RECALL:

$$F: \mathbb{R}^n \to \mathbb{R}^{n_F}$$

$$J_F = \frac{\partial F}{\partial x}$$

FORWARD ALGOR. DIFF (AD): $p \in \mathbb{R}^n$

$$\text{cost}\left(J_F \cdot p\right) \leq 2 \cdot \text{cost}(F)$$

$$\text{cost}(J_F) \leq 2 \cdot n \cdot \text{cost}(F) \qquad p = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \dots \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

BACKWARD AD (REVERSE / ADJOINT): $\lambda \in \mathbb{R}^{n_F}$

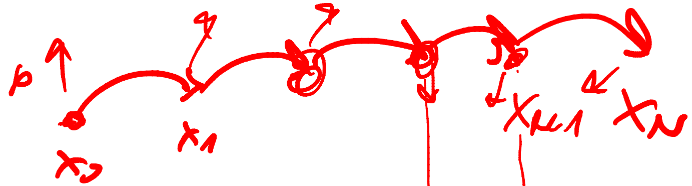
$$\text{cost}\left(\lambda^T J_F\right) \leq 3 \cdot \text{cost}(F)$$

$$\text{cost}(J_F) \leq 3 \cdot n_F \cdot \text{cost}(F) \qquad \lambda = \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{n_F}$$

"CHEAP GRADIENT"

NEED MEMORY

EXERCISE : $F(x) = \phi(\phi(\dots \phi(x_0)))$



$$x_{k+1} = \phi(x_k) \qquad k = 0, \dots, N-1$$

$$F(x_0) := x_N \qquad x_k \in \mathbb{R}^{n_x}$$

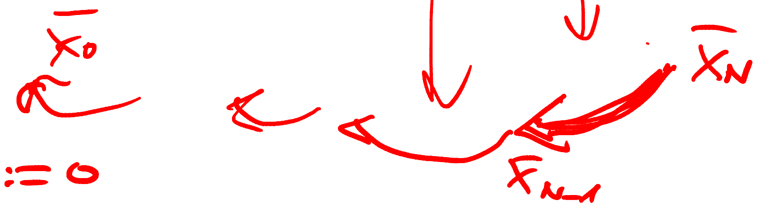
FORWARD AD:

$$J_F(x_0)\cdot p = ?$$

$$\boxed{\dot{x}_0 = p}$$

$$\boxed{x_{k+1} = \phi(x_k) \quad \Big| \quad \dot{x}_{k+1} = \frac{\partial \phi}{\partial x}(x_k)\cdot \dot{x}_k \qquad k = 0, \dots, N-1}$$

$$\dot{x}_N = J_F(x_0)\cdot p$$



BACKWARD AD

$$\lambda^T J_F(x_0) = ?$$

$$\bar{x}_N = \lambda, \quad \bar{x}_0 = \dots = \bar{x}_{N-1} := 0$$

FOR $k = N-1, \dots, 0$

$$\bar{x}_k = \cancel{\bar{x}_k} + \left(\frac{\partial \phi}{\partial x}(x_k)\right)^T \cdot \bar{x}_{k+1}$$

END

$$\bar{x}_N =: \lambda_N \qquad \lambda \equiv \lambda_N$$

$$\bar{x}_k = \lambda_k$$

$$\boxed{\lambda^T J_F(x_0) = \bar{x}_0^T}$$

BACKWARD AD

$$\lambda_N^T \cdot \frac{d x_N}{d x_0} = ?$$

FOR $k=0,\dots,N-1$

$x_{k+1} = \phi(x_k)$

END

FOR $k=N-1,\dots,0$

$$\lambda_k = \frac{\partial \phi}{\partial x}(x_k)^T \cdot \lambda_{k+1}$$

END

$$\lambda_N^T \frac{d x_N}{d x_0} = \lambda_0^T$$

AD-TOOLS: ADOL-C (C++)
CppAD
ADIC (ANSI C)
ADIFOR
TAPENADE
TAFF ...

