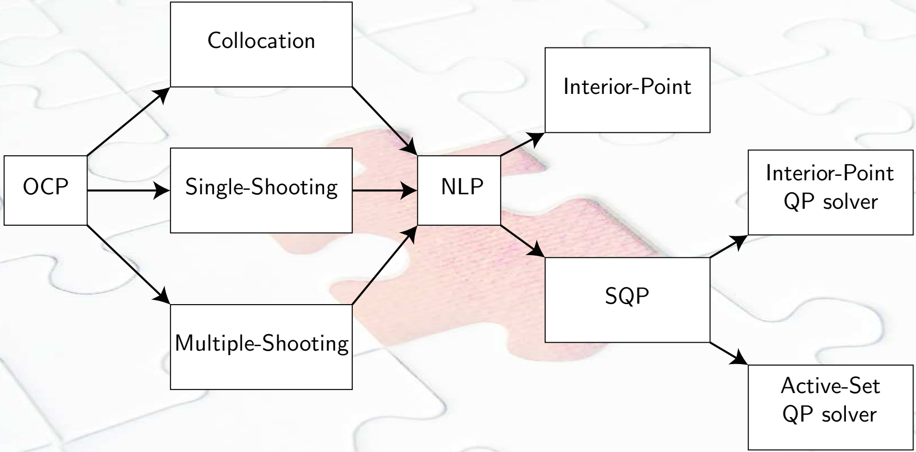
Numerical Optimal Control with DAEs
Lecture 7: Shooting Methods

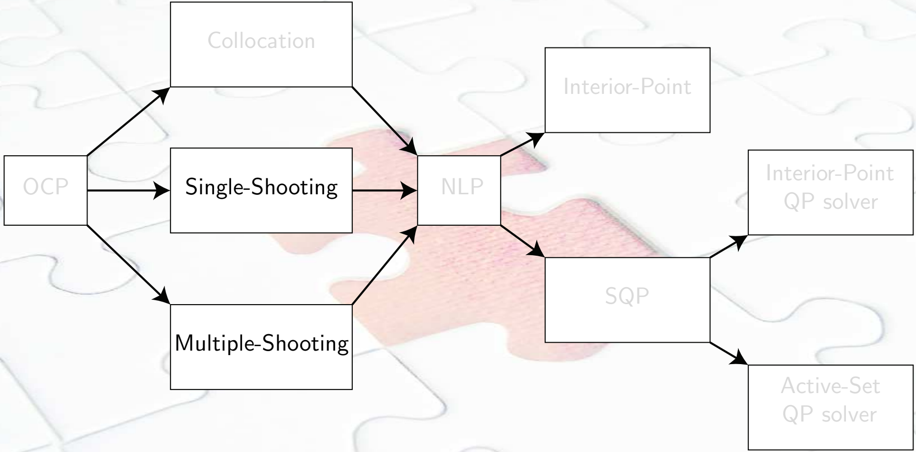
Sébastien Gros

AWESCO PhD course

Survival map of Direct Optimal Control



Survival map of Direct Optimal Control



One way of going from OCP to NLP

Outline

1. Single-Shooting
2. Multiple-Shooting
3. NLP from Multiple-Shooting

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2. Multiple-Shooting
3. NLP from Multiple-Shooting

Discretization - Single-Shooting

Problem:

$$\begin{aligned} \min \quad & \phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\ \text{s.t.} \quad & \dot{\mathbf{x}}\left(t\right) = \mathbf{F}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \\ & \mathbf{x}\left(t_0\right) = \mathbf{x}_0 \\ & \mathbf{h}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \leq 0 \end{aligned}$$

First discretize...

...then optimize:

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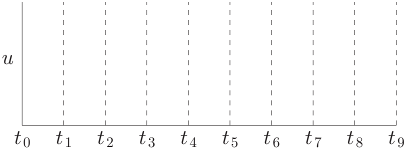

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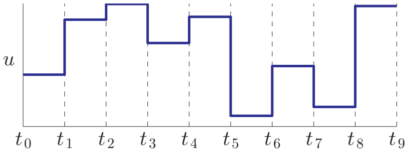

First discretize...

- Usually zero-order hold

$$\mathbf{u}\left(t \in [t_k, t_{k+1}]\right) = \mathbf{u}_k$$

over a time grid $t_0, ..., t_{N-1}$

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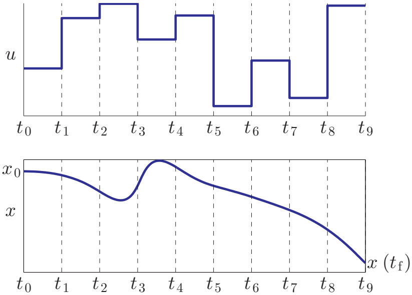

First discretize...

- Usually zero-order hold
$$\mathbf{u}\left(t \in [t_k, t_{k+1}]\right) = \mathbf{u}_k$$
over a time grid $t_0, ..., t_{N-1}$
- See $\mathbf{x}(.)$ as a *function* $\mathbf{f}$ of $\mathbf{w} = \{\mathbf{u}_0, ..., \mathbf{u}_{N-1}\}$, $\mathbf{x}_0$ and t:
$$\mathbf{f}\left(\mathbf{w}, \mathbf{x}_0, t\right) : \mathbf{w}, \mathbf{x}_0, t \mapsto \mathbf{x}(t)$$

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Discretization - Single-Shooting

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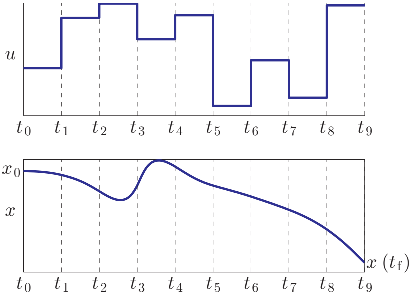

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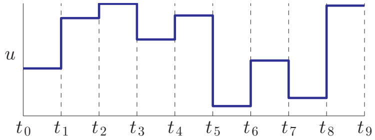

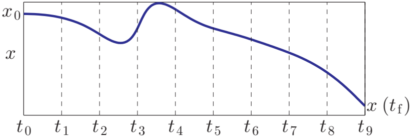

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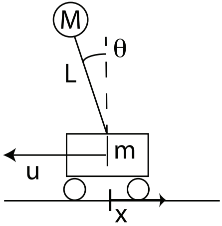
$$\begin{aligned} \min_{\mathbf{w}} \quad & \phi\left(\mathbf{f}\left(\mathbf{w}, \mathbf{x}_0, .\right), \mathbf{w}\right) \\ \text{s.t.} \quad & \mathbf{h}\left(\mathbf{f}\left(\mathbf{w}, \mathbf{x}_0, t_k\right), \mathbf{w}_k\right) \leq 0 \end{aligned}$$

Single Shooting - Example: swing-up of a pendulum

OCP

$$\begin{aligned}
\min_{u_0,\ldots,u_{N-1}} \quad & \sum_{k=0}^{N-1} u_k^2 \\
\text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, u_k\right), \quad \forall t \in [t_k, t_{k+1}] \\
& -20 \leq u_k \leq 20 \\
& \mathbf{x}(0) = \begin{bmatrix} 0 & \pi & 0 & 0 \end{bmatrix}, \quad \mathbf{x}\left(t_\mathrm{f}\right) = 0
\end{aligned}$$



Single Shooting - Example: swing-up of a pendulum

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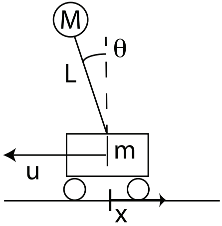

Integrator function provides $\mathbf{x}(t) = \mathbf{f}(\mathbf{u}_0, ..., \mathbf{u}_{N-1}, \mathbf{x}(0), t)$

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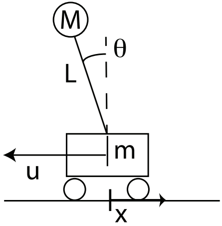

Integrator function provides $\mathbf{x}(t) = \mathbf{f}\left(\mathbf{u}_0, ..., \mathbf{u}_{N-1}, \mathbf{x}(0), t\right)$

An integrator is a function in the most rigorous sense of the term. E.g.

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}_k}\left(\mathbf{u}_0, ..., \mathbf{u}_{N-1}, \mathbf{x}(0), t\right)$$

is well defined and computable

Single Shooting - Example: swing-up of a pendulum

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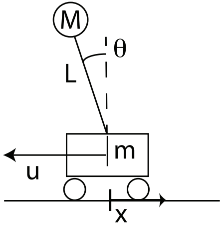
$$\begin{aligned} \min_{u_0,\dots,u_{N-1}} \quad & \sum_{k=0}^{N-1} u_k^2 \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, u_k), \quad \forall t \in [t_k, t_{k+1}] \\ & -20 \leq u_k \leq 20 \\ & \mathbf{x}(0) = \begin{bmatrix} 0 & \pi & 0 & 0 \end{bmatrix}, \quad \mathbf{x}(t_\mathrm{f}) = 0 \end{aligned}$$



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NLP from single shooting

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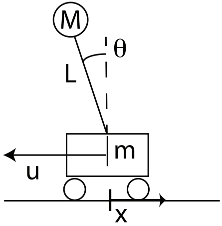

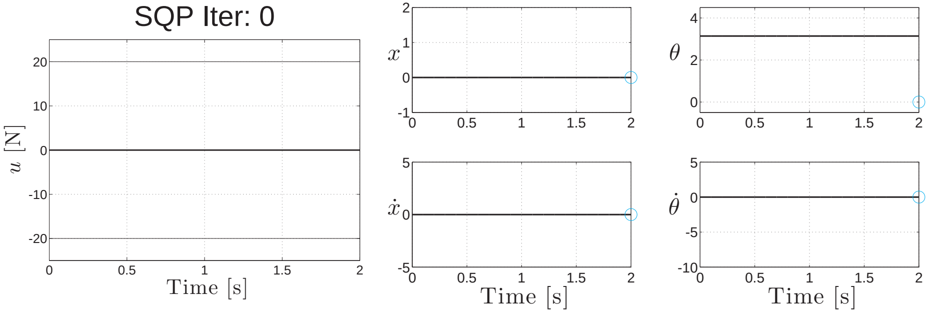

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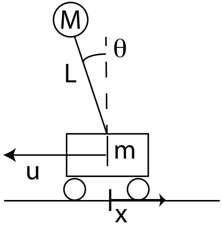

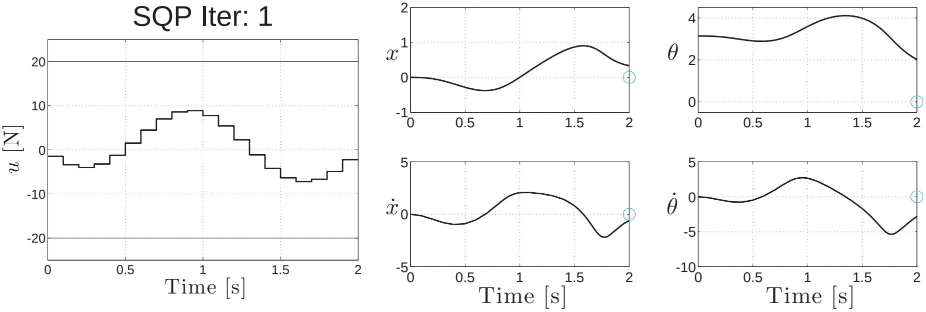

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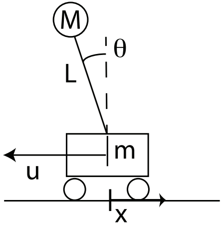

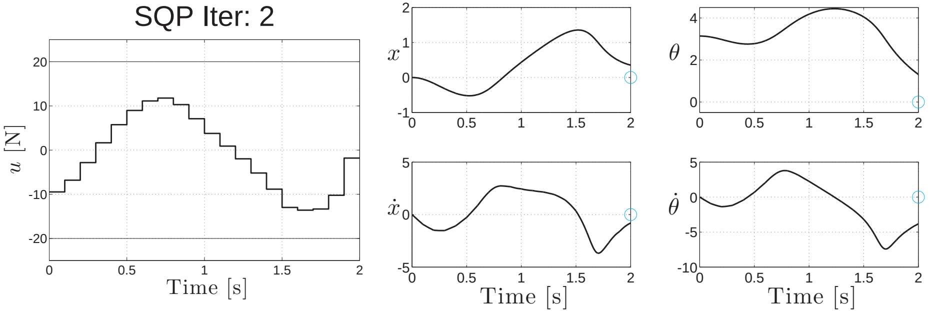

Single Shooting - Example: swing-up of a pendulum

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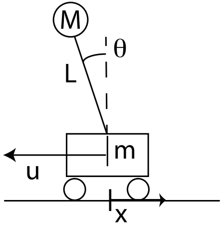

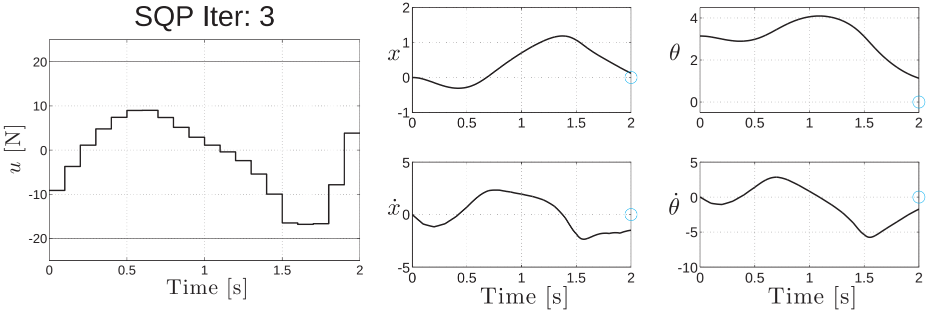

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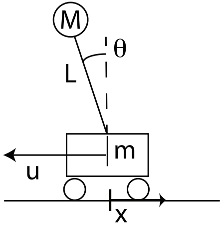

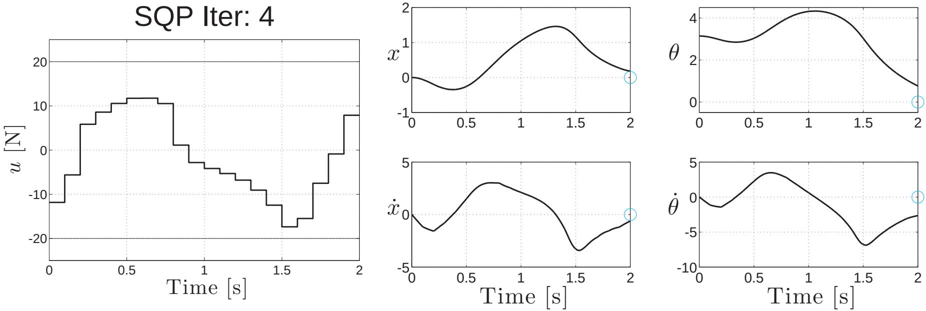

Single Shooting - Example: swing-up of a pendulum

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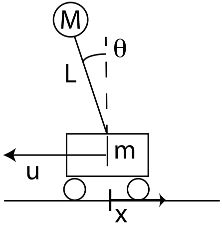

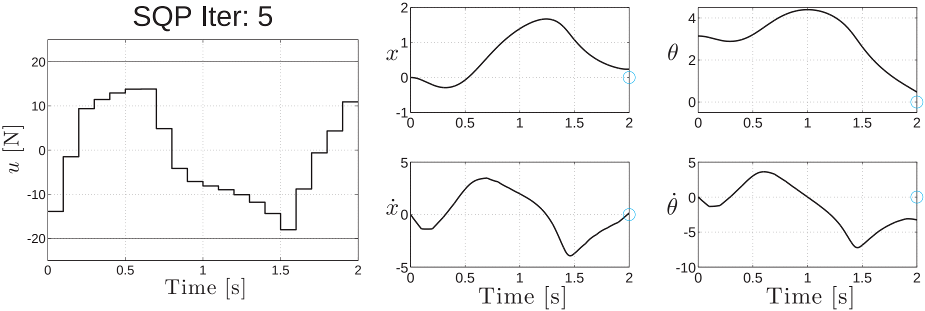

Single Shooting - Example: swing-up of a pendulum

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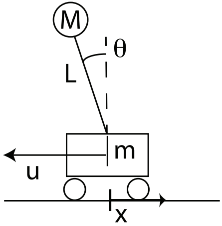

