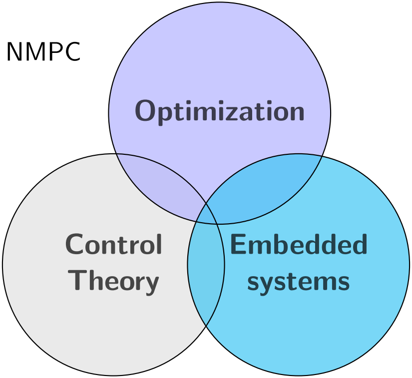
Efficient Numerical Methods for Embedded Optimization in AWE Applications

AWESCO Kickoff Meeting - Andrea Zanelli

System Control and Optimization Laboratory, IMTEK

Few words about me...

- Bachelors in Automation Technologies at Politecnico di Milano
- Masters in Robotics, Systems and Control at ETH Zurich
  - Internship at ABB Corporate Research Center on Embedded Model Predictive Control
  - Master thesis at Embotech on NMPC
  - Research assistant at ETH
- AWESCO fellow at ALUFR



...and my PhD topic

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Challenging control problems:

- strongly nonlinear
- unstable dynamics
- presence of constraints
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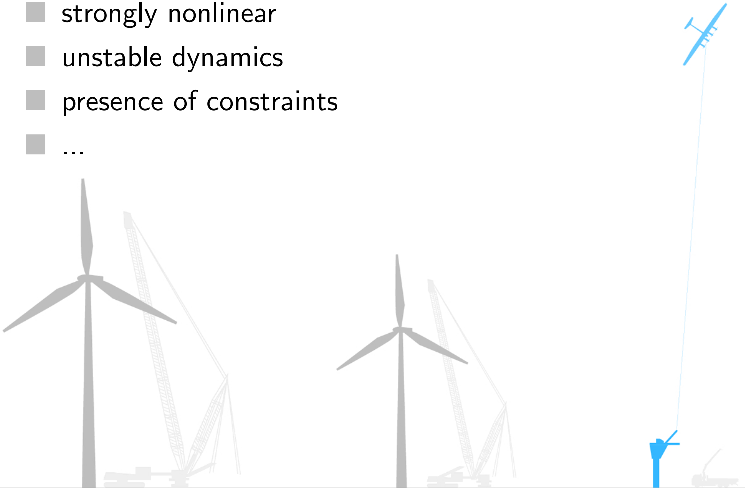

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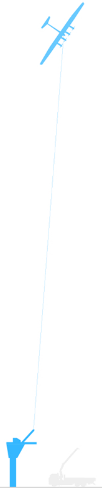
→ need for advanced control techniques

→ optimization-based solutions

→ several open questions and challenges



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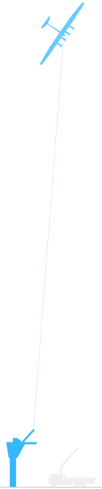
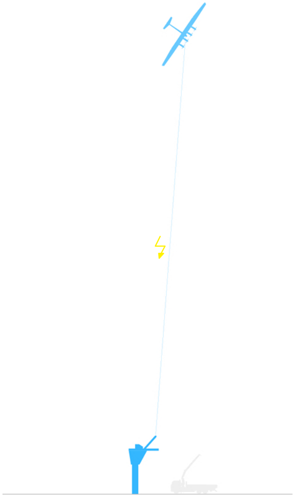
main focus: **efficient numerical methods for embedded optimization**



Efficient Numerical Methods for Embedded Optimization

$$\max_{x} \quad f(x)$$



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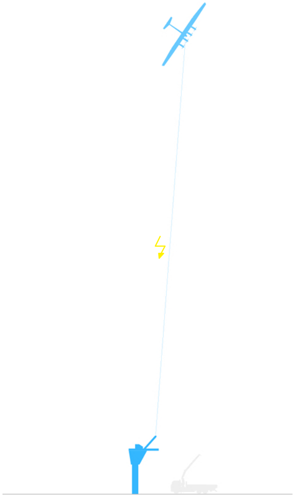

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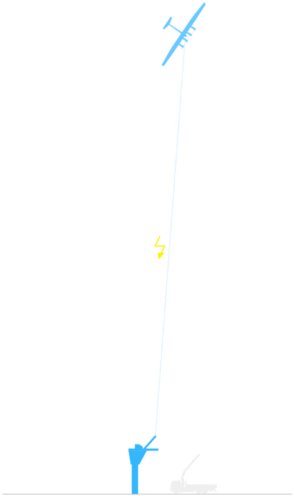