CasADi

ACADO Integrators

CH 5 OPTIMAL CONTROL FORMULATIONS

5.1

$x_k \in \mathbb{R}^{n_x}$

$u_k \in \mathbb{R}^{n_u}$

$f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$

NLP

$$\boxed{\begin{aligned} \underset{x_0, u_0, x_1, u_1, \ldots, x_N}{\text{minimize}} \quad & \sum_{k=0}^{N-1} L(x_k, u_k) + E(x_N) \\ \text{subject to} \quad & x_{k+1} - f(x_k, u_k) = 0, \quad k = 0, \ldots, N-1 \\ & h(x_k, u_k) \leq 0, \quad k = 0, \ldots, N-1 \\ & r(x_0, x_N) = 0 \end{aligned}}$$

(OCP)

Ex:

$r(x_0, x_N) = x_0 - \bar{x}_0$ $\quad$ ($\bar{x}_0 \in \mathbb{R}^{n_x}$ FIXED/CONSTANT)

$r(x_0, x_N) = x_0 - x_N$

$$r(x_0, x_N) = \begin{bmatrix} x_0 - \bar{x}_0 \\ x_N - \bar{x}_N \end{bmatrix}$$

$p_{k+1} = p_k$

$$\tilde{x}_k = \begin{bmatrix} x_k \\ p_k \end{bmatrix}$$

SIMULTANEOUS APPROACH

SOLVE (OCP) BY NLP-SOLVER
(KEEP $x_0, x_1, \ldots x_N$ AS VARIABLES)

TO BE EFFICIENT, NEED
SPARSITY EXPLOITATION

SEQUENTIAL APPROACH

ELIMINATE $x_1, \ldots, x_N$
AS FUNCTION OF $(x_0, u_0, u_1, \ldots u_{N-1})$
BY FORWARD SIMULATION,

SOLVE REDUCED NLP

$$\underset{x_0, \underbrace{u_0, u_1, \ldots, u_{N-1}}_{U}}{\text{minimize}} \quad \ldots \phi(x_0, U)$$

s.t.

$$h(\bar{x}_n(x_0, U), u_n) \leq 0 \quad n = 0, \ldots, N-1$$

$$r(x_0, \bar{x}_N(x_0, U)) = 0$$

BY NLP SOLVER
(ONLY x_0, AND $U = (u_0, \ldots, u_{N-1})$
AS VARIABLES)

OPTIM.

SIMULATION

5.2 ANALYSIS OF OCP (SIMULT.)

$$\boxed{\begin{aligned} \underset{x,u}{\text{minimize}} \quad & \underbrace{\sum L(x_k,u_k) \qquad + E(x_N)}_{=:F(w)} & \\ \text{s.t.} \quad & f(x_k,u_k) - x_{k+1} = 0 & k=0,\dots,N-1 \\ & r(x_0,x_N) = 0 & \end{aligned}}$$

$$w = \begin{pmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ x_{N-1} \\ u_{N-1} \\ x_N \end{pmatrix}$$

$$w \in \mathbb{R}^{n_w}$$

$$n_w = N \cdot (n_x + n_u) + n_x$$

$$G(w) = \begin{bmatrix} f(x_0,u_0) - x_1 \\ \vdots \\ f(x_{N-1},u_{N-1}) - x_N \\ r(x_0,x_N) \end{bmatrix} \qquad \lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \\ \lambda_r \end{bmatrix}$$

$$\boxed{\min_{w} \quad F(w) \quad \text{s.t.} \quad G(w) = 0}$$

$$\mathcal{L}(w,\lambda) = F(w) + \lambda^T G(w)$$

$$= \sum_{k=0}^{N-1} L(x_k,u_k) + E(x_N) + \sum_{k=0}^{N-1} \lambda_{k+1}^T \left(f(x_k,u_k) - x_{k+1}\right) + \lambda_r^T r(x_0,x_N)$$

$$= \sum_{k=0}^{N-1} \left[L(x_k,u_k) + \lambda_{k+1}^T f(x_k,u_k) \right] - \sum_{k=1}^{N} \left(\lambda_k^T x_k\right) + E(.) + \lambda_r^T r(.)$$

$$\mathcal{L}(w,\lambda) = \sum_{k=0}^{N-1} \left[L(x_k,u_k) + \lambda_{k+1}^T f(x_k,u_k) \right] - \sum_{k=1}^{N} \lambda_k^T x_k + E(x_N) + \lambda_r^T r(x_0,x_N)$$

