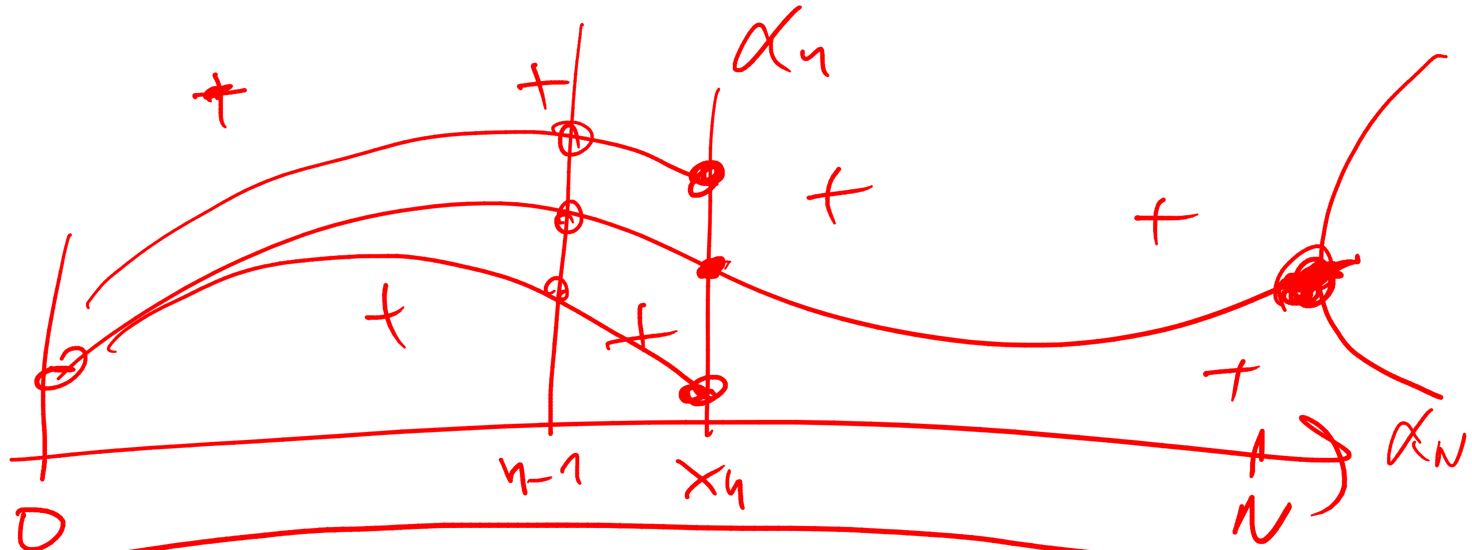
RECALL:



$$\min_{x,w} \; \underbrace{\sum_{n=0}^{N-1} \left[\beta_n(w_n) + \phi_n\left(y_n - g_n(x_n, w_n)\right) \right]}_{=:\, \ell_n(x_n, w_n)} + \alpha_0(x_0)$$

$$x_{n+1} - f_n(x_n, w_n) = 0 \qquad n = 0, \dots, N-1$$

MAP-ESTIMATE OF TRAJECTORY

12.3 OP FOR TR. EST. PROBLEM

DEF: ARRIVAL COST $\alpha_n(x_n)$

$$\boxed{\alpha_n(x_n) = \min_{\substack{x_0,\dots,x_{n-1} \\ w_0,\dots,w_{n-1}}} \alpha_0(x_0) + \sum_{k=0}^{n-1} \varphi_k(x_k, w_k) \quad \text{s.t.} \quad x_{k+1} = f_k(x_k, w_k),\ k=0,\dots,n-1}$$

x_n IS NOT OPT. VARIABLE, BUT PARAMETER (TERMINAL VALUE)

$$x_n = f_{n-1}(x_{n-1}, w_{n-1})$$

CONST.

