cost function often approximated as (rectangular quadrature):

$$T\left(\mathbf{x}\left(\boldsymbol{\theta}_{N-1}, t_{N-1,K}\right)\right) + \sum_{k=0}^{N-1} \left(t_{k+1} - t_k\right) L\left(\boldsymbol{\theta}_{k,0}, \mathbf{u}_k\right)$$

Cost and constraints discretisation in Direct Collocation

OCP:

$$\begin{aligned} \min \quad & T\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) + \int_0^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) \\ & \mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0 \end{aligned}$$



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  but often only on the "shooting" nodes $t_{0,0}, \; t_{1,0}, \; ..., \; t_{N,0}$

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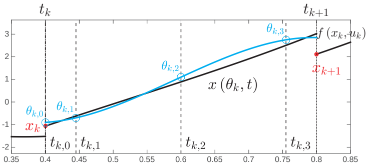
Careful: if you want to use $\boldsymbol{\theta}_{k,i}$ for $i = 1, ..., K$, the time grid is not uniform !!

Cost and constraints discretisation in Direct Collocation

OCP:

$$\begin{aligned} \min \quad & T\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) + \int_0^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) \\ & \mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0 \end{aligned}$$



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  but often only on the "shooting" nodes $t_{0,0}$, $t_{1,0}$, ..., $t_{N,0}$

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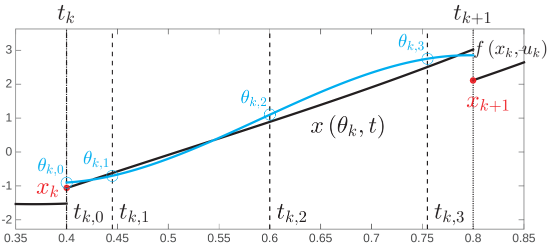
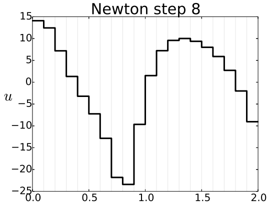
  Careful: if you want to use $\boldsymbol{\theta}_{k,i}$ for $i = 1, ..., K$, the time grid is not uniform !!

- Quadratic term in cost function $L\left(\mathbf{x}, \mathbf{u}\right) = \frac{1}{2}\mathbf{x}^\top Q \mathbf{x} + ...$ can be implemented using:

$$\int_{t_k}^{t_{k+1}} \frac{1}{2}\mathbf{x}\left(t\right)^\top Q \mathbf{x}\left(t\right)\,\mathrm{d}t = \frac{1}{2}\sum_{l=0}^{K}\sum_{j=0}^{K} \boldsymbol{\theta}_{k,l}{}^\top Q \boldsymbol{\theta}_{k,j} \underbrace{\int_{t_k}^{t_{k+1}} P_{k,l}(t) P_{k,j}(t) dt}_{=\alpha_j \delta_{l,j}\ (P\text{:s are orthogonal})} = \frac{1}{2}\sum_{j=0}^{K} \alpha_j \boldsymbol{\theta}_{k,j}{}^\top Q \boldsymbol{\theta}_{k,j}$$

Some remarks



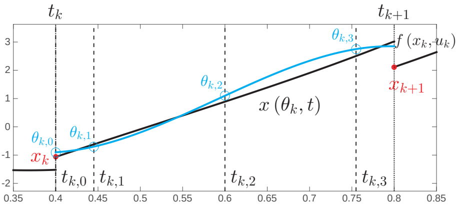


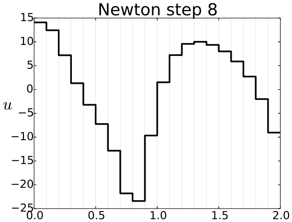
- Direct collocation is a "fully simultanuous" approach, as the integration and the optimization are performed **together** in the NLP solver.
- The decision variables are:

$$\mathbf{w} = \{\boldsymbol{\theta}_{0,0}, ..., \boldsymbol{\theta}_{0,K}, \mathbf{u}_0, ..., \boldsymbol{\theta}_{N-1,0}, ..., \boldsymbol{\theta}_{N-1,K}, \mathbf{u}_{N-1}\}$$

Observe that $\boldsymbol{\theta}_{k,i}$, i.e. the state at the collocation point $t_{k,i}$ of the interval $[t_k,\ t_{k+1}]$ is in $\mathbb{R}^n$ (size of the state). Manipulating these variables properly in a computer code can be tricky.

