CH5 - REP.

SEQUENTIAL

- ELIMINATE ALL STATES
- NLP IN CONTROLS ONLY
- NLP NOT SPARSE (REDUCED NLP)

SIMULTANEOUS

- KEEP STATES IN NLP (FULL SPACE)
- NLP SPARSE (TODAY)

COMPUTE GRADIENT OF OBJECTIVE IN RED. NLP VIA BACKWARD AD $\longrightarrow$ BAR QUANTITIES SAME AS MULTIPLIERS

$$\boxed{\lambda_{k+1}} \quad \Big\| \quad f(x_k, u_k) - x_{k+1} = 0$$

CH6: SPARSITY OF OCP (SIMULTANEOUS)

$$\underset{x_0, u_0, x_1, u_1, \dots, x_N}{\text{minimize}} \quad \sum_{k=0}^{N-1} L_k(x_k, u_k) + E(x_N)$$

$$\cancel{\phi_k(x_k, u_k, x_{k+1}) = 0}$$

λ_{k+1} s.t. $\quad f_k(x_k, u_k) - x_{k+1} = 0, \quad k = 0, \dots, N-1$

λ_r $\quad \sum_{k=0}^{N-1} r_k(x_k, u_k) + r_N(x_N) = 0$ (COUPLED CONSTRAINTS)

μ_k $\quad h_k(x_k, u_k) \leq 0, \quad k = 0, \dots, N-1$

μ_N $\quad h_N(x_N) \leq 0$

DECOUPLED CONSTRAINTS

$$\boxed{\min_w \; F(w) \quad \text{s.t.} \quad G(w) = 0, \; H(w) \leq 0}$$

$$\mathcal{L}(w, \lambda, \mu) = F(w) + \lambda^T G(w) + \mu^T H(w) = \dots$$

EX: $x_0 - x_N = 0 \qquad r_0 = x_0, \; r_1 = 0, \dots, r_{N-1} = 0, \; r_N = -x_N$

6.1 PARTIAL SEPARABILITY OF LAGRANGIAN

DEF 14 (PARTIAL SEPARABILITY): $f:\mathbb{R}^n \to \mathbb{R}$ is <u>PARTIALLY SEPARABLE</u> IFF IT IS A SUM OF $f_j:\mathbb{R}^{n_j} \to \mathbb{R}$ WITH $j=1,\dots,m$ AND $n_j < n$ WITH SUBSETS $I_j \subset \{1,\dots,n\}$ AND SUBVECTORS x_{I_j} SUCH THAT:

$$\boxed{f(x) = \sum_{j=1}^{m} f_j(x_{I_j})}$$

EX:

$$f(x_1,x_2,x_3) = \underline{x_1 \cdot x_2} + \underline{x_1 \cdot x_3} + x_3 \cdot x_2$$

$$= f_1(x_1,x_2) + f_2(x_1,x_3) + f_3(x_2,x_3)$$

$I_1 = \{1,2\}$ $\quad x_{I_1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$I_2 = \{1,3\}$ $\quad x_{I_2} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$

$I_3 = \{2,3\}$ $\quad x_{I_3} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$

$$\nabla^2 f = \nabla^2 f_1 + \nabla^2 f_2 + \nabla^2 f_3$$

$$= \begin{bmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} x & 0 & x \\ 0 & 0 & 0 \\ x & 0 & x \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & x \\ 0 & x & x \end{bmatrix}$$

$$= \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$$

THM: $\mathcal{L}(w,\lambda,\mu)$ IS PARTIALLY SEPARABLE, I.E.

$$\boxed{\mathcal{L}(w,\lambda,\mu) = \sum_{k=0}^{N} \mathcal{L}_k(w_k,\lambda,\mu)} \qquad w = \begin{bmatrix} w_1 \\ \vdots \\ \\ w_{N-1} \\ w_N \end{bmatrix}$$

WITH $w_k = \begin{bmatrix} x_k \\ u_k \end{bmatrix}$ FOR $k=0,\dots,N-1$ — $w_N := x_N$

