HINTS FOR EX 7, TASK 2:

1) LOOK AT $\nabla_{x_0}\left[f(f(f(f(x_0))))\right]$

(LECTURE RECORDING)

2) LOOK AT 5.2.2 IN SCRIPT

3) EXAMPLE $x_2 = x_1^2 + u_1$, $x_1 = x_0^2 + u_0$

$\frac{dx_2}{du_0}$, $\frac{dx_2}{du_1}$ $\quad x_2 = (x_0^2 + u_0)^2 + u_1$

USING REVERSE AD

INPUTS: u_0, u_1

x_0 FIXED

(1) $x_1 = x_0^2 + u_0$

(2) $x_2 = x_1^2 + u_1$

OUTPUT:

x_2

INTERNAL VARS: x_1, x_2
+ INPUTS u_0, u_1

$\bar{u}_0 = 0, \bar{u}_1 = 0, \bar{x}_1 = 0, \bar{x}_2 = 1$

(8.166)

(2): $\bar{x}_1 = \cancel{\bar{x}_1} + \overset{1}{\bar{x}_2} \cdot (2x_1)$

$\bar{u}_1 = \cancel{\bar{u}_1} + \bar{x}_2 \cdot (1)$

(8.16.e)

(1) $\bar{u}_0 = \cancel{\bar{u}_0} + \bar{x}_1 \cdot (1)$

$\frac{dx_2}{dx_1}$

OUTPUT:

$\begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \end{bmatrix} = \nabla_u x_2$

$\frac{dx_1}{d\cancel{x_0}u_0}$ $\frac{dx_2}{du_1}$

$x_1 = x_0^2 + u_0$ $x_2 = x_1^2 + u_1$	FORWARD SWEEP
$\bar{x}_1 = \cancel{\bar{x}_2}\, 2x_1$, $\bar{u}_1 = 1$ $\bar{u}_0 = \bar{x}_1$.	BACKWARD SWEEP

RECALL: SPARSE QP SOLUTION IN

- SIMULTANEOUS APPROACH
- NEWTON TYPE OPT. ALG

A) CONDENSING: ELIMINATE ALL STATES FROM QP (IN LINEARIZED SYSTEM)

B) SPARSE KKT SYSTEM SOLVE ~ RICCATI RECURSION

A) GOOD FOR SHORT HORIZONS

B) GOOD FOR LONG HORIZONS

CH 7, RECAP. & 7.7 / 7.8 (SKIP: 7.5, 7.6)

$$\boxed{\begin{aligned} \min_{\substack{x_0,\dots,x_N \\ u_0,\dots,u_{N-1}}} \quad & \sum_{k=0}^{N-1} L(x_k,u_k) + E(x_N) \\ \text{s.t.} \quad & x_0 = \bar{x}_0 \\ & x_{k+1} = f(x_k,u_k), \quad k=0,\dots,N-1 \end{aligned}}$$

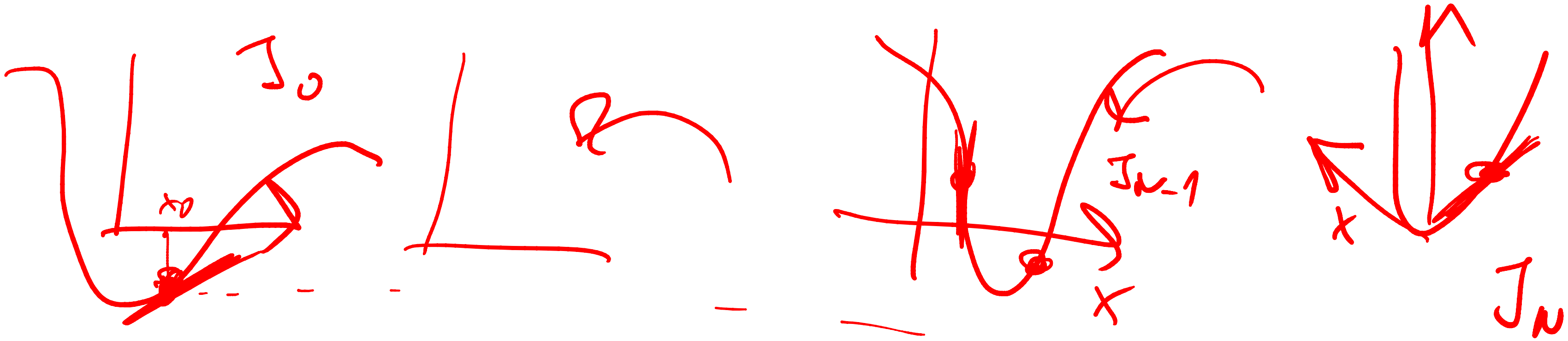
COST-TO-GO J_k

$$J_N(x) = E(x)$$

$$\rightarrow \boxed{J_k(x) = \min_u \underbrace{L(x,u) + J_{k+1}(f(x,u))}_{=:\tilde{J}_k(x,u)}} \quad \text{DP-RECURSION}$$