Refining the input discretization





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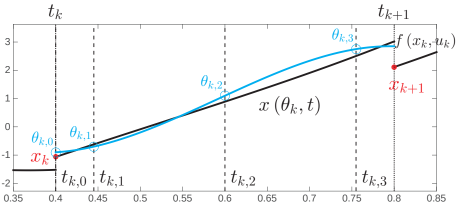

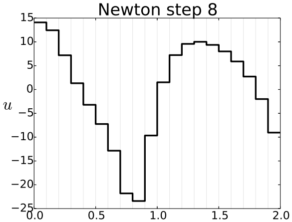

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- However one can pick a different input $\mathbf{u}_{k,i}$ for each collocation time $t_{k,i}$. Gives K input vector per collocation interval, i.e. $\mathbf{u}_{k,1}, ..., \mathbf{u}_{k,K}$

Collocation constraints:

$$\mathbf{x}\left(\boldsymbol{\theta}_k, t_k\right) = \mathbf{x}_k$$

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for $i = 1, ..., K$

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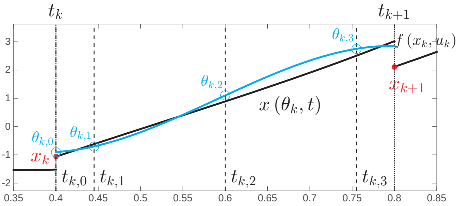

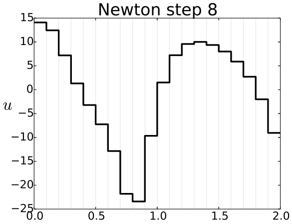

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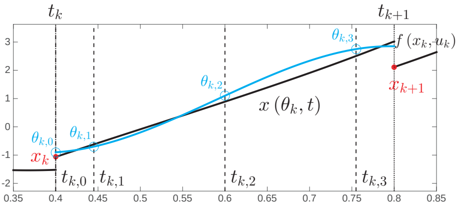

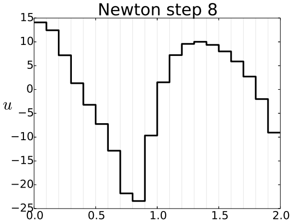

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- Drawbacks: **1.** the input profile can present important "oscillations", **2.** the linear algebra can loose some conditioning

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Outline

Hessian in Direct Collocation

Lagrange function:

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}) = \Phi(\mathbf{w}) + \boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{w}) + \boldsymbol{\mu}^\top \mathbf{h}(\mathbf{w})$$

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Reminder: dynamics yield

$$\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{x}(\boldsymbol{\theta}_0, t_1) - \boldsymbol{\theta}_{1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{0,i}, \mathbf{u}_0) - \sum_{j=0}^{K} \boldsymbol{\theta}_{0,j} \dot{P}_{0,j}(t_{0,i}) \\ \dots \\ \mathbf{x}(\boldsymbol{\theta}_k, t_{k+1}) - \boldsymbol{\theta}_{k+1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) \\ \dots \end{bmatrix}$$

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Contribution of the dynamics:

$$\nabla^2_{\mathbf{w}}\left(\boldsymbol{\lambda}^\top \mathbf{g}\right) = \nabla^2_{\mathbf{w}}\left[\sum_{k=0,\ldots,N-1}\ \sum_{i=1,\ldots,K} \boldsymbol{\lambda}_{k,i}{}^\top \left(\mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j}\dot{P}_{k,j}(t_{k,i})\right)\right]$$

Hessian in Direct Collocation

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$$= \sum_{k=0,\dots,N-1}\sum_{i=1,..,K}\nabla^2_{\mathbf{w}}\left(\boldsymbol{\lambda}_{k,i}^\top\mathbf{F}(\boldsymbol{\theta}_{k,i},\mathbf{u}_k)\right)$$

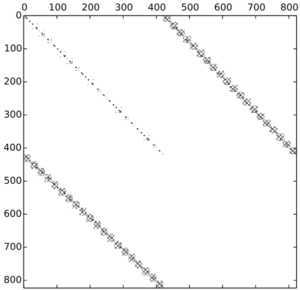
With $\mathbf{w} = \{\boldsymbol{\theta}_{0,0}, ..., \boldsymbol{\theta}_{0,K}, \mathbf{u}_0, ..., \boldsymbol{\theta}_{N-1,0}, ..., \boldsymbol{\theta}_{N-1,K}, \mathbf{u}_{N-1}\}$, the contributions

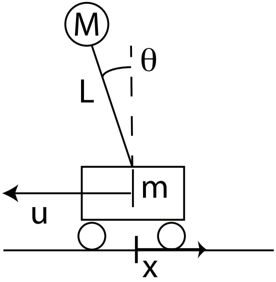
$$\nabla^2_{\mathbf{w}}\left(\boldsymbol{\lambda}_{k,i}^\top\mathbf{F}(\boldsymbol{\theta}_{k,i},\mathbf{u}_k)\right)$$

are sparse and trivial to compute !! (e.g. CasADi)

Sparsity pattern

E.g. for the crane, the KKT matrix M is:





$$M = \begin{bmatrix} H & \nabla \mathbf{g} \\ \nabla \mathbf{g}^\top & 0 \end{bmatrix}$$