Modelling and System Identification – Microexam 1 Solutions

Prof. Dr. Moritz Diehl, IMTEK, Universität Freiburg

November 18, 2014, 8:15-9:15, Freiburg

1. What is the probability density function (PDF) $p_X(x)$ for a normally distributed random variable X with mean μ and variance σ^2? The answer is $p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \ldots$

(a) ☐ $e^{\frac{(x-\mu)^2}{2\sigma}}$	(b) ☐ $e^{-\frac{(x-\mu)^2}{2\sigma}}$	(c) ☐ $e^{\frac{(x-\mu)^2}{2\sigma^2}}$	(d) ☒ $e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

2. What is the PDF of a variable Z with uniform distribution on the interval $[c, d]$? For $x \in [c, d]$ it has the value:

(a) ☐ $p_Z(x) = (d - c)$	(b) ☐ $p_Z(x) = (c - d)^2$	(c) ☐ $p_Z(x) = \frac{x}{\sqrt{d-c}}$	(d) ☒ $p_Z(x) = \frac{1}{d-c}$

3. What is the PDF of an n-dimensional normally distributed variable Z with zero mean and covariance matrix $\Sigma \succ 0$? The answer is $p_Z(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \ldots$

(a) ☐ $e^{-\frac{1}{2}x^T\Sigma x}$	(b) ☒ $e^{-\frac{1}{2}x^T\Sigma^{-1}x}$	(c) ☐ $e^{\frac{1}{2}x^T\Sigma x}$	(d) ☐ $e^{\frac{1}{2}x^T\Sigma^{-1}x}$

4. Regard a random variable $X \in \mathbb{R}^n$ with mean $d \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$. For a fixed $a \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$, regard another random variable Y defined by $Y = a + AX$. What is the mean μ_Y of Y ?

(a) ☐ $Y - a + AX$	(b) ☒ $a + Ad$	(c) ☐ AXX^TA^T	(d) ☐ $a^TAd + d^T\Sigma d$

5. Above in Question 4, what is the covariance matrix of Y?

(a) ☐ $d^T\Sigma d$	(b) ☒ $A\Sigma A^T$	(c) ☐ $A^T\Sigma^{-1}A$	(d) ☐ $A\Sigma^{-1}A^T$

6. Above in Question 4, which statement is true? Cov(Y) =

(a) ☐ $Y^\top Y - \mu_Y^\top \mu_Y$	(b) ☒ $\mathbb{E}\{YY^\top\} - \mu_Y\mu_Y^\top$
(c) ☐ $YY^\top - \mu_Y\mu_Y^\top$	(d) ☐ $\mathbb{E}\{Y^\top Y\} - \mu_Y^\top\mu_Y$

7. (*) Above in Question 4, what is the mean of the matrix valued random variable $Z = YY^T$?

(a) ☐ $(a + Ad)(a + Ad)^T$	(b) ☐ $aa^T + Add^TA^T + A\Sigma A^T$
(c) ☒ $(a + Ad)(a + Ad)^T + A\Sigma A^T$	(d) ☐ $aa^T + Add^TA^T$

8. A scalar random variable has the standard deviation y. What is its variance?

(a) ☐ $\sqrt{y}$	(b) ☒ y^2	(c) ☐ y	(d) ☐ y^{-1}

9. A scalar random variable has the variance w. What is its standard deviation?

(a) ☐ w	(b) ☐ w^{-1}	(c) ☐ w^2	(d) ☒ $\sqrt{w}$

10. Regard a random variable $\beta \in \mathbb{R}$ with zero mean and variance σ^2. What is the mean of the random variable $z = \beta^2$?

(a) ☐ $\beta + \sigma^2$	(b) ☐ σ	(c) ☒ σ^2	(d) ☐ $\beta + \sigma$

11. (*) Regard a random variable $X \in \mathbb{R}^n$ with zero mean and covariance matrix Σ. What is the mean of $Z = X^TX$?

(a) ☐ $\|\Sigma\|_F^2$	(b) ☐ $\det(\Sigma)$	(c) ☐ $\|\Sigma\|_2^2$	(d) ☒ $\mathrm{trace}(\Sigma)$

points on page: 11	

12. What is the minimizer x^* of the convex function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - 2x$?

(a) ☐ $x^* = -1$	(b) ☐ $x^* = 1$	(c) ☒ $x^* = \log_e(2)$	(d) ☐ $x^* = 0$

13. What is the minimizer x^* of the convex function $f : \mathbb{R} \to \mathbb{R}$, $\boxed{f(x) = \alpha + \beta x + \frac{1}{2}\gamma x^2}$ with $\gamma > 0$?