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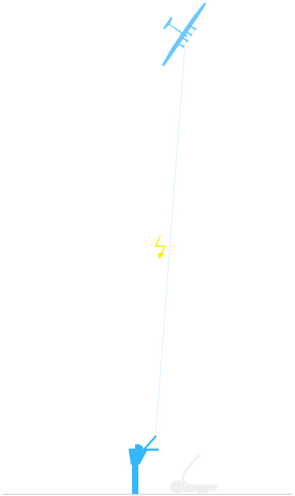

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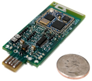

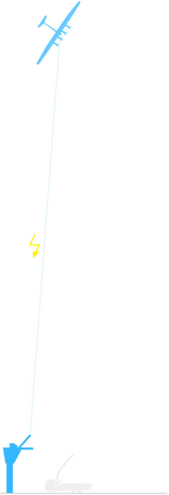

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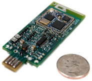

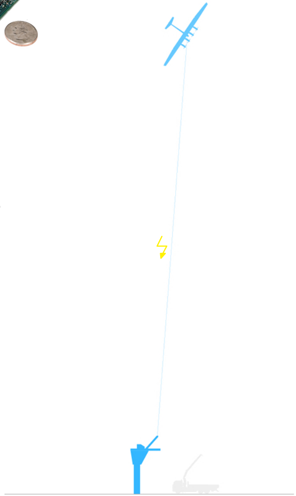

- integrate **smart decision making** into **embedded systems**

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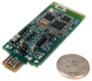

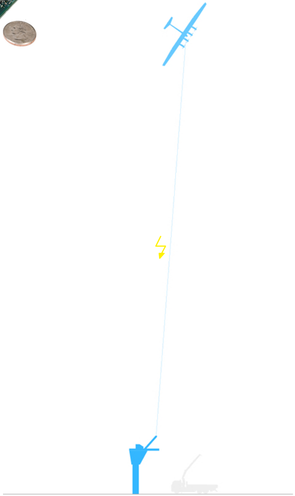

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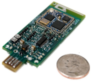

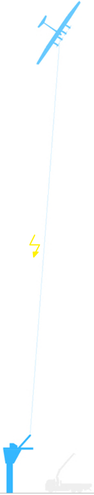

- integrate **smart decision making** into **embedded systems**
- challenges:
  - efficiency at **algorithmic level**
  - efficiency at **implementation level**

Sequential Quadratic Programming

<u>Problem formulation:</u>

$$\min_{\substack{x_0,\cdots,x_N\\u_0,\cdots,u_{N-1}}} \quad \frac{1}{2}\sum_{i=0}^{N-1}\left(x_i^T Q x_i + u_i^T R u_i\right)$$

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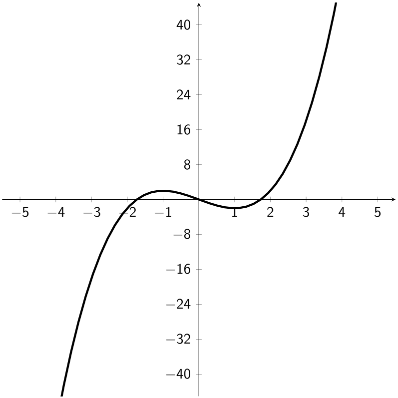

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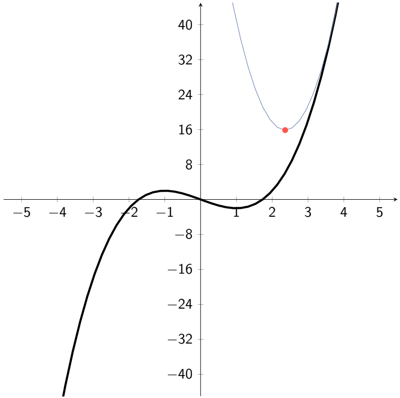

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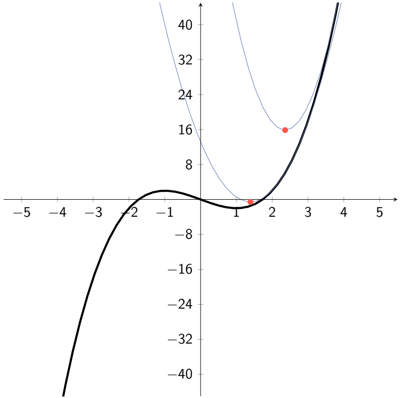