$$\boxed{u_k^*(x) = \arg\min_u \tilde{J}_k(x,u)}$$

7.7 GRADIENT OF J_k



AIM: EVALUATE $J_k(x_k)$ AND $\nabla_x J_k(x_k)$ ON <u>OPTIMAL</u> TRAJECTORY $(x_0, \dots, x_N)$, $(u_0, \dots, u_{N-1})$

$$\boxed{J_k(x_k) = L(x_k, u_k) + J_{k+1}(f(x_k, u_k))}$$

$$= L(x_k, u_k) + J_{k+1}(x_{k+1})$$

$\rightarrow$ IF $L \equiv 0$

$J_k(x_k)$ REMAINS CONSTANT

$$J_n(x) = L(x, u_n^*(x)) + J_{n+1}\left(f(x, u_n^*(x))\right)$$

$$\frac{dJ_n}{dx} = \nabla J_n^T \qquad \boxed{\nabla J_N(x_N) = \nabla E(x_N)} \qquad (7.13)$$

$$\frac{dJ_n}{dx} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial u}\cdot\frac{\partial u^*}{\partial x} + \frac{\partial J_{n+1}}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u^*}{\partial x}\right)$$

$$= \left(\frac{\partial L}{\partial x} + \frac{\partial J_{n+1}}{\partial x}\cdot\frac{\partial f}{\partial x}\right) + \cancel{\underbrace{\left(\frac{\partial L}{\partial u} + \frac{\partial J_{n+1}}{\partial x}\cdot\frac{\partial f}{\partial u}\right)}_{=0 \;=\; \frac{\partial \tilde{J}_n}{\partial u}}\cdot\frac{\partial u^*}{\partial x}}$$

$$\nabla J_n(x_n) = \nabla_x L(x_n, u_n) + \frac{\partial f}{\partial x}(x_n, u_n)^T \cdot \nabla J_{n+1}(x_{n+1})$$

$$\lambda_n \equiv \nabla J_n(x_n)$$

$$u_n^* = \arg\min \underbrace{L(x,u) + J_{n+1}(f(x,u))}_{\tilde{J}_n}$$

$$\Downarrow \quad \frac{\partial \tilde{J}_n}{\partial u} = 0$$

cf. AD& (5.16d)

AT OPTIMAL SOLUTION

$$\boxed{\lambda_k = \nabla J_k(x_k)}, \quad k=0,\dots,N$$

$$\lambda_N = \nabla E(x_N)$$

$$\left.\begin{aligned} \lambda_k &= \nabla_x L(x_k,u_k) + \frac{\partial f}{\partial x}(x_k,u_k)^T \cdot \lambda_{k+1} \\ 0 &= \nabla_u L(x_k,u_k) + \frac{\partial f}{\partial u}(x_k,u_k)^T \cdot \lambda_{k+1} \end{aligned}\right\} k=0,\dots,N-1$$

7.8 NECESSARY OPTIMALITY CONDITIONS:

AT OPT. TRAJ. $x_0, \dots, x_N$, $u_0, \dots, u_{N-1}$

THERE EXIST VECTORS $\lambda_N, \dots, \lambda_0$

(GRADIENTS OF VALUE FUNCTION, MULTIPLIERS, $\bar{x}_0, \dots$)

SO THAT:

$$x_0 = \bar{x}_0$$

$$x_{k+1} = f(x_k, u_k) \qquad k = 0, \dots, N-1$$

$$\lambda_N = \nabla E(x_N)$$

$$\lambda_k = \nabla_x L(x_k, u_k) + \frac{\partial f}{\partial x}(x_k, u_k)^T \cdot \lambda_{k+1} \qquad k = 0, \dots, N-1$$

$$\rightarrow 0 = \nabla_u L(x_k, u_k) + \frac{\partial f}{\partial x}(x_k, u_k)^T \cdot \lambda_{k+1}$$