(a) ☐ $x^* = \frac{2\beta}{\alpha}$	(b) ☒ $x^* = -\frac{\beta}{\gamma}$	(c) ☐ $x^* = -\frac{\beta}{2\gamma}$	(d) ☐ $x^* = -\frac{\beta}{\alpha}$

14. What is the minimizer of the convex function $f : \mathbb{R}^n \to \mathbb{R}$, $\boxed{f(x) = \|y - \Phi x\|_2^2}$ (with Φ of rank n) ? The answer is $x^* = \dots$

(a) ☐ $-(\Phi\Phi^T)^{-1}\Phi^T y$	(b) ☐ $-(\Phi^T\Phi)^{-1}\Phi^T y$	(c) ☒ $(\Phi^T\Phi)^{-1}\Phi^T y$	(d) ☐ $(\Phi\Phi^T)^{-1}\Phi^T y$

15. What is the minimizer of the function $f : \mathbb{R}^n \to \mathbb{R}$, $\boxed{f(x) = \|b + B^T x\|_2^2}$ (with B^T of rank n)? The answer is $x^* = \dots$

(a) ☐ $(BB^T)^{-1}B^T b$	(b) ☒ $-(BB^T)^{-1}Bb$	(c) ☐ $(B^T B)^{-1}B^T b$	(d) ☐ $-(B^T B)^{-1}B^T b$

16. For a matrix $\Phi \in \mathbb{R}^{N\times d}$ with rank d (and $N \geq d$), what is its pseudo-inverse Φ^+ ?

(a) ☐ $(\Phi\Phi^T)^{-1}\Phi^T$	(b) ☐ $(\Phi\Phi^T)^{-1}\Phi$	(c) ☒ $(\Phi^T\Phi)^{-1}\Phi^T$	(d) ☐ $(\Phi^T\Phi)^{-1}\Phi$

17. Given a sequence of numbers $y(1), \dots, y(N)$, what is the minimizer θ^* of the function $\boxed{f(\theta) = \sum_{k=1}^N (y(k) - \theta)^2}$?

(a) ☐ $\frac{1}{N}\sum_{k=1}^N y(k)^2$	(b) ☒ $\frac{\sum_{k=1}^N y(k)}{N}$	(c) ☐ $\frac{1}{N^2}\sum_{k=1}^N y(k)^2$	(d) ☐ $\frac{N}{\sum_{k=1}^N y(k)}$

18. What does "i.i.d." stand for?

(a) ☐ infinite identically disturbed	(b) ☐ infinite identically dependent
(c) ☐ independent identically disturbed	(d) ☒ independent identically distributed

19. Given a sequence of i.i.d. scalar random variables $X(1), \dots, X(N)$, each with mean μ and variance σ^2, what is the expected value of their arithmetic mean, i.e. of the random variable Y defined by $\boxed{Y = \frac{1}{N}\sum_{k=1}^N X(k)}$?

(a) ☒ μ	(b) ☐ $\frac{\mu}{N}$	(c) ☐ $\frac{\mu}{\sigma^2}$	(d) ☐ $\frac{\mu}{\sqrt{\sigma^2}}$

20. In Question 19, what is the variance of the variable Y ?

(a) ☐ $\frac{\sigma}{N}$	(b) ☐ $\frac{\sigma}{N-1}$	(c) ☐ $\frac{\sigma^2}{N^2}$	(d) ☒ $\frac{\sigma^2}{N}$

21. Given a prediction model $\boxed{y(k) = \theta_1 + \theta_2 x(k)^2 + \epsilon(k)}$ with unknown parameter vector $\theta = (\theta_1, \theta_2)^T$, and assuming i.i.d. noise $\epsilon(k)$ with zero mean, and given a sequence of N scalar input and output measurements $x(1), \dots, x(N)$ and $y(1), \dots, y(N)$, we want to compute the linear least squares (LLS) estimate $\hat{\theta}_N$ by minimizing the function $f(\theta) = \|y_N - \Phi_N\theta\|_2^2$. If $y_N = (y(1), \dots, y(N))^T$, how do we need to choose the matrix $\Phi_N \in \mathbb{R}^{N\times 2}$?

(a) ☐ $\begin{bmatrix} x(1)^2 & 1 \\ \vdots & \vdots \\ x(1)^2 & 1 \end{bmatrix}$	(b) ☒ $\begin{bmatrix} 1 & x(1)^2 \\ \vdots & \vdots \\ 1 & x(N)^2 \end{bmatrix}$	(c) ☐ $\begin{bmatrix} 1 & x(1) \\ \vdots & \vdots \\ 1 & x(N) \end{bmatrix}$	(d) ☐ $\begin{bmatrix} 1 & -x(1) \\ \vdots & \vdots \\ 1 & -x(N) \end{bmatrix}$

points on page: 10	