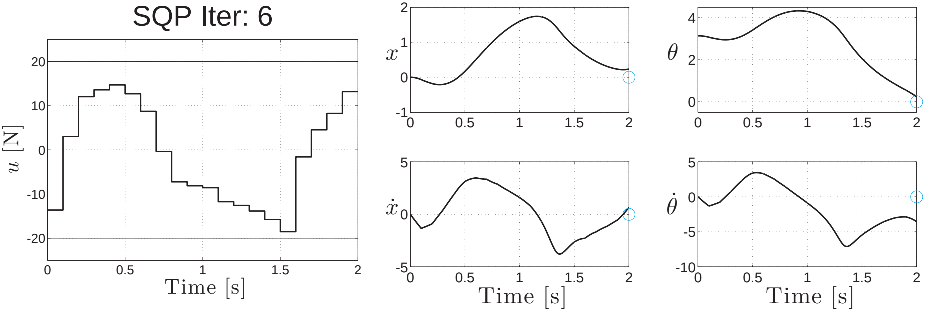

Single Shooting - Example: swing-up of a pendulum

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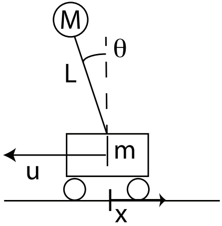

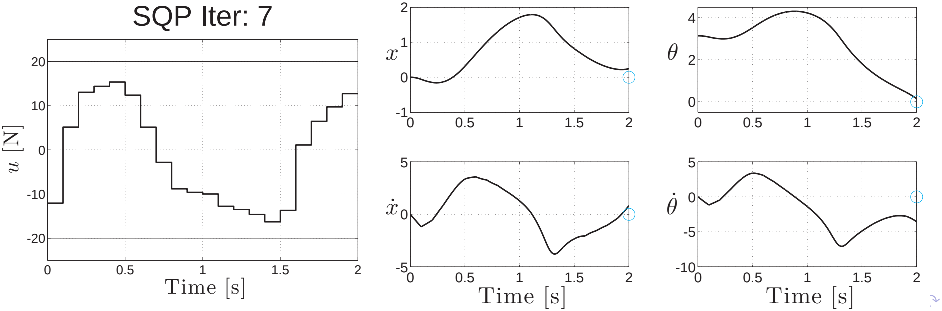

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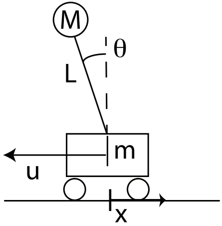

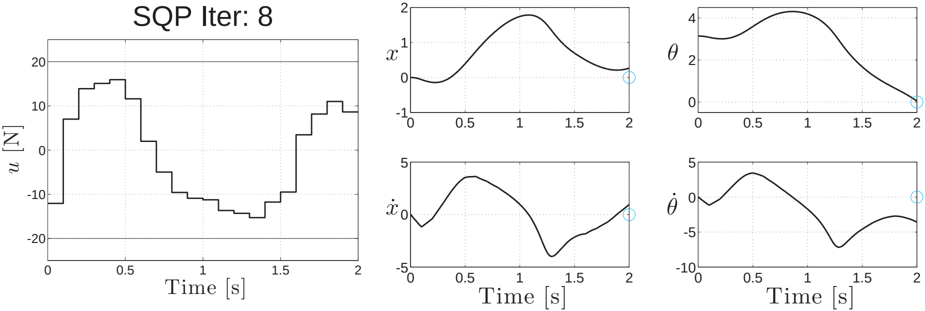

Single Shooting - Example: swing-up of a pendulum

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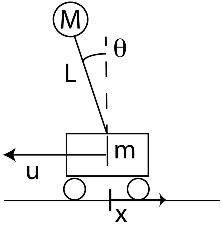

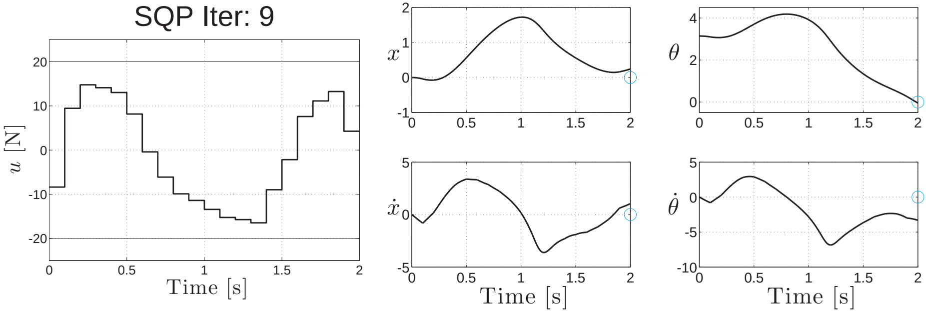

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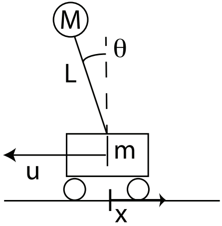
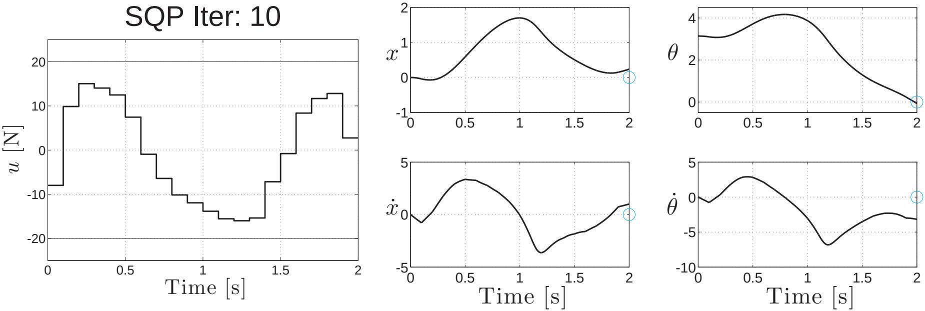
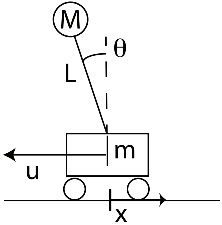


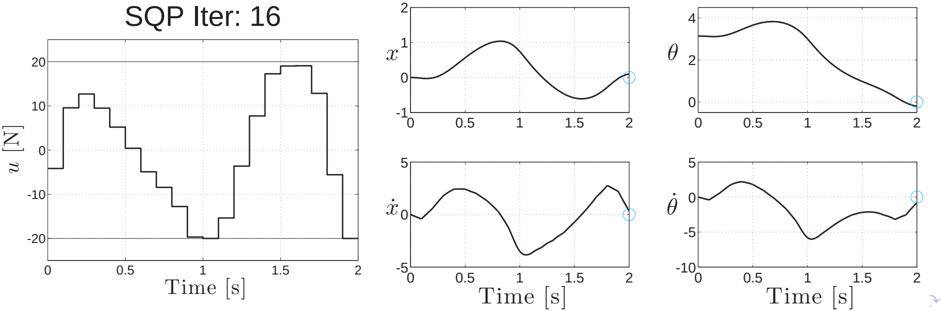


Single Shooting - Example: swing-up of a pendulum

NLP from single shooting

$$\begin{aligned} \min_{u_0,\dots,u_{N-1}} \quad & \sum_{k=0}^{N-1} u_k^2 \\ \text{s.t.} \quad & -20 \leq u_k \leq 20 \\ & \mathbf{f}\left(u_0, \dots, u_{N-1}, \mathbf{x}(0), t_{\mathrm{f}}\right) = 0 \end{aligned}$$





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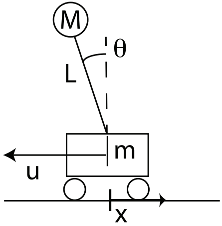

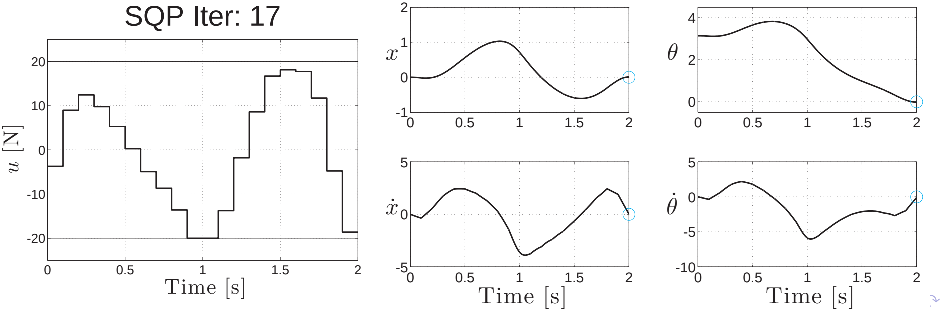

Single-Shooting - Propagating continuous dynamics

Nonlinearity propagation: integrator function $\mathbf{f}$

$$\mathbf{f}\left(\mathbf{u}_0, ..., \mathbf{u}_{N-1}, \mathbf{x}_0, t\right): \quad \mathbf{u}_0, ..., \mathbf{u}_{N-1}, \mathbf{x}_0, t \quad \longmapsto \quad \mathbf{x}(t)$$

tends to become highly nonlinear for large t.

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Example

$$\begin{aligned}\dot{x} &= 10\left(y - x\right)\\ \dot{y} &= x\left(u - z\right) - y\\ \dot{z} &= xy - 3z\end{aligned}$$

State: $\mathbf{x}(.) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

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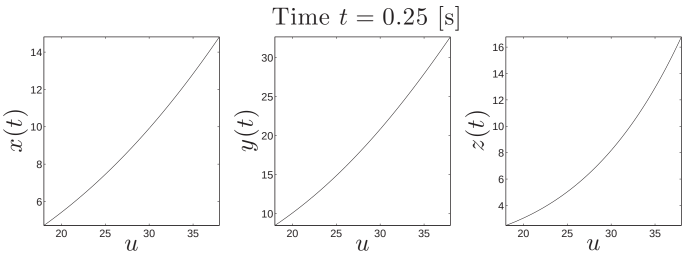
States as a function of u (constant) at time t, for a fixed $\mathbf{x}_0$



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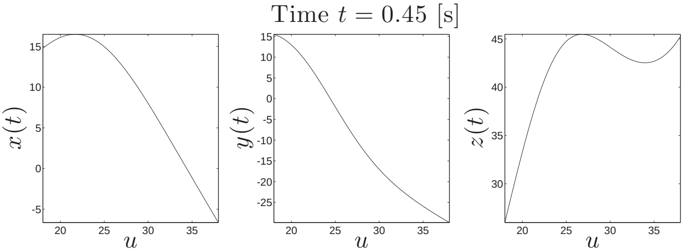

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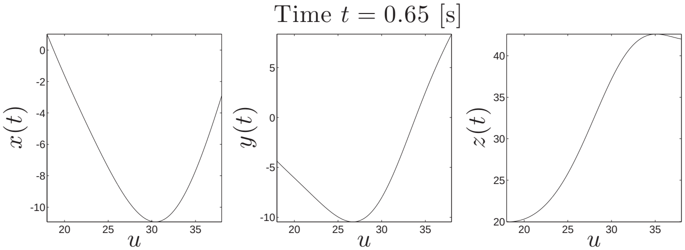

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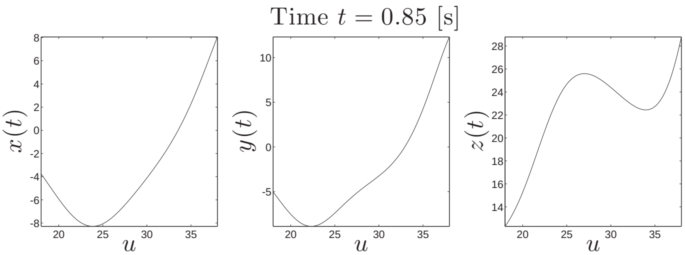

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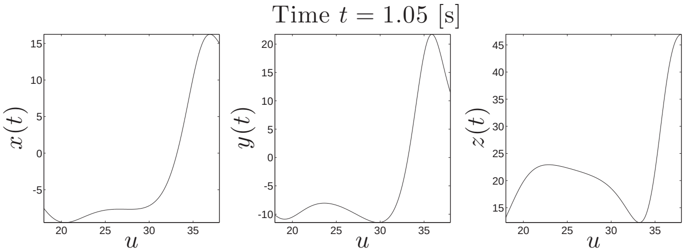

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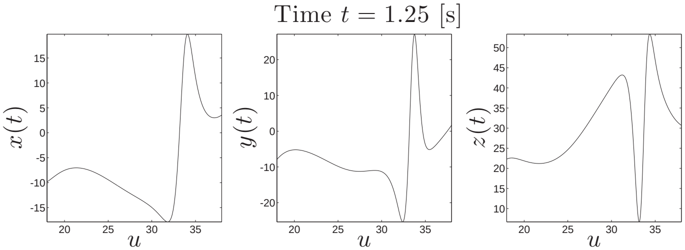

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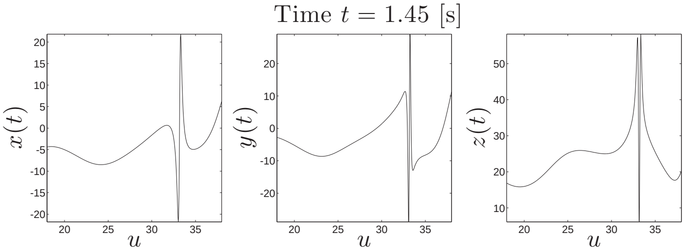

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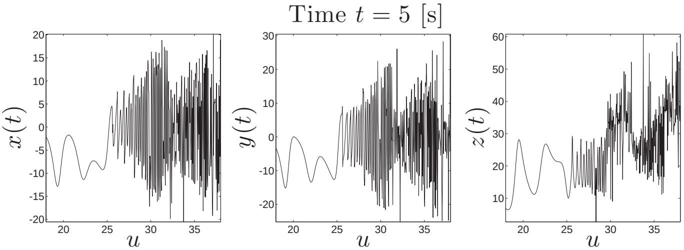

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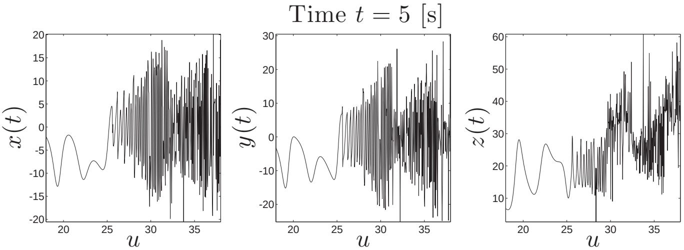

What's this crazy system btw ?!?

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Example

Lorentz attractor ($u = 28$) stable but chaotic

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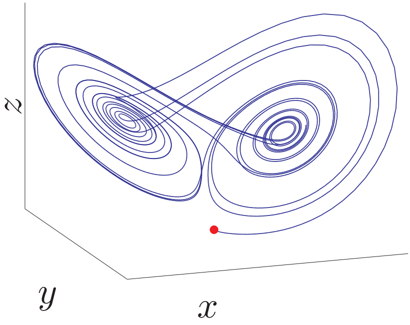

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In optimal control, **don't** simulate a nonlinear/unstable system over a *long* time horizon

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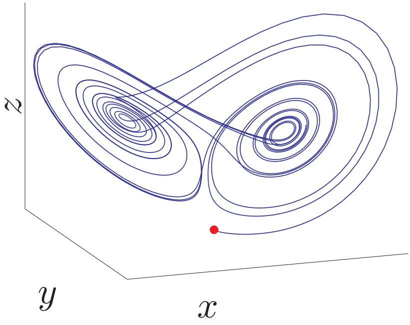

Nonlinearity of the integrator function - A bit of formalism...