DP-RECURSION

$$\boxed{\alpha_{n+1}(x_{n+1}) = \min_{x_n, w_n} \ \alpha_n(x_n) + \varphi_n(x_n, w_n) \quad \text{s.t.} \quad x_{n+1} = f_n(x_n, w_n)}$$

1. START $\alpha_0(\cdot)$
2. COMP. $\alpha_1, \ldots, \alpha_N$
3. $x_N^* = \arg\min_x \alpha_N(x)$
4. $(x_n^*, w_n^*) = \arg\min \ldots$

12.4 LQ TRAJ. ESTIMATION

$$f_k(x_k, w_k) = A_k \cdot x_k + b_k + w_k$$

$$g_k(x_k, w_k) = C_k \cdot x_k$$

$$\beta_k(w_k) = \frac{1}{2} \|w_k\|_R^2$$

$$\Phi_k(v_k) = \frac{1}{2} \|v_k\|_Q^2$$

$$\alpha_0(x_0) = \frac{1}{2} \|x_0 - \bar{x}_0\|_{P_0}^2$$

$$\left[\|w\|_R^2 = w^T R w \right]$$

$$x_0 \sim \mathcal{N}(\bar{x}_0, P_0^{-1})$$

$$\underset{x,\,w}{\text{minimize}} \quad \frac{1}{2}\|x_0-\bar{x}_0\|_{P_0}^2 + \sum_{n=0}^{N-1} \frac{1}{2}\|y_n - C_n x_n\|_Q^2 + \frac{1}{2}\|w_n\|_R^2$$

$$\text{s.t.} \quad x_{n+1} - A_n x_n - b_n - \underline{\underline{w_n}} = 0$$

$$n=0,\dots,N-1$$

$$\Updownarrow$$

$$\min_{x} \quad \underbrace{\frac{1}{2}\|x_0-\bar{x}_0\|_{P_0}^2}_{\alpha_0} + \sum_{n=0}^{N-1} \frac{1}{2}\|y_n - C_n x_n\|_Q^2 + \frac{1}{2}\|x_{n+1} - A_n x_n - b_n\|_R^2$$

$$\alpha_{n+1}(x_{n+1}) = \min_{x_n} \quad \alpha_n(x_n) + \frac{1}{2}\|y_n - C_n \cdot x_n\|_Q^2 + \frac{1}{2}\|x_{n+1} - A_n x_n - b_n\|_R^2$$

FIXED

... IT TURNS OUT ALL α_n ARE QUADRATIC

LEMMA 2 (SCHUR COMPL LEMMA)

IF $R \succ 0$ THEN

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = x^T (Q - S^T R^{-1} S) x + \| R^{-1} S x + u \|_R^2$$

IN PARTICULAR:

$$\min_u \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = x^T (Q - S^T R^{-1} S) x$$

IF IN ADDITION $\begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \succeq 0$ THEN ALSO

$$Q - S^T R^{-1} S \succeq 0$$

$$\begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} = \left[\begin{array}{c|c} Q - S^T R^{-1} S & 0 \\ \hline 0 & 0 \end{array}\right] + \underbrace{\left[\begin{array}{c|c} S^T R^{-1} S & S^T \\ \hline S & R \end{array}\right]}$$

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \left[\begin{array}{c|c} S^T R^{-1} S & S^T \\ \hline S & R \end{array}\right] \begin{bmatrix} x \\ u \end{bmatrix} = \ldots$$

$$= x^T S^T R^{-1} \underbrace{R \quad R^{-1}} S x + 2 u^T \underset{RR^{-1}}{\wedge} S x + u^T R u$$

$$= \| R^{-1} S x + u \|_R^2$$

COR. 3 : LINEAR QUADRATIC (IF $R \succ 0$)

$$\begin{bmatrix} 1 \\ x \\ \hline u \end{bmatrix}^T \left[\begin{array}{cc|c} c & q^T & s^T \\ q & Q & S^T \\ \hline s & S & R \end{array}\right] \begin{bmatrix} 1 \\ x \\ \hline u \end{bmatrix}$$

$$= c - s^T R^{-1} s + 2 x^T (q - S^T R^{-1} s) + x^T (Q - S^T R^{-1} S) x$$

$$\| R^{-1}(s + Sx) + u \|_R^2$$

BACK TO DP:

$$\alpha_{n+1}(x_{n+1}) = \min_{x_n} \overbrace{\frac{1}{2}\|x_n - \bar{x}_n\|^2_{P_n} + \frac{1}{2}\|y_n - C_n x_n\|^2_{Q} + \frac{1}{2}\|x_{n+1} - A_n x_n - b_n\|^2_{R}}^{=*}$$