AND FOR $k=1,\dots,N-1$

$$\mathcal{L}_k(w_k,\lambda,\mu) = L_k(x_k,u_k) + \lambda_{k+1}^T f_k(x_k,u_k) - \lambda_k^T x_k + \mu_k^T h_k(x_k,u_k) + \lambda_r^T r_k(x_k,u_k)$$

$$\mathcal{L}_0(w_0 \dots) = \quad '' \qquad '' \qquad \cancel{} \qquad ''$$

$$\mathcal{L}_N(w_N,\ ) = E(x_N) - \lambda_N^T x_N + \mu_N^T h_N(x_N) + \lambda_r^T r_N(x_N)$$

$$\nabla_w^2 \mathcal{L} = \sum_{k=0}^{N} \nabla_w^2 \mathcal{L}_k = \begin{bmatrix} * & \boxed{*} & 0 & \cdots \\ \boxed{*} & * & & \\ 0 & & 0 & \\ \vdots & & & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \\ 0 & 0 & & 0 \cdots \\ 0 & * & * & 0 \cdots \\ & * & * & \\ 0 & 0 & 0 & \ddots \end{bmatrix} + \dots + \begin{bmatrix} 0 & \\ & \boxed{*} \end{bmatrix}$$

$$= \nabla_w^2 \mathcal{L}_0 + \nabla_w^2 \mathcal{L}_1 + \dots + \nabla_w^2 \mathcal{L}_N$$

$$\boxed{\frac{\partial^2 \mathcal{L}}{\partial w_k \partial w_j} = 0 \quad \text{For } k \neq j}$$

$$\begin{bmatrix} \boxed{Q_1} & & 0 \\ & \boxed{Q_2} & \\ & & \ddots & \\ 0 & & & \boxed{Q_N} \end{bmatrix} = \begin{bmatrix} \boxed{+} & & & & \\ & \boxed{+} & & & \\ & & \boxed{+} & & \\ & & & \boxed{+} & \\ & & & & \ddots \; \boxed{+} \\ & & & & & \boxed{*} \end{bmatrix}$$

$$\boxed{Q_k := \nabla_{w_k}^2 \mathcal{L}_k(w_k, \lambda, \mu)}$$

$$Q_k \in \mathbb{R}^{n_{w_k} \times n_{w_k}}$$

$$n_{w_k} = n_x + n_u \quad (k = 0, \dots, N-1), \quad n_{w_N} = n_x$$

NO LECTURE THIS FRIDAY

(FREE, READ CH 1 - 6)
+ 7

NEXT MONDAY: LECTURER

DIMITRIS KOUZOURIS

(SPARSE QP SOLVERS, 6.3)

6.2 SPARSE QP (IN SQP FOR OCP)

RECALL: SQP

$$\begin{aligned} \min_{w} \quad & F(w) \\ \text{s.t.} \quad & G(w) = 0 \\ & H(w) \leq 0 \end{aligned}$$

$$\begin{bmatrix} w \\ \lambda \\ \mu \end{bmatrix} \longrightarrow$$

QP

$$\begin{aligned} \min_{\Delta w} \quad & \nabla F(w)^T \Delta w + \frac{1}{2} \Delta w^T \overbrace{\nabla_w^2 \mathcal{L}(w,\lambda,\mu)}^{B_k} \Delta w \\ \text{s.t.} \quad & G(w) + \nabla G(w)^T \Delta w = 0 \\ & H(w) + \nabla H(w)^T \Delta w \leq 0 \end{aligned}$$

$$\begin{bmatrix} w^+ \\ \lambda^+ \\ \mu^+ \end{bmatrix} = \begin{bmatrix} w + \Delta w^{QP} \\ \lambda^{QP} \\ \mu^{QP} \end{bmatrix} \longleftarrow \begin{bmatrix} \Delta w^{QP} \\ \lambda^{QP} \\ \mu^{QP} \end{bmatrix}$$

$$\nabla_w^2 \mathcal{L} = \begin{bmatrix} Q_0 & & \\ & \ddots & \\ & & Q_N \end{bmatrix}$$

$$G(w) = \begin{bmatrix} f_0(x_0, u_0) - x_1 \\ \vdots \\ f_{N-1}(\cdots) - x_N \\ \hline \sum r_k(x_k,u_k) + c_N \end{bmatrix} \qquad H(w) = \begin{bmatrix} h_0 \\ \vdots \\ h_N \end{bmatrix}$$

$$\underline{\underline{\nabla H(w)^T}} = \frac{\partial H}{\partial w}(w) = \begin{bmatrix} \nabla_w h_0(x_0,u_0)^T \\ \nabla_w h_1(x_1,u_1)^T \\ \vdots \\ \nabla_w h_N(x_N)^T \end{bmatrix} = \begin{array}{cccccc} x_0 & u_0 & x_1 & u_1 & \cdots & x_N \\ \end{array}\begin{bmatrix} \boxed{* \; *} & 0 & 0 & \cdots & & \\ 0 & 0 & * & * & 0 & \cdots \\ & & & & & \\ - & & - & - & \cdot & 0 \;|\; * \end{bmatrix}$$