Consider the dynamics $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ and the corresponding **integrator function** $\mathbf{f}(\mathbf{x}, t)$ that maps the initial conditions $\mathbf{x} \in \mathbb{R}^n$ onto a trajectory $\mathbf{x}(t) \in \mathbb{R}^n$, i.e.

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Remarks:

- Here we omit the input, but $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is still general. Indeed, one can always rewrite $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u})$ as:

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}, \qquad \dot{\mathbf{z}} = \begin{bmatrix} \mathbf{F}(\mathbf{z}) \\ 0 \end{bmatrix}$$

  then the contorl input is "hidden" in the initial conditions for $\mathbf{z}$.

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- How to measure the nonlinearity of $\mathbf{f}(\mathbf{x}, t)$ in $\mathbf{x}$? What about:

$$\left\| \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} - \frac{\partial \mathbf{f}(\mathbf{y}, t)}{\partial \mathbf{y}} \right\| \leq L \left\| \mathbf{x} - \mathbf{y} \right\|$$

  If $\mathbf{f}(\mathbf{x}, t)$ is affine in $\mathbf{x}$, i.e. $\mathbf{f}(\mathbf{x}, t) = A(t)\mathbf{x} + \mathbf{b}(t)$, then $L = 0$... If L large, then $\mathbf{f}(\mathbf{x}, t)$ is very nonlinear...

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Proposition

Assume:

- Lipschitz ODE: $\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \leq L_0 \|\mathbf{x} - \mathbf{y}\|$
- Lipschitz sensitivity of the dynamics: $\left\| \frac{\partial \mathbf{F}(\mathbf{x})}{\partial \mathbf{x}} - \frac{\partial \mathbf{F}(\mathbf{y})}{\partial \mathbf{y}} \right\| \leq L_1 \|\mathbf{x} - \mathbf{y}\|$
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Then the following holds:

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- **Bounded nonlinearity**: $\left\| \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} - \frac{\partial \mathbf{f}(\mathbf{y}, t)}{\partial \mathbf{y}} \right\| \leq \frac{L_1}{L_0} e^{\beta t} \left(e^{L_0 t} - 1 \right) \|\mathbf{x} - \mathbf{y}\|$

Proof

Two useful mathematical tricks:

- The inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{a}\right\| \leq \left\|\dot{\mathbf{a}}\right\|$$

holds on any vector space equipped with a differentiable norm $\|.\|$ (e.g. 2-norm, Frobenius).

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- Gronwall Lemma: if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{a}(t)\right\| \leq \alpha(t)\left\|\mathbf{a}(t)\right\| + \beta(t)$$

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$$\|\mathbf{a}(t)\| \leq \|\mathbf{a}(0)\| e^{\int_0^t \alpha(s)\mathrm{d}s} + \int_0^t e^{\int_s^t \alpha(s)}\beta(s)\mathrm{d}s$$

  holds for all t.

Proof

Bounded divergence of solutions: let $\mathbf{e}(\mathbf{x}, \mathbf{y}, t) = \mathbf{f}(\mathbf{x}, t) - \mathbf{f}(\mathbf{y}, t)$, then:

$$\|\dot{\mathbf{e}}\| = \|\mathbf{F}(\mathbf{f}(\mathbf{x}, t)) - \mathbf{F}(\mathbf{f}(\mathbf{y}, t))\| \leq L_0 \|\mathbf{f}(\mathbf{x}, t) - \mathbf{f}(\mathbf{y}, t)\| = L_0\|\mathbf{e}\|$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}\| \leq L_0\|\mathbf{e}\|$$

Using $\mathbf{e}(\mathbf{x}, \mathbf{y}, 0) = \mathbf{x} - \mathbf{y}$, **Gronwall Lemma** ensures that

$$\|\mathbf{e}(\mathbf{x}, \mathbf{y}, t)\| \leq e^{L_0 t}\|\mathbf{x} - \mathbf{y}\|$$

i.e.:

$$\|\mathbf{f}(\mathbf{x}, t) - \mathbf{f}(\mathbf{y}, t)\| \leq e^{L_0 t}\|\mathbf{x} - \mathbf{y}\|$$

Proof

Let us write $\frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}} = A(\mathbf{x},t)$. Here we will use the fact that $A(\mathbf{x},t)$ is given by

$$\dot{A}(\mathbf{x},t) = \frac{\partial \mathbf{F}(\mathbf{f}(\mathbf{x},t))}{\partial \mathbf{x}} A(\mathbf{x},t)$$

Bounded sensitivities: we use the Frobenius matrix norm and observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A(\mathbf{x},t)\| \leq \left\|\dot{A}(\mathbf{x},t)\right\| = \left\|\frac{\partial \mathbf{F}(\mathbf{f}(\mathbf{x},t))}{\partial \mathbf{x}} A(\mathbf{x},t)\right\| \leq \left\|\frac{\partial \mathbf{F}(\mathbf{f}(\mathbf{x},t))}{\partial \mathbf{x}}\right\| \|A(\mathbf{x},t)\|$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A(\mathbf{x},t)\| \leq \beta \|A(\mathbf{x},t)\|$$

Using $A(\mathbf{x},0) = I$, the **Gronwall Lemma** then ensures that:

$$\|A(\mathbf{x},t)\| \leq e^{\beta t}$$

Proof

Bounded nonlinearity: let us write $E(\mathbf{x},\mathbf{y},t) = A(\mathbf{x},t) - A(\mathbf{y},t)$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|E\| \leq \left\|\dot{E}\right\| = \left\|\frac{\partial \mathbf{F}(\mathbf{f}(\mathbf{x},t))}{\partial \mathbf{x}} A(\mathbf{x},t) - \frac{\partial \mathbf{F}(\mathbf{f}(\mathbf{y},t))}{\partial \mathbf{y}} A(\mathbf{y},t)\right\|$$

Let us use the short notation $\xi_{.} = \frac{\partial \mathbf{F}(\mathbf{f}(.,t))}{\partial .}$. Then:

$$\left\|\dot{E}\right\| = \|\xi_x (A(\mathbf{x},t) - A(\mathbf{y},t)) + (\xi_x - \xi_y) A(\mathbf{y},t)\| \leq$$
$$\|\xi_x\| \|A(\mathbf{x},t) - A(\mathbf{y},t)\| + \|\xi_x - \xi_y\| \|A(\mathbf{y},t)\| \leq \beta \|E\| + \|\xi_x - \xi_y\| e^{\beta t}$$

Then we observe that:

$$\|\xi_x - \xi_y\| \leq L_1 \|\mathbf{f}(\mathbf{x},t) - \mathbf{f}(\mathbf{y},t)\| \leq L_1 e^{L_0 t} \|\mathbf{x} - \mathbf{y}\|$$

We use $E(\mathbf{x},\mathbf{y},0) = 0$ and the **Gronwall Lemma** to conclude that:

$$\|E(\mathbf{x},\mathbf{y},t)\| \leq \int_0^t e^{\int_s^t \beta} L_1 \|\mathbf{x} - \mathbf{y}\| e^{(\beta + L_0)s} \mathrm{d}s = \int_0^t e^{\beta(t-s)} L_1 \|\mathbf{x} - \mathbf{y}\| e^{(\beta + L_0)s} \mathrm{d}s =$$
$$e^{\beta t} L_1 \|\mathbf{x} - \mathbf{y}\| \int_0^t e^{L_0 s} \mathrm{d}s = \frac{L_1}{L_0} e^{\beta t} \|\mathbf{x} - \mathbf{y}\| \left. e^{L_0 s}\right|_0^t = \frac{L_1}{L_0} e^{\beta t} \|\mathbf{x} - \mathbf{y}\| \left(e^{L_0 t} - 1\right)$$

Outline

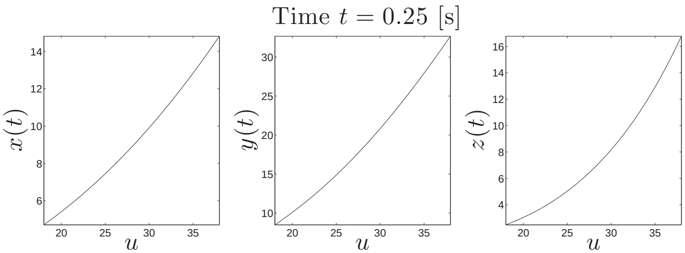
1. Single-Shooting
2. Multiple-Shooting
3. NLP from Multiple-Shooting

Multiple-Shooting - Key idea

Example

$$\dot{x} = 10\,(y - x)$$
$$\dot{y} = x\,(u - z) - y$$
$$\dot{z} = xy - 3z$$

States as a function of u at time t



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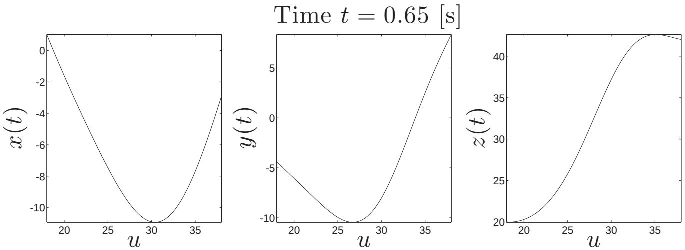

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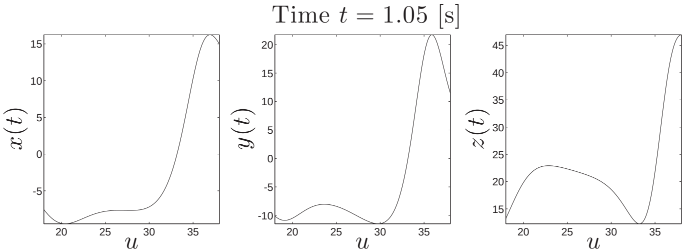

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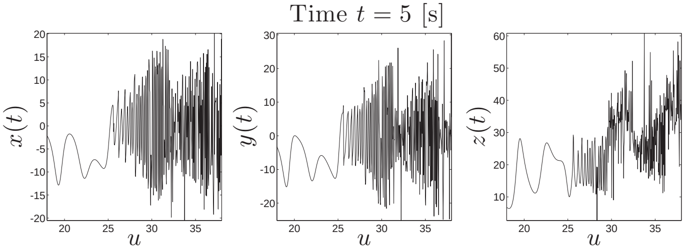

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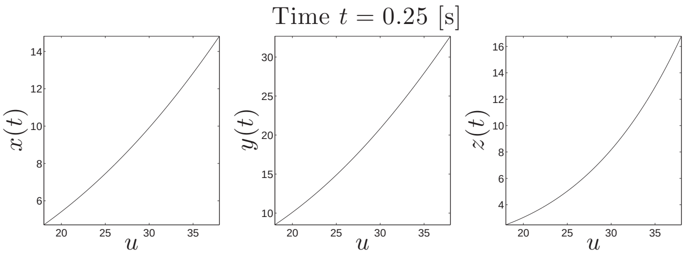

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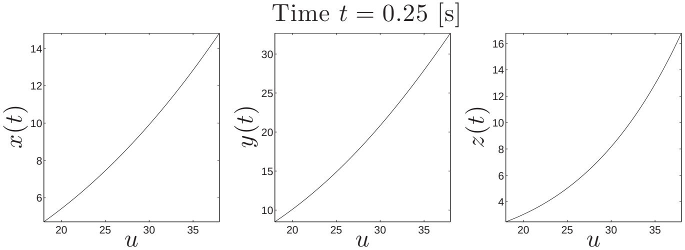

For any smooth dynamics $\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right)$ differentiable and Lipschitz continuous, with bounded & Lipschitz continuous sensitivities

$$\frac{\partial \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right)}{\partial \mathbf{x}}, \ \frac{\partial \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right)}{\partial \mathbf{u}}$$

the integration function $\mathbf{f}\left(\mathbf{x}, \mathbf{u}, t\right)$ can be made "arbitrarily linear" by reducing the integration time t !!

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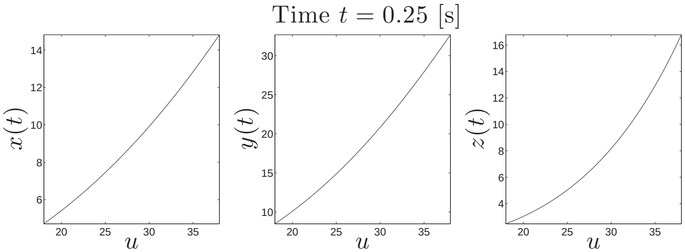

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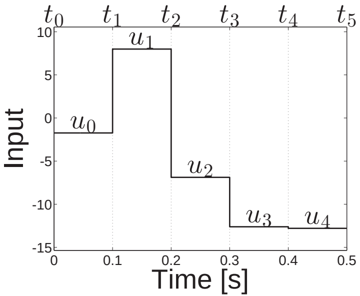
Multiple-shooting breaks down the system integration into short time intervals !!

Multiple-Shooting - key idea

Input ...

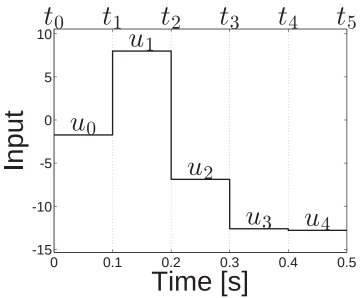


... discretised on the time grid

$\{t_0, t_1, \ldots, t_N\}$

Multiple-Shooting - key idea

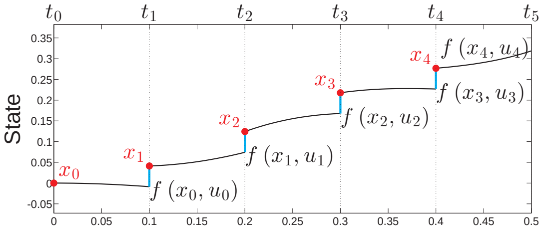
Input ...



... discretised on the time grid

$\{t_0,\ t_1,\ ...,\ t_N\}$

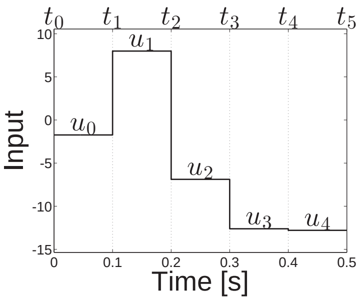
State...



...integration on the time intervals $[t_k,\ t_{k+1}]$

Multiple-Shooting - key idea

Input ...



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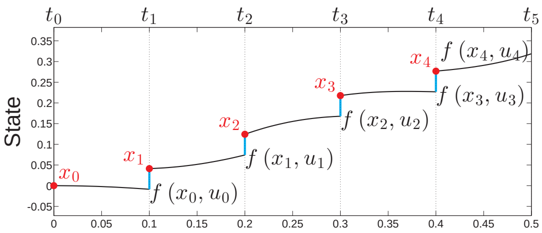
State...

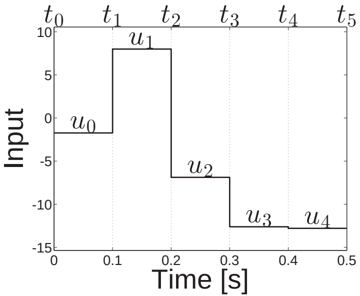


...integration on the time intervals $[t_k, \ t_{k+1}]$

- short integrations starting from *given* $\mathbf{x}_k$

Multiple-Shooting - key idea

Input ...



... discretised on the time grid

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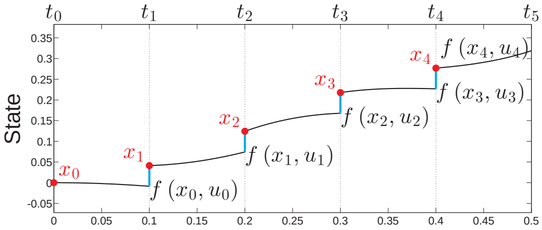
State...