$$* = \text{const} + \frac{1}{2}\begin{bmatrix} 1 \\ x_{n+1} \\ x_n \end{bmatrix}^T \left[\begin{array}{c|cc} \text{const} & -b_n^T R & \\ -Rb_n & \boxed{R} & \boxed{-RA_n} \\ \hline -(P_n\bar{x}_n + C_n^T Q y_n + A_n^T R b_n) & \boxed{-A_n^T R} & \boxed{P_n + C_n^T Q C_n + A_n^T R A_n} \end{array}\right] \begin{bmatrix} 1 \\ x_{n+1} \\ x_n \end{bmatrix}$$

$$\alpha_{n+1}(x_{n+1}) = \min_{x_n} \begin{bmatrix} 1 \\ x_{n+1} \\ x_n \end{bmatrix}^T \left[\begin{array}{c|c} \ddots & \\ \hline & \oslash \end{array}\right] \begin{bmatrix} 1 \\ x_{n+1} \\ x_n \end{bmatrix}$$

$$= x_{n+1}^T \left(R - (\dots)\right) x_{n+1} + x_{n+1}^T g_n + \text{const}$$

$$= \|x_{n+1} - \bar{x}_{n+1}\|^2_{P_{n+1}} + \text{const}$$

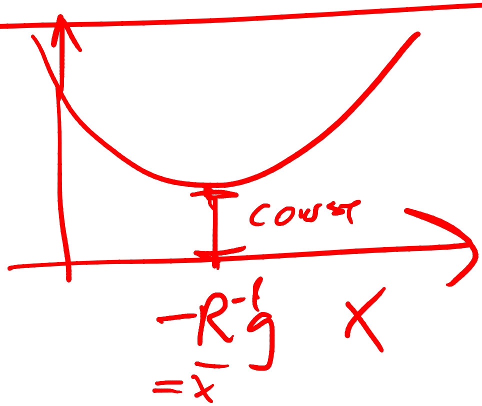
$P_{n+1} = R - \dots$ (Schur compl)

$\bar{x}_{n+1} = ?$

Lemma 3: $\frac{1}{2} x^T P x + g^T x$

IF $P \succ 0$ $\quad = \text{const} + \frac{1}{2} \| x - \bar{x} \|_P^2$

with $\bar{x} = -P^{-1} g$



$Px + g = 0$

$$\boxed{\alpha_{n+1}(x_{n+1}) = \frac{1}{2} \| x_{n+1} - \bar{x}_{n+1} \|_{P_{n+1}}^2 + \text{const}}$$

12.5 RECURSIVE BAYESIAN ESTIMATION

IDEALLY, WE WANT TO KNOW:

$$P(x_{n+1} \mid y_0, \ldots, y_n)$$

IDEA:

COMPUTE IT FROM $P(x_n \mid y_0, \ldots, y_{n-1})$

AND y_n

$$\boxed{P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}}$$

$B = y_n$

$$P(x_{n+1} \mid y_0, \ldots, y_n) = \int P(x_{n+1} \mid x_n)\, P(x_n \mid y_0, \ldots, \underline{y_n})\, dx_n \qquad (12.72)$$

$$= \iint P(x_{n+1} \mid x_n, w_n)\, P(x_n, w_n \mid y_0, \ldots, y_n)\, dx_n\, dw_n$$

$$= \iint_{x_{n+1} = f_n(x_n, w_n)} P(x_n, w_n \mid y_0, \ldots, \underline{\underline{y_n}})\, dx_n\, dw_n$$

$$= \iint_{x_{n+1} = f_n(x_n, w_n)} \frac{P(x_n, w_n \mid y_0, \ldots, y_{n-1}) \cdot P(y_n \mid x_n, w_n)}{\underbrace{P(y_n \mid y_0, \ldots, y_{n-1})}_{\text{const}}}\, dx_n\, dw_n$$

$$= \text{const.} \iint_{x_{n+1} = f_n(x_n, w_n)} \underline{P(w_n)} \cdot P(x_n \mid y_0, \ldots, y_{n-1}) \cdot P(y_n \mid x_n, w_n)\, dx_n\, dw_n$$

$$= \text{const} \cdot \int \underbrace{e^{-\rho_n(v_n)}}_{f_n(x_n, w_n) = x_{n+1}} \cdot \underline{\underline{p(x_n | y_0, \dots, y_{n-1})}} \cdot e^{-\phi_n(y_n - g_n(x_n))} \, dw_n \cdot dx_n$$

CURSE OF DIMENSIONALITY

12.6 ESTIMATION OF LAST STATE FOR LIN. SYS. WITH GAUSSIAN NOISE