AWESCO Winter School on Numerical Optimal Control with DAE - University of Freiburg

Exercise 11: High Index DAEs and Index Reduction

Rien Quirynen, Joel Andersson, Joris Gillis, Sebastien Gros, Moritz Diehl

A first toy example: Solutions

Let us consider the following system of Differential Algebraic Equations (DAE):

$$\begin{aligned} \dot{x}_1(t) - x_3(t) &= 0 \\ x_2(t)(1 - x_2(t)) &= 0 \\ x_1(t)x_2(t) + x_3(t)(1 - x_2(t)) &= t \end{aligned} \tag{1}$$

The following tasks should be carried out on pen and paper, so without using CasADi:

1.1 What is the differential index of the DAE system above?

The index is at least > 0*, since it is not an ODE system. The exact index will however depend on the value of* $x_2(t)$ *as discussed further. We could apply the definition of the differential index and start differentiating the equations with respect to time* t *until we obtain a pure ODE system.*

1.2 Does the index depend on the initial condition $[x_1(0), x_2(0), x_3(0)]$? If yes, how does the behavior of the system change exactly with respect to that initial condition?

The second equation reads:

$$x_2(t)(1 - x_2(t)) = 0,$$

such that x_2 *is either equal to* 0 *or* 1*. When* $x_2(t) = 0$*, the DAE can be rewritten as:*

$$\begin{aligned} \dot{x}_1(t) - x_3(t) &= 0 \\ x_3(t) &= t, \end{aligned}$$

which is of index 1*. When* $x_2(t) = 1$*, the system reads as:*

$$\begin{aligned} \dot{x}_1(t) - x_3(t) &= 0 \\ x_1(t) &= t, \end{aligned}$$

which is of index 2*.*

1.3 Derive the equivalent index-1 DAE system from the above set of equations, by differentiating with respect to time.

In case $x_2(t) = 0$*, the above DAE is already of index* 1 *so nothing needs to be done. However, we need to apply index reduction to the system in case* $x_2(t) = 1$*. Since the system is semi-explicit, we can only differentiate the algebraic equation* $x_1(t) = t$ *with respect to time and obtain:*

$$\begin{aligned} \dot{x}_1(t) &= x_3(t) \\ x_3(t) &= 1, \end{aligned}$$

after substituting the expression $\dot{x}_1(t) = x_3(t)$*.*

1.4 Additionally, write down the corresponding consistency conditions (if there are any) which are necessary to keep your DAE model equivalent to the original system.

For the latter case where $x_2(t) = 1$*, the equivalent DAE system reads as:*

$$\begin{aligned} \dot{x}_1(t) &= x_3(t) \\ x_2(t) &= 1 \\ x_3(t) &= 1, \end{aligned}$$

where additionally the consistency condition $x_1(t) = t$ *is needed. This means that there is no additional degree of freedom to choose an initial condition for this system and the solution reads* $[x_1(t), x_2(t), x_3(t)] = [t, 1, 1]$*. When* $x_2(t) = 0$*, the initial condition* $x_1(0)$ *for the index-*1 *DAE can still be chosen freely.*