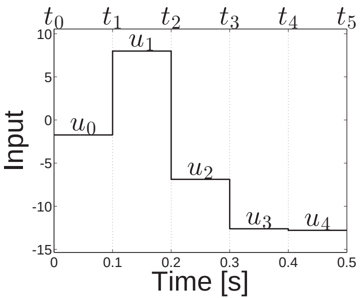


...integration on the time intervals $[t_k,\ t_{k+1}]$

- short integrations starting from *given* $\mathbf{x}_k$
- function $\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ moderately nonlinear

Multiple-Shooting - key idea

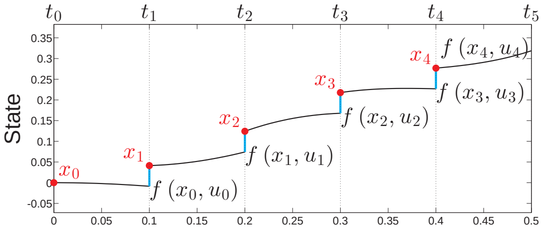
Input ...



... discretised on the time grid

$\{t_0,\ t_1,\ \ldots,\ t_N\}$

State...



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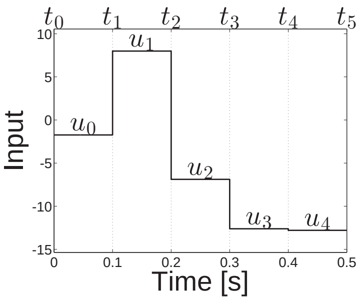
- short integrations starting from *given* $\mathbf{x}_k$
- function $\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ moderately nonlinear
- the trajectory is physically meaningful when the shooting gaps are closed, i.e.

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) - \mathbf{x}_{k+1} = 0$$

Multiple-Shooting - key idea

Input ...

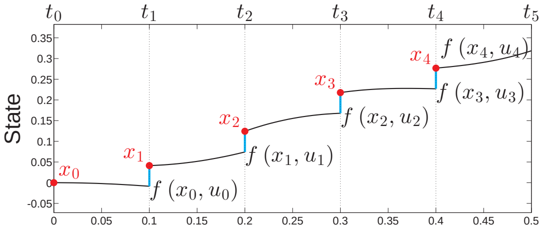


... discretised on the time grid

$\{t_0, \ t_1, \ ..., \ t_N\}$

- The $\mathbf{x}_k$ will become decision variables in the NLP
- The shooting gaps will be constraints in the NLP

State...



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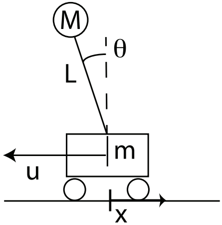
Multiple-shooting - Example

$$\min_{u,\mathbf{x}} \quad \sum_{k=0}^{N-1} u_k^2$$

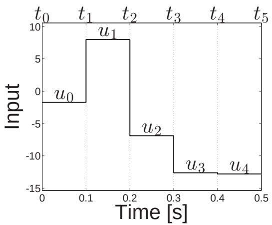
$$\text{s.t.} \quad \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) - \mathbf{x}_{k+1} = 0$$

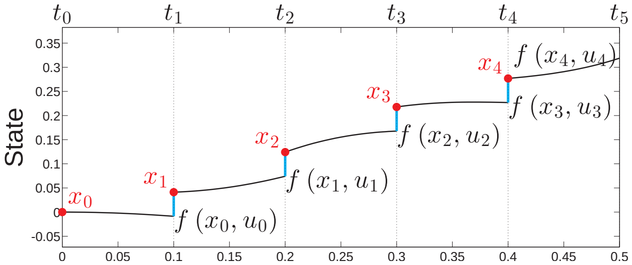
$$-20 \leq u_k \leq 20$$

$$\mathbf{x}_0 = \begin{bmatrix} 0 & \pi & 0 & 0 \end{bmatrix}, \quad \mathbf{x}_N = 0$$



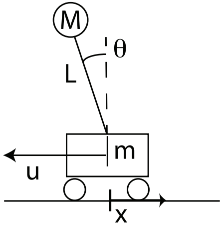
$\mathbf{f}(\mathbf{x}_k, \mathbf{u}_k)$ integrates the dynamics over the time interval $[t_k, t_{k+1}]$





Multiple-shooting - Example

$$\begin{aligned}\min_{u,\mathbf{x}} \quad & \sum_{k=0}^{N-1} u_k^2 \\ \text{s.t.} \quad & \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) - \mathbf{x}_{k+1} = 0 \\ & -20 \leq u_k \leq 20 \\ & \mathbf{x}_0 = \begin{bmatrix} 0 & \pi & 0 & 0 \end{bmatrix}, \quad \mathbf{x}_N = 0\end{aligned}$$



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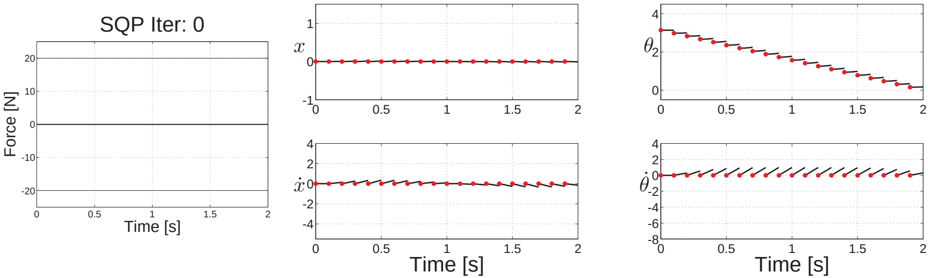

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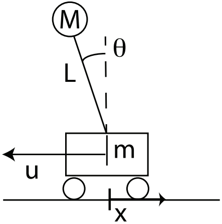

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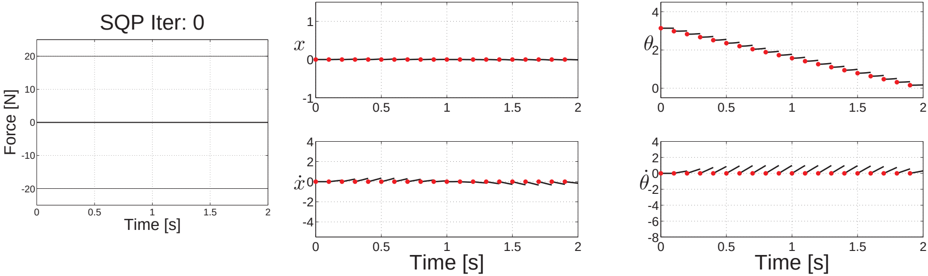

Note: one can provide a guess for the state trajectories in the form of the $\mathbf{x}_{0,\ldots,N}$!!

Multiple-shooting - Example

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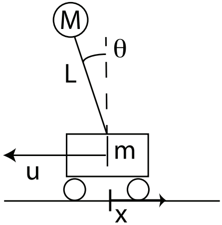

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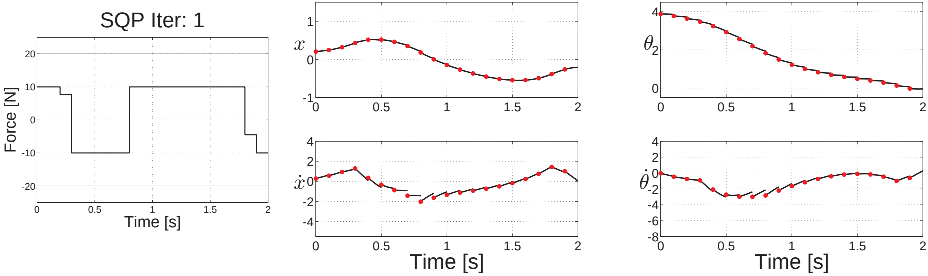

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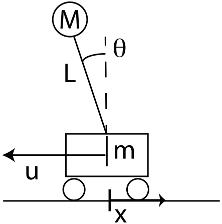

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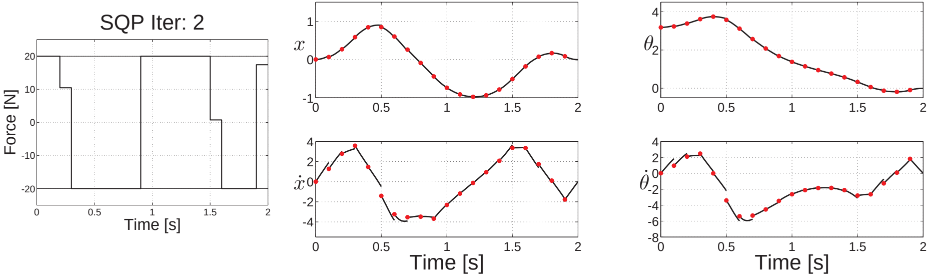

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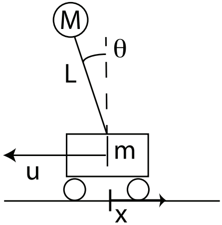

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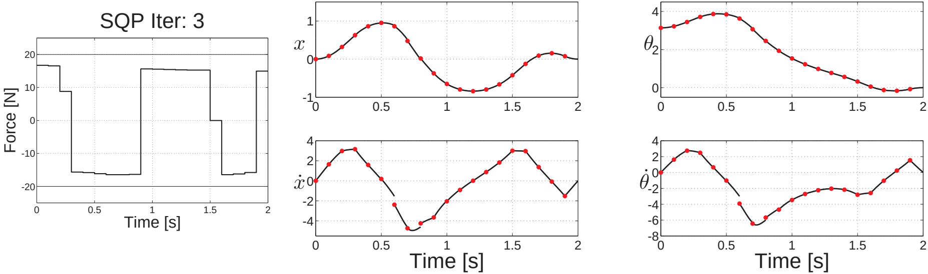

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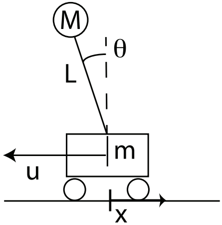

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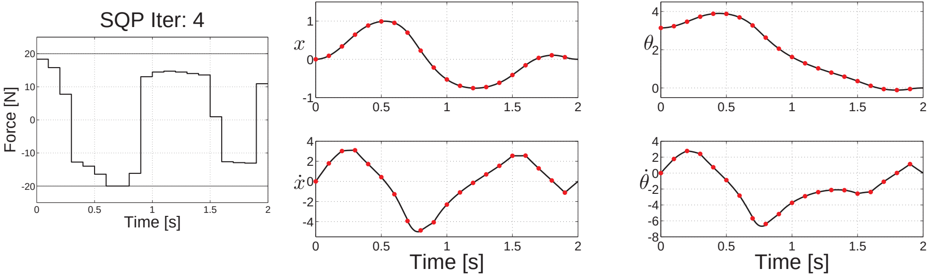

Multiple-shooting - Example

$$
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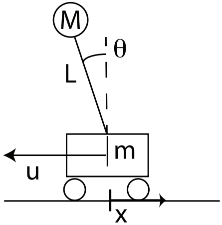

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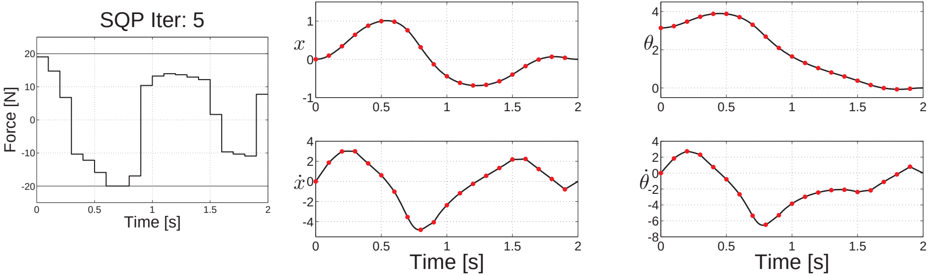

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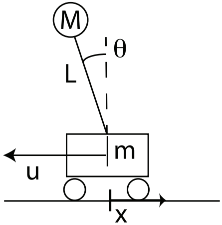

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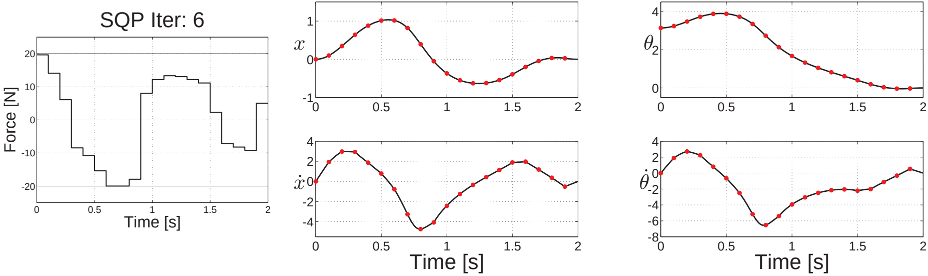

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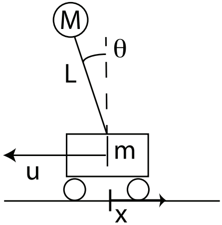
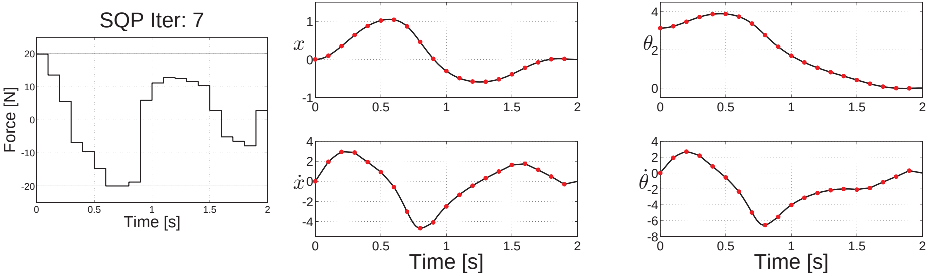


$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ integrates the dynamics over the time interval $[t_k, t_{k+1}]$



Multiple-shooting - Example

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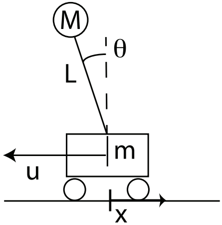

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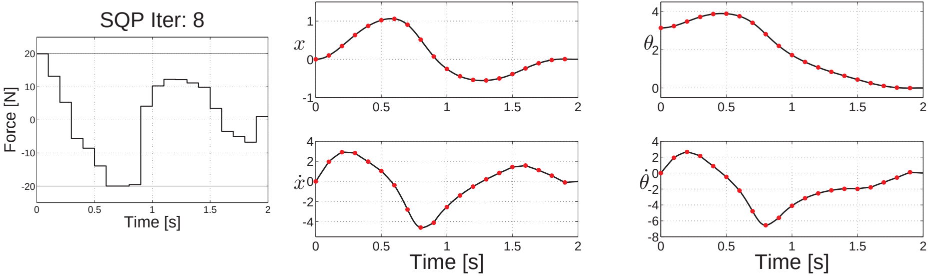

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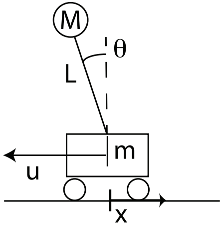

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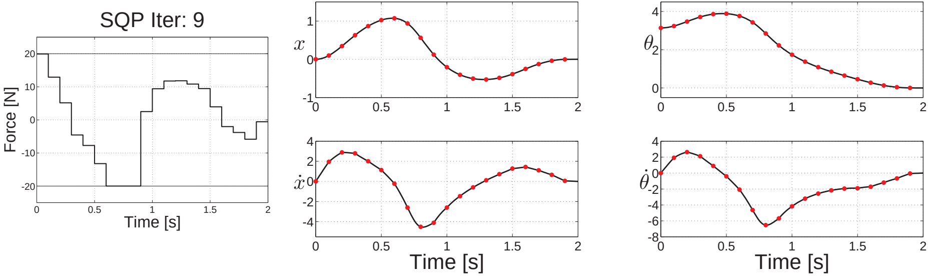

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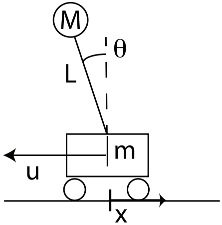

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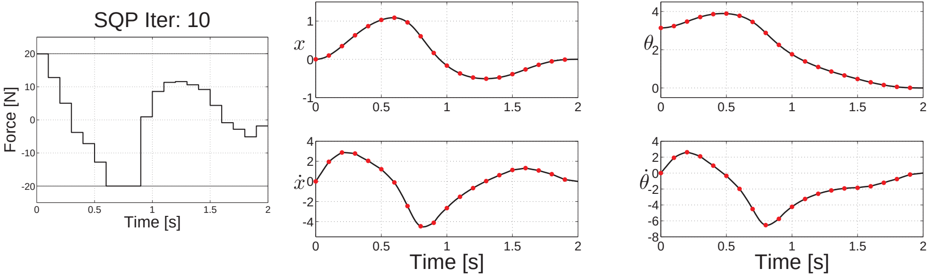

Remember Single-shooting ?