OR

$$\rightarrow \left[u_k = \arg\min_u L(x_k, u) + \lambda_{k+1}^T \cdot f(x_k, u) \right]$$

CH 8: CONT. TIME OCP

$$\underset{x(\cdot),\,u(\cdot)}{\text{minimize}} \quad \int_0^T L(x(t), u(t))\, dt + E(x(T))$$

subject to

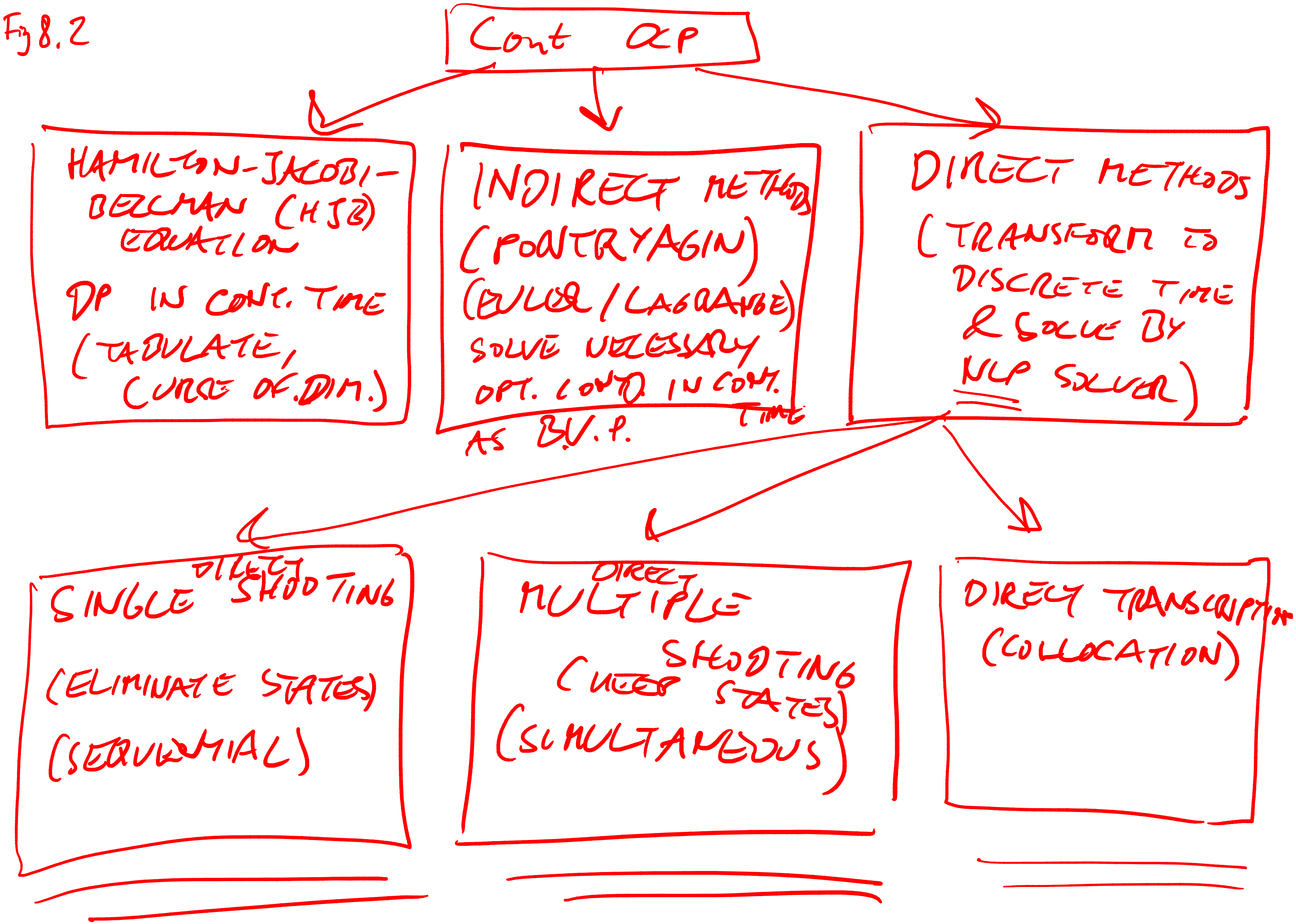
$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [0, T]$$