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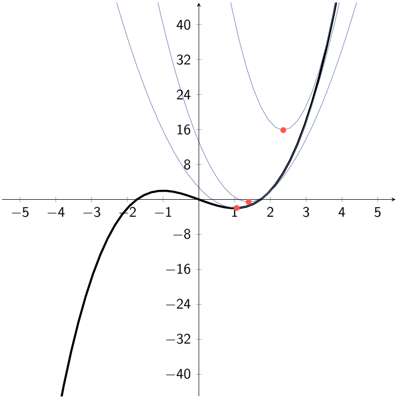

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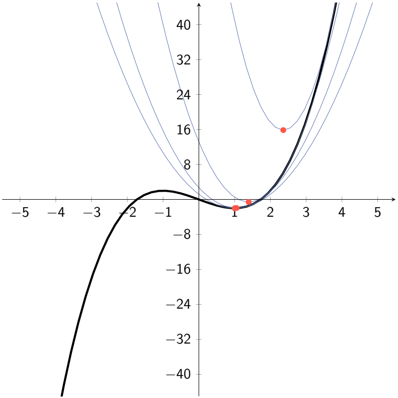

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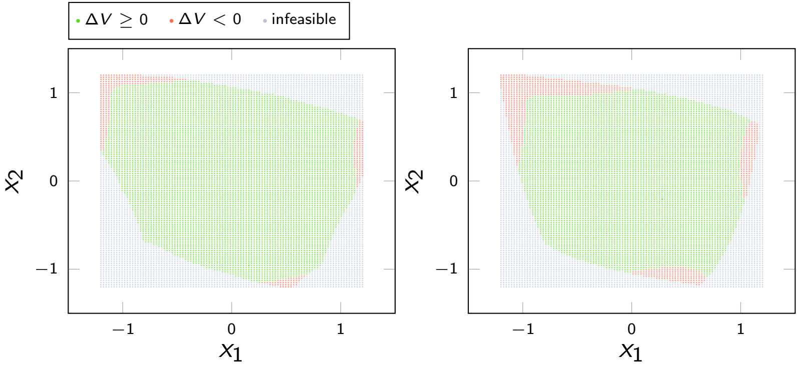
$\rightarrow$ **how is stability affected?**

Result: stability can be guaranteed



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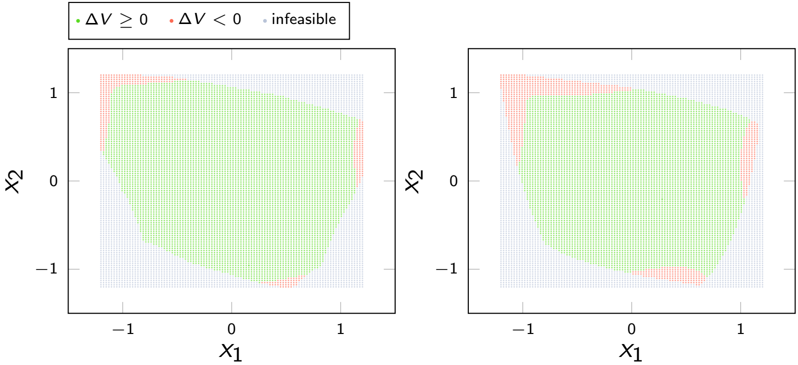

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→ **feasibility guaranteed**

→ **computational burden** reduced up to 70%

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- main operations boil down to **linear algebra** routines:
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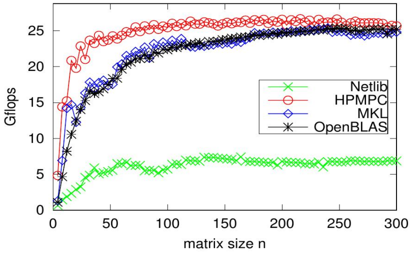
- how much do we gain? **a lot!** [Frison, 2013]



Scope and Project Plan

Objectives:

- tackle challenging control problems in AWE applications
- development of novel **numerical methods**
- **efficiency** at both algorithmic and implementation level

Secondments:

- Chalmers - 2 months
- Makani - 2 months
- ETH Zurich - 2 months