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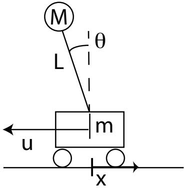

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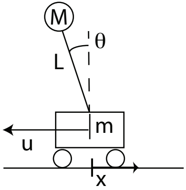

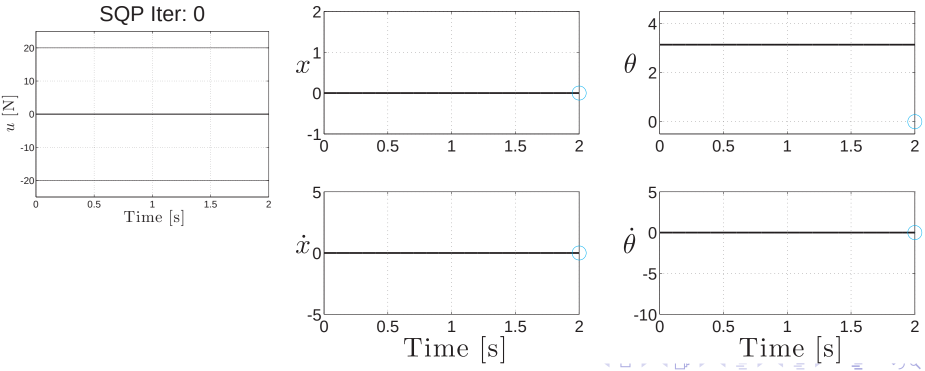

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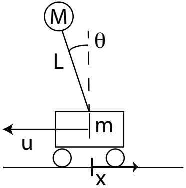

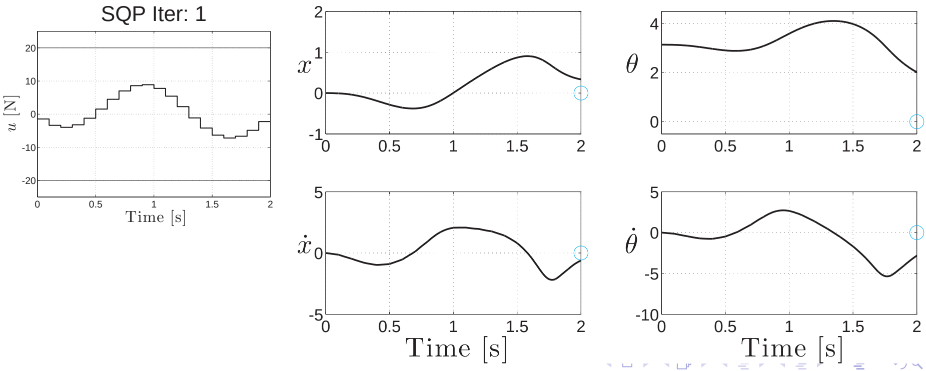

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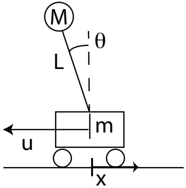

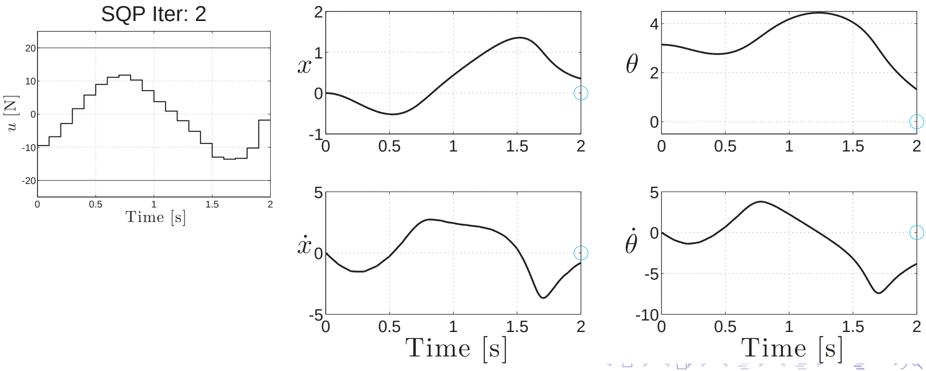

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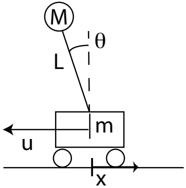

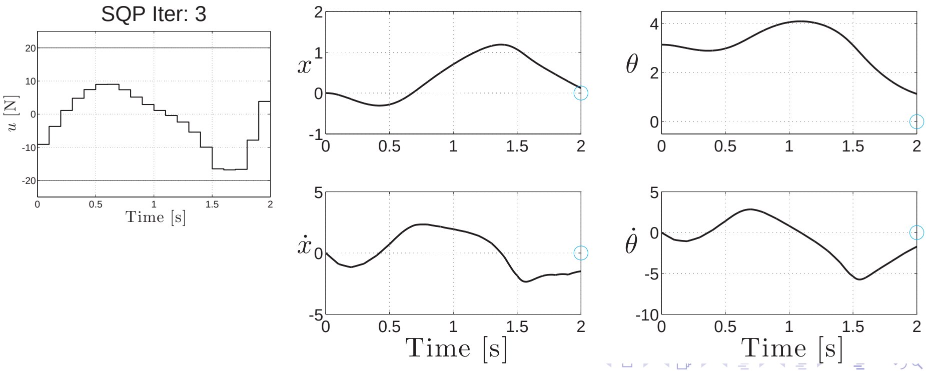

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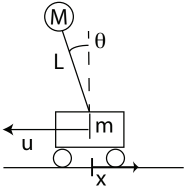

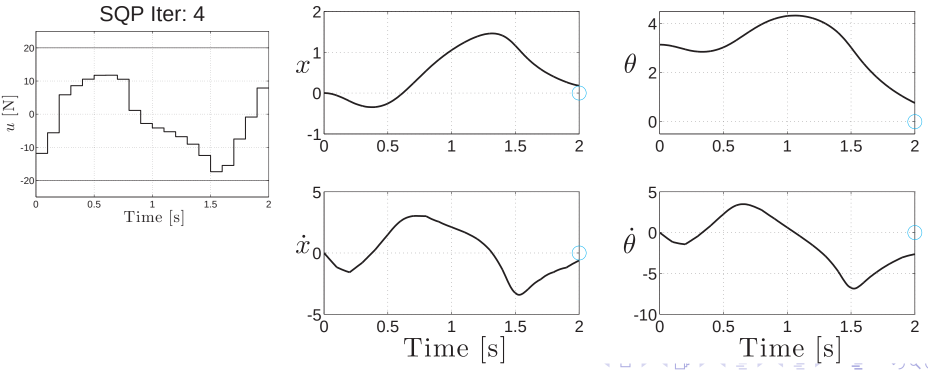

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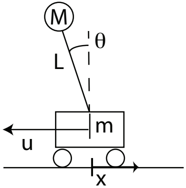

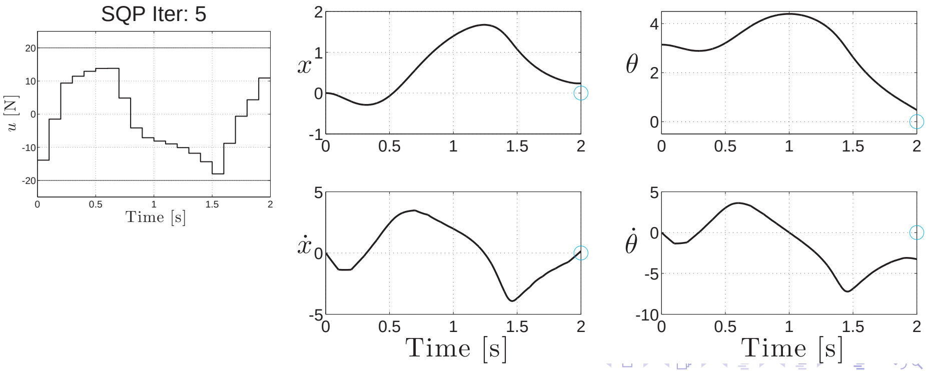

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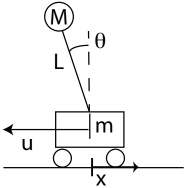

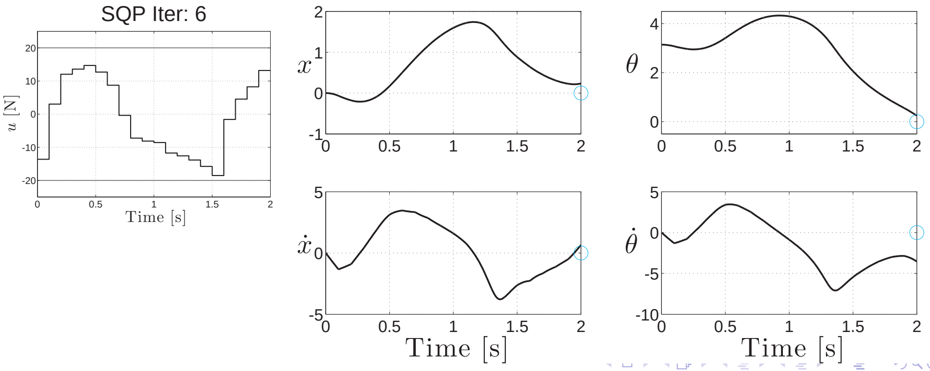

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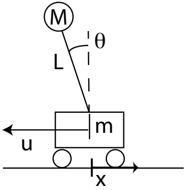

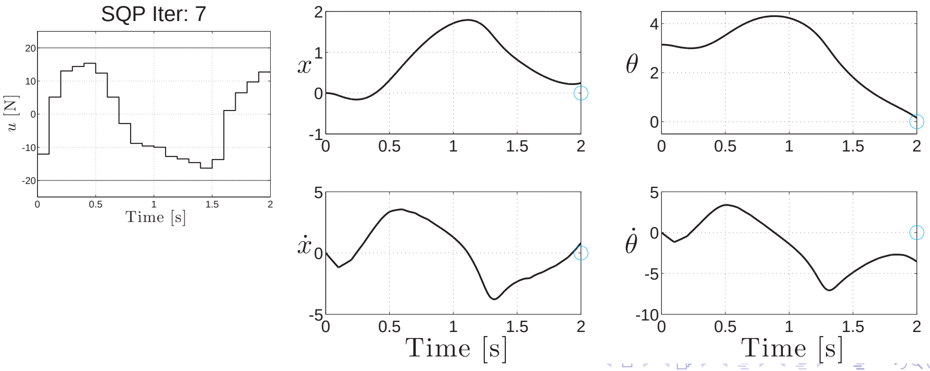

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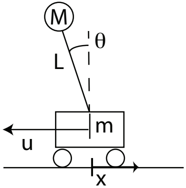

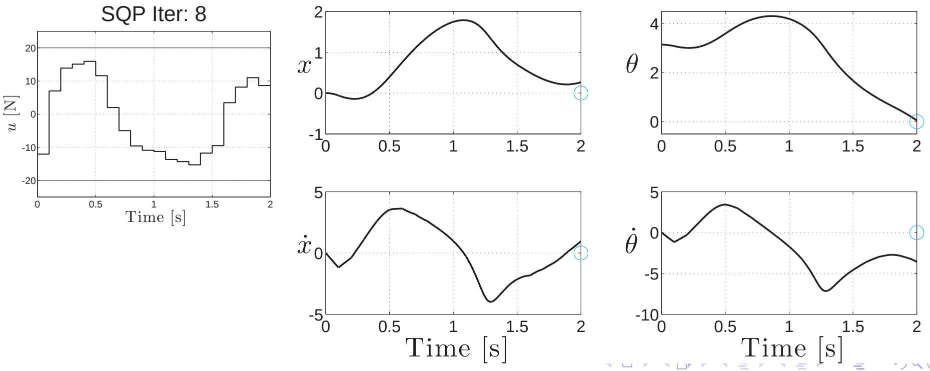

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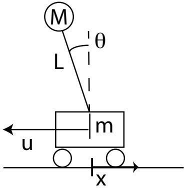

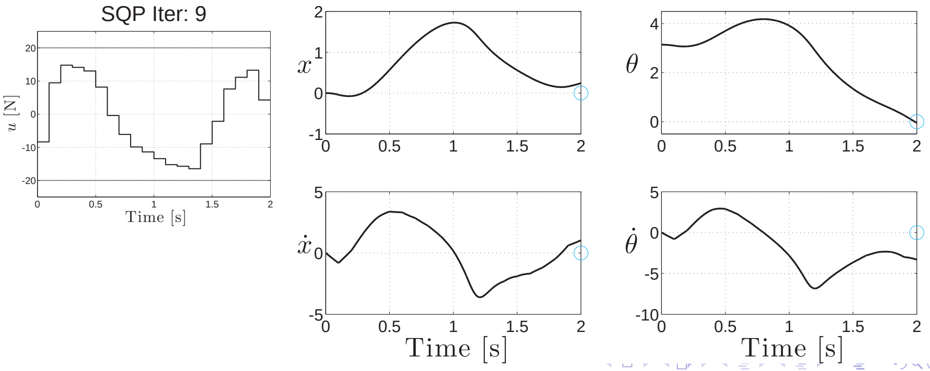

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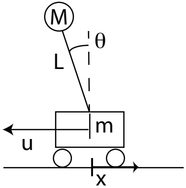

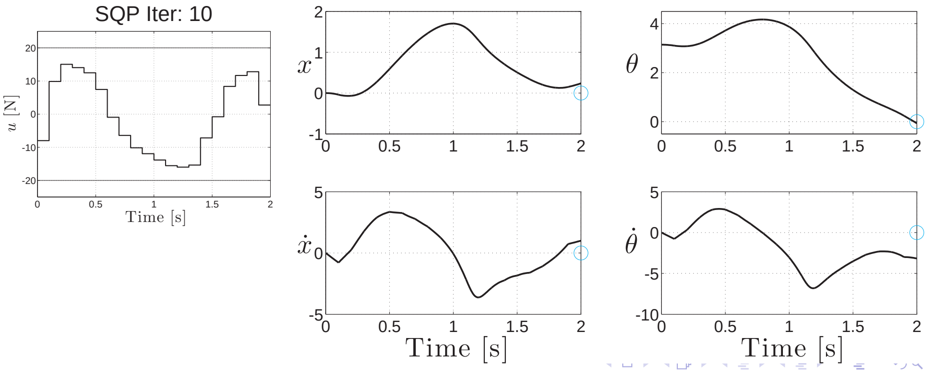

Multiple-Shooting is a Lifted Single-Shooting

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Lifting: reformulate a function with more variables so as to make it less nonlinear...