$$x(0) = \bar{x}_0$$

$$h(x(t), u(t)) \le 0 \qquad t \in [0, T]$$

$$r(x(T)) \le 0$$

Fig 8.2


Cont OCP
HAMILTON-JACOBI-BELLMAN (HJB) EQUATION
DP IN CONT. TIME
(TABULATE, CURSE OF DIM.)
INDIRECT METHODS
(PONTRYAGIN)
(EULER/LAGRANGE)
SOLVE NECESSARY OPT. COND. IN CONT. TIME AS B.V.P.
DIRECT METHODS
(TRANSFORM TO DISCRETE TIME & SOLVE BY NLP SOLVER)
SINGLE DIRECT SHOOTING
(ELIMINATE STATES)
(SEQUENTIAL)
DIRECT MULTIPLE SHOOTING
(KEEP STATES)
(SIMULTANEOUS)
DIRECT TRANSCRIPTION
(COLLOCATION)

CH 9: 9.1 & 9.2 "SHOOTING METHODS"

IDEA:

TRANSFORM ODE & COST INTEGRAL INTO DISCRETE TIME

$$\dot{x} = f_c(x,u)$$

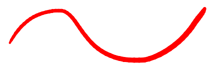
$$\downarrow$$

$$x_{n+1} = f_d(x_n, u_n)$$

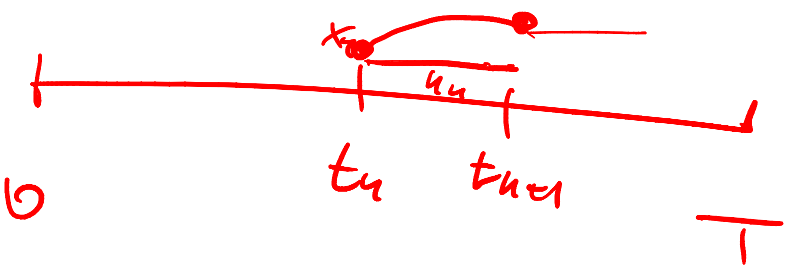
(1) DISCRETIZE CONTROLS

E.G. PIECEWISE CONSTANT ON A GRID

(OR POLYNOMIALS, FOURIER SERIES, ...)

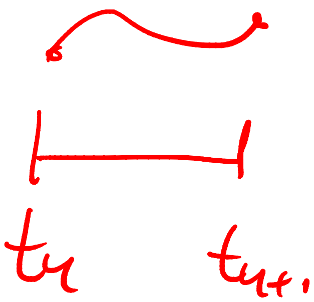


$$0 = t_0, t_1, \dots, t_N = T$$



(2) INTEGRATE STATE/ODE FROM START TO END TO GET $f_d(x_n, u_n)$ & $x(t; x_n, u_n)$ $t \in [t_n, t_{n+1}]$

$$L_d(x_k,u_k) := \int_{t_k}^{t_{k+1}} L(x(t;x_k,u_k),\,u_k)\,dt$$



$$c(t_k) = 0$$

$$\dot{c} = L(x,u)$$

$$c(t_{k+1}) = L_d(x_k,u_k)$$

OCP (*)

$$\min_{\substack{x_0,\ldots,x_N \\ u_0,\ldots,u_{N-1}}} \sum_{k=0}^{N-1} L_d(x_k,u_k) + E(x_N)$$

$$x_0 = \bar{x}_0$$

$$x_{k+1} = f_d(x_k,u_k), \quad k=0\ldots N-1$$

$$h(x_k,u_k) \leq 0, \quad k=0,\ldots,N-1$$

$$r(x_N) \leq 0$$

OCP IN DISCRETE TIME (SIMULTANEOUS)

(BOCK & PLITT 1984) $\equiv$ <u>DIRECT MULTIPLE SHOOTING</u>

SOLVE OCP (*) WITH SEQUENTIAL APPROACH :

DIRECT SINGLE SHOOTING

(SARGENT & RAY 1979)

$$\boxed{\begin{aligned} &\min_{u_0,\ldots,u_{N-1}} \quad \phi(u_0,\ldots,u_{N-1}) \\ & h(x(t_k; u_0,\ldots,u_{N-1}), u_k) \le 0 \qquad k=0,\ldots,N-1 \\ & r(x(t_N; u_0,\ldots,u_{N-1})) \le 0 \end{aligned}}$$