KKT-OPTIMALITY COND:

$$\boxed{\begin{aligned} \nabla_w \mathcal{L}(w,\lambda) &= 0 \\ G(w) &= 0 \end{aligned}}$$

$$\nabla_w \mathcal{L} = \begin{bmatrix} \nabla_{x_0} \mathcal{L} \\ \nabla_{u_0} \mathcal{L} \\ \vdots \\ \nabla_{u_{N-1}} \mathcal{L} \\ \nabla_{x_N} \mathcal{L} \end{bmatrix}$$

$\nabla_w \mathcal{L} = ?$

$$\nabla_{x_0} \mathcal{L}(w,\lambda) = \nabla_{x_0} L(x_0,u_0) + \left(\lambda_1^T \frac{\partial f}{\partial x}(x_0,u_0)\right)^T + \left(\lambda_r^T \frac{\partial r}{\partial x_0}(x_0,x_N)\right)^T \overset{!}{=} 0 \quad (1)$$

$k = 1, \dots, N-1$

$$\nabla_{x_k} \mathcal{L} = \nabla_{x_k} L(x_k,u_k) + \frac{\partial f}{\partial x}(x_k,u_k)^T \cdot \lambda_{k+1} - \lambda_k \overset{!}{=} 0 \quad (2)*$$

$$\nabla_{x_N} \mathcal{L} = -\lambda_N + \nabla E(x_N) + \frac{\partial r}{\partial x_N}(x_0,x_N)^T \cdot \lambda_r \overset{!}{=} 0 \quad (3)$$

$$\nabla_{u_k} \mathcal{L} = \nabla_{u_k} L(x_k,u_k) + \frac{\partial f}{\partial u}(x_k,u_k)^T \lambda_{k+1} \overset{!}{=} 0 \quad k = 0, \dots, N-1 \quad (4)*$$

$f(x_k,u_k) - x_{k+1} = 0 \qquad k = 0, \dots, N-1 \qquad (5)*$

$r(x_0,x_N) = 0 \qquad (6)$

$r(x_0, x_N) = x_0 - \bar{x}_0$ (SIMPLY FIXED INITIAL VALUE)

REORDER KKT-CONDITIONS

$$\left.\begin{array}{lll} (6) & x_0 = \bar{x}_0 & \\ (5)^* & x_{k+1} = f(x_k, u_k) & k = 0, \dots, N-1 \end{array}\right\} \text{FORWARD}$$

$$\left.\begin{array}{lll} (3) & \lambda_N = \nabla E(x_N) & \\ (2)^* & \lambda_k = \nabla_x L(x_k, u_k) + \left(\frac{\partial f}{\partial x}(x_k, u_k)\right)^T \cdot \lambda_{k+1}, & k = N-1, \dots, 1 \\ (1) & -\lambda_r = \nabla_x L(x_0, u_0) + \frac{\partial f}{\partial x}(x_0, u_0)^T \lambda_1 & \boxed{\lambda_0 := -\lambda_r} \end{array}\right\} \text{BACKW AD}$$

$$(4)^* \quad 0 = \nabla_u L(x_k, u_k) + \frac{\partial f}{\partial u}(x_k, u_k)^T \lambda_{k+1}, \quad k = 0, \dots, N-1$$

SIMULTANEOUS:
SOLVE ABOVE EQS WITH NEWTON-TYPE OPTIMIZATION

SEQUENTIAL: ELIMINATE $x_0, x_1, \dots, x_N$ BY (6) & (5)*

VARS: U

ELIM: $\lambda_N, \dots, \lambda_0$ BY (3) & (2)* & (1)

(4)* $\boxed{R(U) = 0}$ $\quad R(U) = \nabla_U \tilde{F}(U)$ (BY REVERSE AD)

$$\boxed{\tilde{F}(\bar{U}) = \sum_{k=0}^{N-1} L(\bar{x}_k(\bar{U}), u_k) + E(\bar{x}_N(\bar{U}))}$$ SEQ.

$$\nabla \tilde{F}(U) = \begin{bmatrix} \nabla_{u_0} \mathcal{L} \\ \vdots \\ \nabla_{u_{N-1}} \mathcal{L} \end{bmatrix}$$