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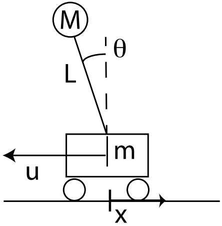

where $\tilde{\mathbf{f}}\left(\mathbf{x}(0), u_0, ..., u_{N-1}\right)$ integrates over $[0, t_{\mathrm{f}}]$

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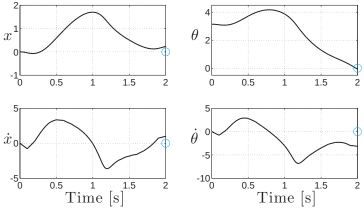

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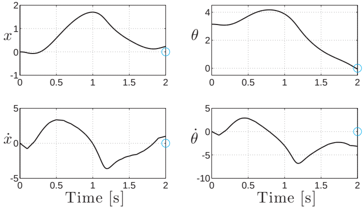

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Using $\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ as an integrator over $[t_k, t_{k+1}]$, one can construe *single shooting*:

$$\tilde{\mathbf{f}}\left(\mathbf{x}\left(0\right), \mathbf{u}_0, ..., \mathbf{u}_{N-1}\right): \quad \mathbf{x}\left(0\right), \mathbf{u}_0, \dots, \mathbf{u}_{N-1} \quad \longmapsto \quad \mathbf{x}\left(t_\mathrm{f}\right)$$

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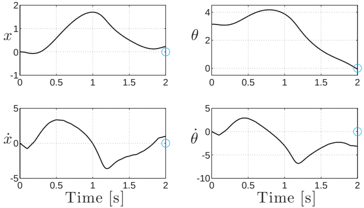

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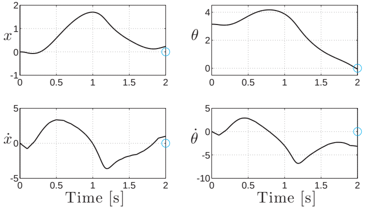

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as... ? $\quad \mathbf{x}\left(t_2\right) = \mathbf{f}\left(\mathbf{f}\left(\mathbf{x}\left(0\right), \mathbf{u}_0\right), \mathbf{u}_1\right)$

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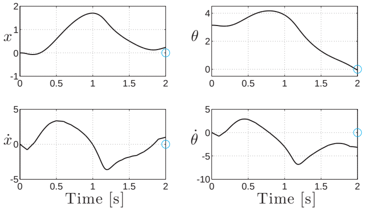

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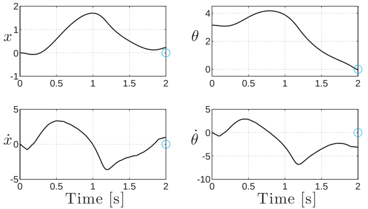

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$$\tilde{\mathbf{f}}\left(\mathbf{x}\left(0\right), \mathbf{u}_0, \dots, \mathbf{u}_{N-1}\right): \quad \mathbf{x}\left(0\right), \mathbf{u}_0, \dots, \mathbf{u}_{N-1} \quad \longmapsto \quad \mathbf{x}\left(t_\mathrm{f}\right)$$

as $\quad \mathbf{x}\left(t_\mathrm{f}\right) = \mathbf{f}\left(\dots\,\mathbf{f}\left(\mathbf{f}\left(\mathbf{f}\left(\mathbf{x}\left(0\right), \mathbf{u}_0\right), \mathbf{u}_1\right), \mathbf{u}_2\right)\dots, \mathbf{u}_{N-1}\right) = \tilde{\mathbf{f}}\left(\mathbf{x}\left(0\right), \mathbf{u}_0, \dots, \mathbf{u}_{N-1}\right)$

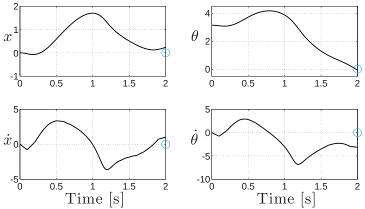
Multiple-Shooting is a Lifted Single-Shooting

Lifting: reformulate a function with more variables so as to make it less nonlinear...

$$\min_{u_0,\ldots,u_{N-1}} \quad \sum_{k=0}^{N-1} u_k^2$$

$$\text{s.t.} \quad -20 \leq u_k \leq 20$$

$$\tilde{\mathbf{f}}\left(\mathbf{x}(0), u_0, \ldots, u_{N-1}\right) = 0$$



where $\tilde{\mathbf{f}}\left(\mathbf{x}(0), u_0, \ldots, u_{N-1}\right)$ integrates over $[0, t_\mathrm{f}]$

Using $\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ as an integrator over $[t_k, t_{k+1}]$, one can construe *single shooting*:

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as $\quad \mathbf{x}(t_\mathrm{f}) = \mathbf{f}\left(\ldots \mathbf{f}\left(\mathbf{f}\left(\mathbf{f}\left(\mathbf{x}(0), \mathbf{u}_0\right), \mathbf{u}_1\right), \mathbf{u}_2\right) \ldots, \mathbf{u}_{N-1}\right) = \tilde{\mathbf{f}}\left(\mathbf{x}(0), \mathbf{u}_0, \ldots, \mathbf{u}_{N-1}\right)$

The nonlinearity of $\tilde{\mathbf{f}}$ follows from the N recursive calls of the integrator $\mathbf{f}$.

Multiple-Shooting is a Lifted Single-Shooting

Lifting: reformulate a function with more variables so as to make it less nonlinear...

$$\min_{u_0,\dots,u_{N-1}} \quad \sum_{k=0}^{N-1} u_k^2$$

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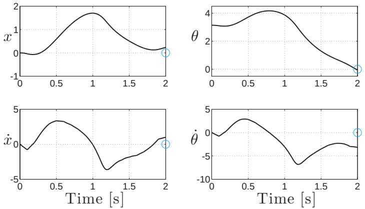
$$\tilde{\mathbf{f}}\left(\mathbf{x}(0), u_0, \dots, u_{N-1}\right) = 0$$



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Multiple-Shooting **introduces** the variables $\mathbf{x}_0, \dots, \mathbf{x}_N$ and **lifts** the recursion as:

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right) \\ \mathbf{x}_2 &= \mathbf{f}\left(\mathbf{x}_1, \mathbf{u}_1\right) \\ &\dots \\ \mathbf{x}_N &= \mathbf{f}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) \end{aligned}$$

Multiple-Shooting is a Lifted Single-Shooting

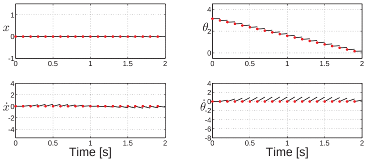
Lifting: reformulate a function with more variables so as to make it less nonlinear...

$$\min_{u,\mathbf{x}} \quad \sum_{k=0}^{N-1} u_k^2$$

$$\text{s.t.} \quad \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) - \mathbf{x}_{k+1} = 0$$

$$-20 \leq u_k \leq 20$$

$$\mathbf{x}_0 = \begin{bmatrix} 0 & \pi & 0 & 0 \end{bmatrix}, \quad \mathbf{x}_N = 0$$



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$$\tilde{\mathbf{f}}\left(\mathbf{x}(0), \mathbf{u}_0, ..., \mathbf{u}_{N-1}\right): \quad \mathbf{x}(0), \mathbf{u}_0, \ldots, \mathbf{u}_{N-1} \quad \longmapsto \quad \mathbf{x}(t_\mathrm{f})$$

as $\quad \mathbf{x}(t_\mathrm{f}) = \mathbf{f}\left(\ldots\ \mathbf{f}\left(\mathbf{f}\left(\mathbf{f}\left(\mathbf{x}(0), \mathbf{u}_0\right), \mathbf{u}_1\right), \mathbf{u}_2\right) \ldots, \mathbf{u}_{N-1}\right) = \tilde{\mathbf{f}}\left(\mathbf{x}(0), \mathbf{u}_0, ..., \mathbf{u}_{N-1}\right)$

The nonlinearity of $\tilde{\mathbf{f}}$ follows from the N recursive calls of the integrator $\mathbf{f}$.
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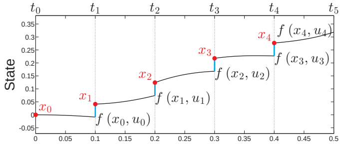
$$\mathbf{x}_1 = \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right)$$
$$\mathbf{x}_2 = \mathbf{f}\left(\mathbf{x}_1, \mathbf{u}_1\right)$$
$$\ldots$$
$$\mathbf{x}_N = \mathbf{f}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right)$$

Cost and constraints discretisation in Multiple-shooting

OCP:

$$\begin{aligned}
\min_{\mathbf{x},\mathbf{u}} \quad & T\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right) + \int_0^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\
\text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \quad \mathbf{x}\left(0\right) = \bar{\mathbf{x}}_0 \\
& \mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0
\end{aligned}$$



Cost and constraints discretisation in Multiple-shooting

OCP:

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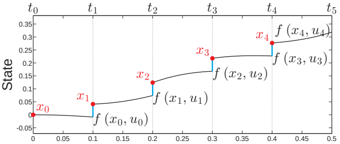

- Inequality constraints: $\mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0$ are enforced on the the shooting nodes:

$$\mathbf{h}\left(\mathbf{x}_k, t_k, \mathbf{u}_k\right) \leq 0, \quad \forall\, k = 0, ..., N-1$$

... what happens in between ?!?

Cost and constraints discretisation in Multiple-shooting

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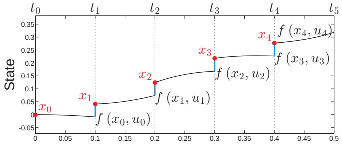
$$\begin{aligned}\min_{\mathbf{x},\mathbf{u}} \quad & T\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) + \int_0^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \quad \mathbf{x}\left(0\right) = \bar{\mathbf{x}}_0 \\ & \mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0\end{aligned}$$



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... what happens in between ?!?

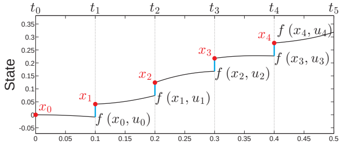
- Cost function often approximated as (rectangular quadrature):

$$T\left(\mathbf{x}_N\right) + \sum_{k=0}^{N-1} \left(t_{k+1} - t_k\right) L\left(\mathbf{x}_k, \mathbf{u}_k\left(t\right)\right)$$

Cost and constraints discretisation in Multiple-shooting

OCP:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{u}} \quad & T\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) + \int_0^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right), \quad \mathbf{x}\left(0\right) = \bar{\mathbf{x}}_0 \\ & \mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0 \end{aligned}$$



- Inequality constraints: $\mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0$ are enforced on the the shooting nodes:

$$\mathbf{h}\left(\mathbf{x}_k, t_k, \mathbf{u}_k\right) \leq 0, \quad \forall\, k = 0, ..., N-1$$

... what happens in between ?!?

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$$T\left(\mathbf{x}_N\right) + \sum_{k=0}^{N-1} \left(t_{k+1} - t_k\right) L\left(\mathbf{x}_k, \mathbf{u}_k\left(t\right)\right)$$

- Alternatively, Lagrange term $L\left(\mathbf{x}, \mathbf{u}\right)$ can be implemented via a *dynamic extension*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c} \mathbf{x} \\ \rho \end{array}\right] = \left[\begin{array}{c} \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) \\ L\left(\mathbf{x}, \mathbf{u}\right) \end{array}\right], \quad \left[\begin{array}{c} \mathbf{x}(0) \\ \rho(0) \end{array}\right] = \left[\begin{array}{c} \bar{\mathbf{x}}_0 \\ 0 \end{array}\right], \qquad \Phi\left(\mathbf{w}\right) = T\left(\mathbf{x}_N\right) + \rho_N$$

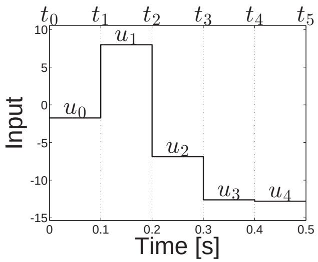
Outline

1. Single-Shooting
2. Multiple-Shooting
3. NLP from Multiple-Shooting

NLP from Multiple-Shooting

OCP:

$$
\begin{aligned}
\min \quad & \Phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\
\text{s.t.} \quad & \dot{\mathbf{x}}\left(t\right) = \mathbf{F}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \\
& \mathbf{h}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) \leq 0 \\
& \mathbf{x}\left(t_0\right) = \bar{\mathbf{x}}_0
\end{aligned}
$$

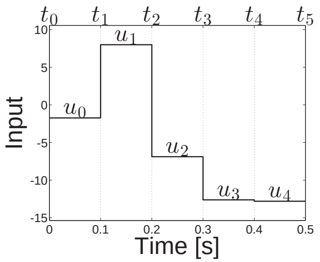


NLP from Multiple-Shooting

OCP:

$$\begin{aligned} \min \quad & \Phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\ \text{s.t.} \quad & \dot{\mathbf{x}}\left(t\right) = \mathbf{F}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \\ & \mathbf{h}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) \leq 0 \\ & \mathbf{x}\left(t_0\right) = \bar{\mathbf{x}}_0 \end{aligned}$$

$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ integrates the dynamics $\mathbf{F}$ over the time interval $[t_k,\ t_{k+1}]$

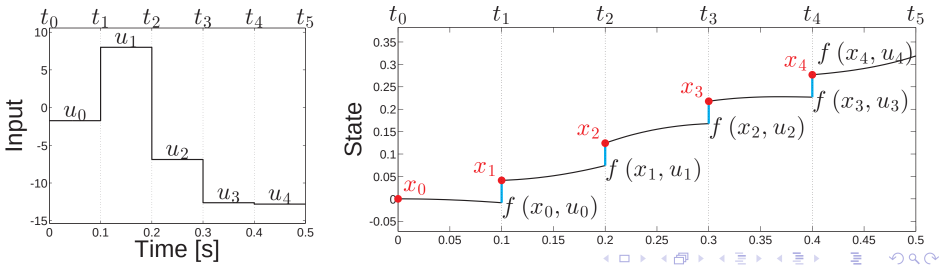


NLP from Multiple-Shooting

OCP:

$$\begin{aligned} \min \quad & \Phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\ \text{s.t.} \quad & \dot{\mathbf{x}}\left(t\right) = \mathbf{F}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \\ & \mathbf{h}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) \leq 0 \\ & \mathbf{x}\left(t_0\right) = \bar{\mathbf{x}}_0 \end{aligned}$$

$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ integrates the dynamics $\mathbf{F}$ over the time interval $[t_k,\ t_{k+1}]$



NLP from Multiple-Shooting

OCP:

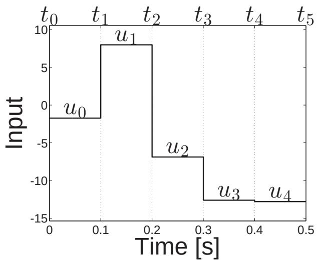
$$\begin{aligned} \min \quad & \Phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\ \text{s.t.} \quad & \dot{\mathbf{x}}\left(t\right) = \mathbf{F}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \\ & \mathbf{h}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) \leq 0 \\ & \mathbf{x}\left(t_0\right) = \bar{\mathbf{x}}_0 \end{aligned}$$

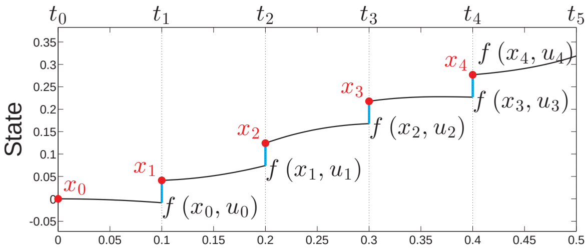
NLP with $\mathbf{w} = \{\mathbf{x}_0, \mathbf{u}_0, ..., \mathbf{x}_{N-1}, \mathbf{u}_{N-1}, \mathbf{x}_N\}$

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

s.t.