$$H(w) + \nabla H(w)^T \Delta w = \begin{bmatrix} h_0(x_0,u_0) + H_0 \cdot \Delta w_0 \\ \vdots \\ h_n(x_n,u_n) + H_n \Delta w_n \\ \vdots \\ h_N(x_N) + H_N \Delta x_N \end{bmatrix} = \begin{bmatrix} \boxed{H_0} & & & \\ & \boxed{H_1} & & 0 \\ 0 & & \boxed{H_{N-1}} & \\ & & & \boxed{H_N} \end{bmatrix}$$

$$\Delta w_n = \begin{bmatrix} \Delta x_n \\ \Delta u_n \end{bmatrix}$$

BLOCK DIAGONAL

$$\boxed{h_n(x_n,u_n) + H_n \begin{bmatrix} \Delta x_n \\ \Delta u_n \end{bmatrix} \leq 0}$$

$k = 0, \ldots, N-1$

$$G(w) + DG(w)^T \Delta w = \begin{bmatrix} \underbrace{\left(f_0(x_0,u_0) - x_1\right)}_{=:a_0} + A_0 \cdot \Delta x_0 + B_0 \cdot \Delta u_0 - I \cdot \Delta x_1 \\ \vdots \\ a_{N-1} + A_{N-1} \Delta x_{N-1} + B_{N-1} \Delta u_{N-1} - I \cdot \Delta x_N \\ \hline \underbrace{\left(\sum r_k + r_N\right)}_{=:r} + \sum_{k=0}^{N-1} R_k \cdot \Delta w_k + R_N \cdot \Delta w_N \end{bmatrix}$$

$$A_k := \frac{\partial f_k}{\partial x_k}(x_k, u_k)$$

$$B_k = \frac{\partial f_k}{\partial u_k}(x_k, u_k)$$

$$R_k = \frac{\partial r_k}{\partial w_k}(x_k, u_k)$$

$$R_N = \frac{\partial r_N}{\partial x_N}(x_N)$$

$$DG^T = \begin{array}{c} \begin{array}{cccccc} x_0 & u_0 & x_1 & u_1 & \cdots & x_N \end{array} \\ \begin{bmatrix} A_0 & B_0 & -I & & & & \\ & & A_1 & B_1 & -I & & \\ & & & & \ddots & & \\ & & & & A_{N-1} & B_{N-1} & -I \\ \hline \boxed{R_0} & & \boxed{R_1} & & \cdots & & \boxed{R_N} \end{bmatrix} \end{array}$$

(QP - DETAILED FORM)

$$\underset{\Delta x_0, \Delta u_0, \ldots, \Delta x_N}{\text{minimize}} \quad \sum_{k=0}^{N-1} \left(\begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix}^T g_k + \frac{1}{2} \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix}^T Q_k \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix} \right)$$

$$+ g_N^T \Delta x_N + \frac{1}{2} \Delta x_N^T Q_N \Delta x_N$$

s.t.

$$a_k + A_k \Delta x_k + B_k \Delta u_k - \Delta x_{k+1} = 0 \quad , k = 0, \ldots, N-1$$

$$r + \sum_{k=0}^{N-1} R_k \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix} + R_N \Delta x_N = 0$$

$$h_k + H_k \cdot \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix} \leq 0$$

$$h_N + H_N \Delta x_N \leq 0$$

$$r = \sum_{k=0}^{N-1} r_k(x_k, u_k) + r_N(x_N)$$

$$g_k = \nabla_{w_k} L(x_k, u_k)$$

$$
\begin{bmatrix} Q_0 \cdot \Delta w_0 \\ Q_1 \cdot \Delta w_1 \\ \vdots \\ Q_N \cdot \Delta w_N \end{bmatrix}
=
\underbrace{\begin{bmatrix} Q_0 & & \\ & \ddots & \\ & & Q_N \end{bmatrix}}_{\nabla^2 \mathcal{L}}
\begin{bmatrix} \Delta w_0 \\ \vdots \\ \Delta w_N \end{bmatrix}
$$

$$
\Delta w^T \nabla^2 \mathcal{L}\, \Delta w =
\begin{bmatrix} \Delta w_0 \\ \vdots \\ \Delta w_N \end{bmatrix}^T
\begin{bmatrix} Q_0\, \Delta w_0 \\ \vdots \\ Q_N\, \Delta w_N \end{bmatrix}
= \sum_{k=0}^{N} \Delta w_k^T Q_k \Delta w_k
$$