$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ integrates the dynamics $\mathbf{F}$ over the time interval $[t_k,\ t_{k+1}]$





NLP from Multiple-Shooting

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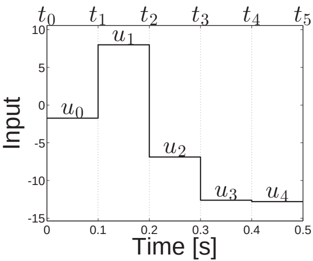
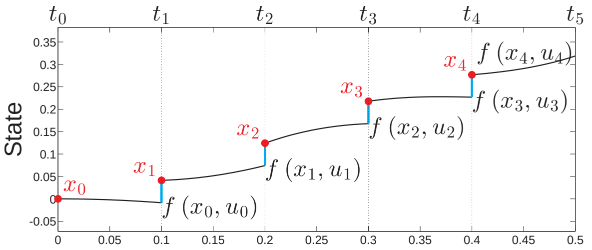
$$\begin{aligned} \min \quad & \Phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\ \text{s.t.} \quad & \dot{\mathbf{x}}\left(t\right) = \mathbf{F}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \\ & \mathbf{h}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) \leq 0 \\ & \mathbf{x}\left(t_0\right) = \bar{\mathbf{x}}_0 \end{aligned}$$

$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ integrates the dynamics $\mathbf{F}$ over the time interval $[t_k,\ t_{k+1}]$

NLP with $\mathbf{w} = \left\{\mathbf{x}_0, \mathbf{u}_0, ..., \mathbf{x}_{N-1}, \mathbf{u}_{N-1}, \mathbf{x}_N\right\}$

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi\left(\mathbf{w}\right) \\ \text{s.t.} \quad & \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \bar{\mathbf{x}}_0 - \mathbf{x}_0 \\ \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}\left(\mathbf{x}_N, \mathbf{u}_{N-1}\right) - \mathbf{x}_{N-1} \end{bmatrix} = 0 \end{aligned}$$





NLP from Multiple-Shooting

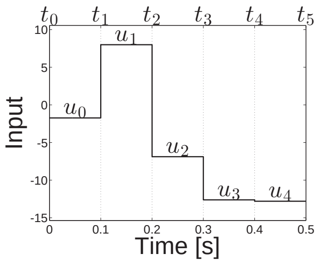
OCP:

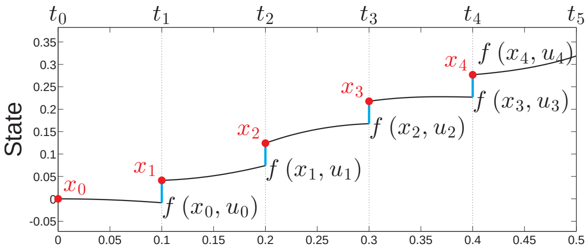
$$\begin{aligned} \min \quad & \Phi\left(\mathbf{x}(.), \mathbf{u}(.)\right) \\ \text{s.t.} \quad & \dot{\mathbf{x}}(t) = \mathbf{F}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \\ & \mathbf{h}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) \leq 0 \\ & \mathbf{x}(t_0) = \bar{\mathbf{x}}_0 \end{aligned}$$

$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ integrates the dynamics $\mathbf{F}$ over the time interval $[t_k,\ t_{k+1}]$

NLP with $\mathbf{w} = \{\mathbf{x}_0, \mathbf{u}_0, ..., \mathbf{x}_{N-1}, \mathbf{u}_{N-1}, \mathbf{x}_N\}$

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_0 - \mathbf{x}_0 \\ \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right) - \mathbf{x}_1 \\ \cdots \\ \mathbf{f}\left(\mathbf{x}_N, \mathbf{u}_{N-1}\right) - \mathbf{x}_{N-1} \end{bmatrix} = 0 \\ & \mathbf{h}(\mathbf{w}) = \begin{bmatrix} \mathbf{h}\left(\mathbf{x}_0, \mathbf{u}_0\right) \\ \cdots \\ \mathbf{h}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) \\ \mathbf{h}\left(\mathbf{x}_N\right) \end{bmatrix} \leq 0 \end{aligned}$$





QP structure from Multiple-Shooting

NLP:

$$\min_{\mathbf{w}} \quad \phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_0 - \mathbf{x}_0 \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{bmatrix} = 0$$

$$\mathbf{h}(\mathbf{w}) = \begin{bmatrix} \mathbf{h}(\mathbf{x}_0, \mathbf{u}_0) \\ \dots \\ \mathbf{h}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}) \\ \mathbf{h}(\mathbf{x}_N) \end{bmatrix} \leq 0$$

SQP recursively solves the QPs:

$$\begin{aligned} \min_{\Delta\mathbf{w}} \quad & \frac{1}{2}\Delta\mathbf{w}^\top H \Delta\mathbf{w} + \nabla\Phi^\top \Delta\mathbf{w} \\ \text{s.t.} \quad & \nabla\mathbf{g}^\top \Delta\mathbf{w} + \mathbf{g} = 0 \\ & \nabla\mathbf{h}^\top \Delta\mathbf{w} + \mathbf{h} \leq 0 \end{aligned}$$

QP structure from Multiple-Shooting

NLP:

$$\min_{\mathbf{w}} \quad \phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_0 - \mathbf{x}_0 \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{bmatrix} = 0$$

$$\mathbf{h}(\mathbf{w}) = \begin{bmatrix} \mathbf{h}(\mathbf{x}_0, \mathbf{u}_0) \\ \dots \\ \mathbf{h}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}) \\ \mathbf{h}(\mathbf{x}_N) \end{bmatrix} \leq 0$$

SQP recursively solves the QPs:

$$\min_{\Delta\mathbf{w}} \quad \frac{1}{2}\Delta\mathbf{w}^{\mathsf{T}} H \Delta\mathbf{w} + \nabla\Phi^{\mathsf{T}}\Delta\mathbf{w}$$

$$\text{s.t.} \quad \nabla\mathbf{g}^{\mathsf{T}}\Delta\mathbf{w} + \mathbf{g} = 0$$

$$\nabla\mathbf{h}^{\mathsf{T}}\Delta\mathbf{w} + \mathbf{h} \leq 0$$

Let's have a look at matrices H, $\nabla\mathbf{g}^\top$ and $\nabla\mathbf{h}^\top$ for this specific type of NLP

Constraints Jacobian - Dynamics

Constraints: $$\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \mathbf{c}(\mathbf{x}_0) \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \mathbf{f}(\mathbf{x}_1, \mathbf{u}_1) - \mathbf{x}_2 \\ \mathbf{f}(\mathbf{x}_2, \mathbf{u}_2) - \mathbf{x}_3 \\ \mathbf{f}(\mathbf{x}_3, \mathbf{u}_3) - \mathbf{x}_4 \\ \mathbf{f}(\mathbf{x}_4, \mathbf{u}_4) - \mathbf{x}_5 \end{bmatrix} \quad \text{with} \quad \mathbf{w} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \\ \mathbf{x}_1 \\ \mathbf{u}_1 \\ \mathbf{x}_2 \\ \mathbf{u}_2 \\ \mathbf{x}_3 \\ \mathbf{u}_3 \\ \mathbf{x}_4 \\ \mathbf{u}_4 \\ \mathbf{x}_5 \end{bmatrix}$$

Constraints Jacobian - Dynamics

Constraints: $$\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \mathbf{c}(\mathbf{x}_0) \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \mathbf{f}(\mathbf{x}_1, \mathbf{u}_1) - \mathbf{x}_2 \\ \mathbf{f}(\mathbf{x}_2, \mathbf{u}_2) - \mathbf{x}_3 \\ \mathbf{f}(\mathbf{x}_3, \mathbf{u}_3) - \mathbf{x}_4 \\ \mathbf{f}(\mathbf{x}_4, \mathbf{u}_4) - \mathbf{x}_5 \end{bmatrix} \quad \text{with} \quad \mathbf{w} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \\ \mathbf{x}_1 \\ \mathbf{u}_1 \\ \mathbf{x}_2 \\ \mathbf{u}_2 \\ \mathbf{x}_3 \\ \mathbf{u}_3 \\ \mathbf{x}_4 \\ \mathbf{u}_4 \\ \mathbf{x}_5 \end{bmatrix}$$

Let's denote $\mathbf{f}_k = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k)$, then the constraints derivative reads as:

$$\nabla \mathbf{g}(\mathbf{w})^\top = \begin{bmatrix} \frac{\partial \mathbf{c}}{\partial \mathbf{x}_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial \mathbf{f}_0}{\partial \mathbf{x}_0} & \frac{\partial \mathbf{f}_0}{\partial \mathbf{u}_0} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{f}_1}{\partial \mathbf{u}_1} & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_2} & \frac{\partial \mathbf{f}_2}{\partial \mathbf{u}_2} & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{f}_3}{\partial \mathbf{x}_3} & \frac{\partial \mathbf{f}_3}{\partial \mathbf{u}_3} & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{f}_4}{\partial \mathbf{x}_4} & \frac{\partial \mathbf{f}_4}{\partial \mathbf{u}_4} & -I \end{bmatrix}$$

Constraints Jacobian - Dynamics

Constraints: $\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \mathbf{c}(\mathbf{x}_0) \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \mathbf{f}(\mathbf{x}_1, \mathbf{u}_1) - \mathbf{x}_2 \\ \mathbf{f}(\mathbf{x}_2, \mathbf{u}_2) - \mathbf{x}_3 \\ \mathbf{f}(\mathbf{x}_3, \mathbf{u}_3) - \mathbf{x}_4 \\ \mathbf{f}(\mathbf{x}_4, \mathbf{u}_4) - \mathbf{x}_5 \end{bmatrix}$ with $\mathbf{w} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \\ \mathbf{x}_1 \\ \mathbf{u}_1 \\ \mathbf{x}_2 \\ \mathbf{u}_2 \\ \mathbf{x}_3 \\ \mathbf{u}_3 \\ \mathbf{x}_4 \\ \mathbf{u}_4 \\ \mathbf{x}_5 \end{bmatrix}$

Let's denote $\mathbf{f}_k = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k)$, then the constraints derivative reads as:

$$\nabla \mathbf{g}(\mathbf{w})^\top = \begin{bmatrix} \frac{\partial \mathbf{c}}{\partial \mathbf{x}_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial \mathbf{f}_0}{\partial \mathbf{x}_0} & \frac{\partial \mathbf{f}_0}{\partial \mathbf{u}_0} & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{f}_1}{\partial \mathbf{u}_1} & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_2} & \frac{\partial \mathbf{f}_2}{\partial \mathbf{u}_2} & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{f}_3}{\partial \mathbf{x}_3} & \frac{\partial \mathbf{f}_3}{\partial \mathbf{u}_3} & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{f}_4}{\partial \mathbf{x}_4} & \frac{\partial \mathbf{f}_4}{\partial \mathbf{u}_4} & -I \end{bmatrix}$$

Observe the **banded** structure of the **Jacobian** $\nabla \mathbf{g}(\mathbf{w})^\top$.
Note that this structure hinges on the ordering in $\mathbf{w}$ and $\mathbf{g}$!!!

Constraints Jacobian - Bounds

Bounds: $\mathbf{h}(\mathbf{w}) = \begin{bmatrix} \mathbf{h}_0(\mathbf{x}_0, \mathbf{u}_0) \\ \mathbf{h}_1(\mathbf{x}_1, \mathbf{u}_1) \\ \mathbf{h}_2(\mathbf{x}_2, \mathbf{u}_2) \\ \mathbf{h}_3(\mathbf{x}_3, \mathbf{u}_3) \\ \mathbf{h}_4(\mathbf{x}_4, \mathbf{u}_4) \\ \mathbf{h}_5(\mathbf{x}_5) \end{bmatrix}$ with $\mathbf{w} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \\ \mathbf{x}_1 \\ \mathbf{u}_1 \\ \mathbf{x}_2 \\ \mathbf{u}_2 \\ \mathbf{x}_3 \\ \mathbf{u}_3 \\ \mathbf{x}_4 \\ \mathbf{u}_4 \\ \mathbf{x}_5 \end{bmatrix}$

Constraints Jacobian - Bounds

Bounds: $\mathbf{h}(\mathbf{w}) = \begin{bmatrix} \mathbf{h}_0(\mathbf{x}_0, \mathbf{u}_0) \\ \mathbf{h}_1(\mathbf{x}_1, \mathbf{u}_1) \\ \mathbf{h}_2(\mathbf{x}_2, \mathbf{u}_2) \\ \mathbf{h}_3(\mathbf{x}_3, \mathbf{u}_3) \\ \mathbf{h}_4(\mathbf{x}_4, \mathbf{u}_4) \\ \mathbf{h}_5(\mathbf{x}_5) \end{bmatrix}$ with $\mathbf{w} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \\ \mathbf{x}_1 \\ \mathbf{u}_1 \\ \mathbf{x}_2 \\ \mathbf{u}_2 \\ \mathbf{x}_3 \\ \mathbf{u}_3 \\ \mathbf{x}_4 \\ \mathbf{u}_4 \\ \mathbf{x}_5 \end{bmatrix}$

Then:

$$\frac{\partial \mathbf{h}}{\partial \mathbf{w}} = \begin{bmatrix} \frac{\partial \mathbf{h}_0}{\partial \mathbf{x}_0} & \frac{\partial \mathbf{h}_0}{\partial \mathbf{u}_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{h}_1}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{h}_1}{\partial \mathbf{u}_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \mathbf{h}_2}{\partial \mathbf{x}_2} & \frac{\partial \mathbf{h}_2}{\partial \mathbf{u}_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{h}_3}{\partial \mathbf{x}_3} & \frac{\partial \mathbf{h}_3}{\partial \mathbf{u}_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{h}_4}{\partial \mathbf{x}_4} & \frac{\partial \mathbf{h}_4}{\partial \mathbf{u}_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{h}_5}{\partial \mathbf{x}_5} \end{bmatrix}$$

Constraints Jacobian - Bounds

Bounds: $\mathbf{h}(\mathbf{w}) = \begin{bmatrix} \mathbf{h}_0(\mathbf{x}_0, \mathbf{u}_0) \\ \mathbf{h}_1(\mathbf{x}_1, \mathbf{u}_1) \\ \mathbf{h}_2(\mathbf{x}_2, \mathbf{u}_2) \\ \mathbf{h}_3(\mathbf{x}_3, \mathbf{u}_3) \\ \mathbf{h}_4(\mathbf{x}_4, \mathbf{u}_4) \\ \mathbf{h}_5(\mathbf{x}_5) \end{bmatrix}$ with $\mathbf{w} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \\ \mathbf{x}_1 \\ \mathbf{u}_1 \\ \mathbf{x}_2 \\ \mathbf{u}_2 \\ \mathbf{x}_3 \\ \mathbf{u}_3 \\ \mathbf{x}_4 \\ \mathbf{u}_4 \\ \mathbf{x}_5 \end{bmatrix}$

Then:

$$\frac{\partial \mathbf{h}}{\partial \mathbf{w}} = \begin{bmatrix} \frac{\partial \mathbf{h}_0}{\partial \mathbf{x}_0} & \frac{\partial \mathbf{h}_0}{\partial \mathbf{u}_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{h}_1}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{h}_1}{\partial \mathbf{u}_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \mathbf{h}_2}{\partial \mathbf{x}_2} & \frac{\partial \mathbf{h}_2}{\partial \mathbf{u}_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{h}_3}{\partial \mathbf{x}_3} & \frac{\partial \mathbf{h}_3}{\partial \mathbf{u}_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{h}_4}{\partial \mathbf{x}_4} & \frac{\partial \mathbf{h}_4}{\partial \mathbf{u}_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{h}_5}{\partial \mathbf{x}_5} \end{bmatrix}$$

Observe the **banded** structure of the **Jacobian** $\nabla \mathbf{h}(\mathbf{w})^\top$.
Note that this structure hinges on the ordering in $\mathbf{w}$ and $\mathbf{g}$!!!

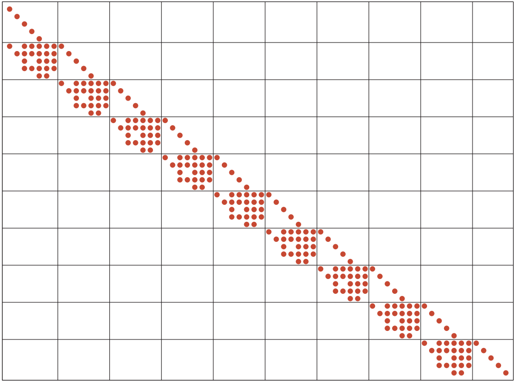
Constraints Jacobian sparsity pattern - Illustration

$$\min_{\Delta \mathbf{w}} \quad \frac{1}{2}\Delta\mathbf{w}^{\mathsf{T}} B \Delta\mathbf{w} + \nabla\Phi^{\mathsf{T}}\Delta\mathbf{w}$$

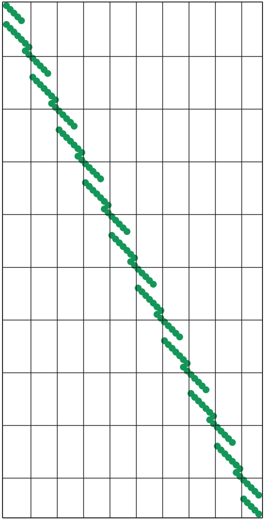
$$\text{s.t.} \quad \nabla\mathbf{g}^{\mathsf{T}}\Delta\mathbf{w} + \mathbf{g} = 0$$

$$\nabla\mathbf{h}^{\mathsf{T}}\Delta\mathbf{w} + \mathbf{h} \leq 0$$

$\nabla\mathbf{g}(\mathbf{w})^{\mathsf{T}}$



$\nabla\mathbf{h}(\mathbf{w})^{\mathsf{T}}$



Lagrange function in Multiple-Shooting

NLP:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \left[\begin{array}{c} \bar{\mathbf{x}}_0 - \mathbf{x}_0 \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{array}\right] = 0$$

$$\mathbf{h}(\mathbf{w}) = \left[\begin{array}{c} \mathbf{h}(\mathbf{x}_0, \mathbf{u}_0) \\ \dots \\ \mathbf{h}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}) \\ \mathbf{h}(\mathbf{x}_N) \end{array}\right] \leq 0$$

Lagrange function in Multiple-Shooting

NLP:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_0 - \mathbf{x}_0 \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{bmatrix} = 0$$

$$\mathbf{h}(\mathbf{w}) = \begin{bmatrix} \mathbf{h}(\mathbf{x}_0, \mathbf{u}_0) \\ \dots \\ \mathbf{h}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}) \\ \mathbf{h}(\mathbf{x}_N) \end{bmatrix} \leq 0$$

Lagrange function:

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \Phi(\mathbf{w}) + \boldsymbol{\lambda}^\mathsf{T} \mathbf{g}(\mathbf{w}) + \boldsymbol{\mu}^\mathsf{T} \mathbf{h}(\mathbf{w})$$

Lagrange function in Multiple-Shooting

NLP:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_0 - \mathbf{x}_0 \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{bmatrix} = 0$$

$$\mathbf{h}(\mathbf{w}) = \begin{bmatrix} \mathbf{h}(\mathbf{x}_0, \mathbf{u}_0) \\ \dots \\ \mathbf{h}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}) \\ \mathbf{h}(\mathbf{x}_N) \end{bmatrix} \leq 0$$

Lagrange function:

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \Phi(\mathbf{w}) + \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{g}(\mathbf{w}) + \boldsymbol{\mu}^{\mathsf{T}} \mathbf{h}(\mathbf{w})$$

Then write:

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \underbrace{T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k, \mathbf{u}_k)}_{\Phi(\mathbf{w})}$$

Lagrange function in Multiple-Shooting

NLP:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_0 - \mathbf{x}_0 \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \ldots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{bmatrix} = 0$$

$$\mathbf{h}(\mathbf{w}) = \begin{bmatrix} \mathbf{h}(\mathbf{x}_0, \mathbf{u}_0) \\ \ldots \\ \mathbf{h}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}) \\ \mathbf{h}(\mathbf{x}_N) \end{bmatrix} \leq 0$$

Lagrange function:

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \Phi(\mathbf{w}) + \boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{w}) + \boldsymbol{\mu}^\top \mathbf{h}(\mathbf{w})$$

Then write:

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \underbrace{T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k, \mathbf{u}_k)}_{\Phi(\mathbf{w})} + \underbrace{\boldsymbol{\lambda}_0^\top (\bar{\mathbf{x}}_0 - \mathbf{x}_0) + \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top (\mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) - \mathbf{x}_{k+1})}_{\boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{w})}$$

Lagrange function in Multiple-Shooting

NLP:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_0 - \mathbf{x}_0 \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}(\mathbf{x}_N, \mathbf{u}_{N-1}) - \mathbf{x}_{N-1} \end{bmatrix} = 0$$

$$\mathbf{h}(\mathbf{w}) = \begin{bmatrix} \mathbf{h}(\mathbf{x}_0, \mathbf{u}_0) \\ \dots \\ \mathbf{h}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}) \\ \mathbf{h}(\mathbf{x}_N) \end{bmatrix} \leq 0$$

Lagrange function:

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \Phi(\mathbf{w}) + \boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{w}) + \boldsymbol{\mu}^\top \mathbf{h}(\mathbf{w})$$

Then write:

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \underbrace{T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k, \mathbf{u}_k)}_{\Phi(\mathbf{w})} + \underbrace{\boldsymbol{\lambda}_0^\top (\bar{\mathbf{x}}_0 - \mathbf{x}_0) + \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top (\mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) - \mathbf{x}_{k+1})}_{\boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{w})}$$

$$+ \underbrace{\boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \sum_{k=0}^{N-1} \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k)}_{\boldsymbol{\mu}^\top \mathbf{h}(\mathbf{w})}$$

Separability of the Lagrange function

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \underbrace{T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k, \mathbf{u}_k)}_{\Phi(\mathbf{w})} + \underbrace{\boldsymbol{\lambda}_0^\top (\bar{\mathbf{x}}_0 - \mathbf{x}_0) + \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top (\mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) - \mathbf{x}_{k+1})}_{\boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{w})}$$

$$+ \underbrace{\boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \sum_{k=0}^{N-1} \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k)}_{\boldsymbol{\mu}^\top \mathbf{h}(\mathbf{w})}$$

Separability of the Lagrange function

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k, \mathbf{u}_k) + \boldsymbol{\lambda}_0^\top (\bar{\mathbf{x}}_0 - \mathbf{x}_0) + \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top (\mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) - \mathbf{x}_{k+1})$$
$$+ \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \sum_{k=0}^{N-1} \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k)$$

Separability of the Lagrange function

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k, \mathbf{u}_k) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_0^\top \mathbf{x}_0 + \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) - \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{x}_{k+1}$$
$$+ \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \sum_{k=0}^{N-1} \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k)$$

Separability of the Lagrange function

$$\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) = T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_0^\top \mathbf{x}_0 + \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{x}_{k+1}$$

$$+ \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \sum_{k=0}^{N-1} \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k) =$$

$$T(\mathbf{x}_N) + \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_N^\top \mathbf{x}_N$$

$$+ \sum_{k=0}^{N-1} \Big(L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \boldsymbol{\lambda}_k^\top \mathbf{x}_k + \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k) \Big)$$

Separability of the Lagrange function

$$\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) = T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_0^\top \mathbf{x}_0 + \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{x}_{k+1}$$
$$+ \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \sum_{k=0}^{N-1} \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k) =$$
$$T(\mathbf{x}_N) + \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_N^\top \mathbf{x}_N$$
$$+ \sum_{k=0}^{N-1} \Big(L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \boldsymbol{\lambda}_k^\top \mathbf{x}_k + \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k) \Big)$$

Define:

$$\mathcal{L}_k(\mathbf{w}_k,\boldsymbol{\lambda},\boldsymbol{\mu}) = L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \boldsymbol{\lambda}_k^\top \mathbf{x}_k + \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k), \qquad k = 1,\ldots,N-1$$

Separability of the Lagrange function

$$\begin{aligned}\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) = & T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_0^\top \mathbf{x}_0 + \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{x}_{k+1} \\ & + \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \sum_{k=0}^{N-1} \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k) = \\ & T(\mathbf{x}_N) + \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_N^\top \mathbf{x}_N \\ & + \sum_{k=0}^{N-1} \Big(L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \boldsymbol{\lambda}_k^\top \mathbf{x}_k + \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k) \Big)\end{aligned}$$

Define:

$$\mathcal{L}_k(\mathbf{w}_k,\boldsymbol{\lambda},\boldsymbol{\mu}) = L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \boldsymbol{\lambda}_k^\top \mathbf{x}_k + \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k), \qquad k = 1, ..., N-1$$

$$\mathcal{L}_0(\mathbf{w}_0,\boldsymbol{\lambda},\boldsymbol{\mu}) = L(\mathbf{x}_0,\mathbf{u}_0) + \boldsymbol{\lambda}_1^\top \mathbf{f}(\mathbf{x}_0,\mathbf{u}_0) - \boldsymbol{\lambda}_0^\top \mathbf{x}_0 + \boldsymbol{\mu}_0^\top \mathbf{h}(\mathbf{x}_0,\mathbf{u}_0) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0$$

Separability of the Lagrange function

$$\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) = T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_0^\top \mathbf{x}_0 + \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{x}_{k+1}$$

$$+ \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \sum_{k=0}^{N-1} \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k) =$$

$$T(\mathbf{x}_N) + \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_N^\top \mathbf{x}_N$$

$$+ \sum_{k=0}^{N-1} \Big(L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \boldsymbol{\lambda}_k^\top \mathbf{x}_k + \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k) \Big)$$

Define:

$$\mathcal{L}_k(\mathbf{w}_k,\boldsymbol{\lambda},\boldsymbol{\mu}) = L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \boldsymbol{\lambda}_k^\top \mathbf{x}_k + \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k), \qquad k = 1,\dots,N-1$$

$$\mathcal{L}_0(\mathbf{w}_0,\boldsymbol{\lambda},\boldsymbol{\mu}) = L(\mathbf{x}_0,\mathbf{u}_0) + \boldsymbol{\lambda}_1^\top \mathbf{f}(\mathbf{x}_0,\mathbf{u}_0) - \boldsymbol{\lambda}_0^\top \mathbf{x}_0 + \boldsymbol{\mu}_0^\top \mathbf{h}(\mathbf{x}_0,\mathbf{u}_0) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0$$

$$\mathcal{L}_N(\mathbf{w}_N,\boldsymbol{\lambda},\boldsymbol{\mu}) = T(\mathbf{x}_N) - \boldsymbol{\lambda}_N^\top \mathbf{x}_N + \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N)$$

Separability of the Lagrange function

$$\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) = T(\mathbf{x}_N) + \sum_{k=0}^{N-1} L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_0^\top \mathbf{x}_0 + \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^\top \mathbf{x}_{k+1}$$

$$+ \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \sum_{k=0}^{N-1} \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k) =$$

$$T(\mathbf{x}_N) + \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0 - \boldsymbol{\lambda}_N^\top \mathbf{x}_N$$

$$+ \sum_{k=0}^{N-1} \Big(L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \boldsymbol{\lambda}_k^\top \mathbf{x}_k + \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k) \Big)$$

Define:

$$\mathcal{L}_k(\mathbf{w}_k,\boldsymbol{\lambda},\boldsymbol{\mu}) = L(\mathbf{x}_k,\mathbf{u}_k) + \boldsymbol{\lambda}_{k+1}^\top \mathbf{f}(\mathbf{x}_k,\mathbf{u}_k) - \boldsymbol{\lambda}_k^\top \mathbf{x}_k + \boldsymbol{\mu}_k^\top \mathbf{h}(\mathbf{x}_k,\mathbf{u}_k), \qquad k = 1, ..., N-1$$

$$\mathcal{L}_0(\mathbf{w}_0,\boldsymbol{\lambda},\boldsymbol{\mu}) = L(\mathbf{x}_0,\mathbf{u}_0) + \boldsymbol{\lambda}_1^\top \mathbf{f}(\mathbf{x}_0,\mathbf{u}_0) - \boldsymbol{\lambda}_0^\top \mathbf{x}_0 + \boldsymbol{\mu}_0^\top \mathbf{h}(\mathbf{x}_0,\mathbf{u}_0) + \boldsymbol{\lambda}_0^\top \bar{\mathbf{x}}_0$$

$$\mathcal{L}_N(\mathbf{w}_N,\boldsymbol{\lambda},\boldsymbol{\mu}) = T(\mathbf{x}_N) - \boldsymbol{\lambda}_N^\top \mathbf{x}_N + \boldsymbol{\mu}_N^\top \mathbf{h}(\mathbf{x}_N)$$

Then use $\mathbf{w}_k = \{\mathbf{x}_k, \mathbf{u}_k\}$ for $k = 0, ..., N-1$, and $\mathbf{w}_N \equiv \mathbf{x}_N$, so that

$$\boxed{\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) = \sum_{k=0}^{N} \mathcal{L}_k(\mathbf{w}_k,\boldsymbol{\lambda},\boldsymbol{\mu})}$$

Sparsity of the exact Hessian

Separability of the Lagrange function

$$\mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) = \sum_{k=0}^{N} \mathcal{L}_k\left(\mathbf{w}_k, \boldsymbol{\lambda}, \boldsymbol{\mu}\right)$$

where $\mathbf{w}_k = \{\mathbf{x}_k, \mathbf{u}_k\}$ for $k = 0, ..., N-1$, and $\mathbf{w}_N \equiv \mathbf{x}_N$.

Sparsity of the exact Hessian

Separability of the Lagrange function

$$\mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) = \sum_{k=0}^{N} \mathcal{L}_k\left(\mathbf{w}_k, \boldsymbol{\lambda}, \boldsymbol{\mu}\right)$$

where $\mathbf{w}_k = \{\mathbf{x}_k, \mathbf{u}_k\}$ for $k = 0, ..., N-1$, and $\mathbf{w}_N \equiv \mathbf{x}_N$. Hence:

$$\frac{\partial^2 \mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\right)}{\partial \mathbf{w}_i \partial \mathbf{w}_j} = 0, \quad \forall i \neq j$$

Sparsity of the exact Hessian

Separability of the Lagrange function

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{k=0}^{N} \mathcal{L}_k(\mathbf{w}_k, \boldsymbol{\lambda}, \boldsymbol{\mu})$$

where $\mathbf{w}_k = \{\mathbf{x}_k, \mathbf{u}_k\}$ for $k = 0, ..., N-1$, and $\mathbf{w}_N \equiv \mathbf{x}_N$. Hence:

$$\frac{\partial^2 \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial \mathbf{w}_i \partial \mathbf{w}_j} = 0, \quad \forall i \neq j$$

Hence the Hessian is **block diagonal**, i.e.

$$\nabla^2_{\mathbf{w}} \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \begin{bmatrix} \nabla^2_{\mathbf{w}_0} \mathcal{L}_0(\mathbf{w}_0, \boldsymbol{\lambda}, \boldsymbol{\mu}) & 0 & 0 & 0 \\ 0 & \nabla^2_{\mathbf{w}_1} \mathcal{L}_1(\mathbf{w}_1, \boldsymbol{\lambda}, \boldsymbol{\mu}) & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \nabla^2_{\mathbf{w}_N} \mathcal{L}_N(\mathbf{w}_N, \boldsymbol{\lambda}, \boldsymbol{\mu}) \end{bmatrix}$$

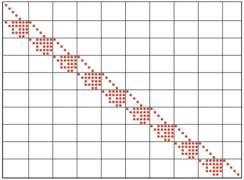
Sparsity pattern - Illustration

$$\min_{\Delta \mathbf{w}} \quad \frac{1}{2}\Delta \mathbf{w}^\top B \Delta \mathbf{w} + \nabla \Phi^\top \Delta \mathbf{w}$$

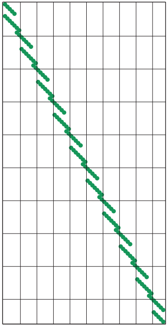
$$\text{s.t.} \quad \nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0$$

$$\nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} \leq 0$$

$\nabla \mathbf{g}(\mathbf{w})^\top$



$\nabla \mathbf{h}(\mathbf{w})^\top$



$B \equiv \nabla^2_{\mathbf{w}} \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda})$