2 OPTIMIZATION

$$\min_{x \in \mathbb{R}^n} f(x)$$
$$\text{s.t.} \quad g(x) = 0$$
$$h(x) \le 0$$

CONSTRAINT h_i is <u>ACTIVE</u> at a FEASIBLE $\bar{x}$ IFF

$$h_i(\bar{x}) = 0$$

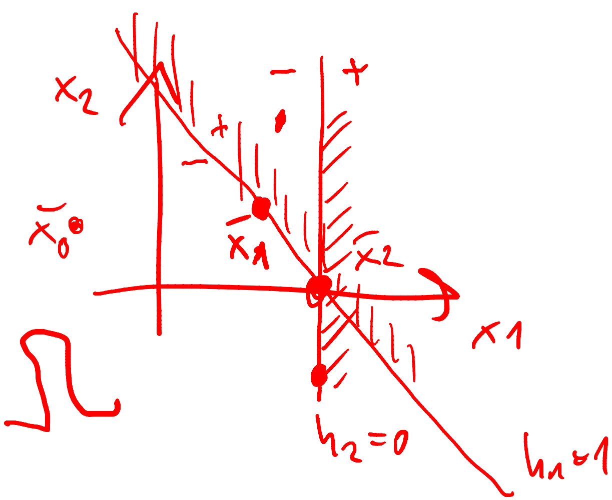
SET OF ALL i SO THAT h_i IS ACTIVE IS CALLED <u>ACTIVE SET</u>

$$\mathcal{A} := \{ i \mid h_i(\bar{x}) = 0 \}$$

$$\mathcal{A}(\bar{x})$$

$$h_1(x) = x_1 + x_2 - 1$$

$$h_2(x) = x_1 - 1$$



$$\mathcal{A}(\bar{x}_0) = \{ \quad \} = \emptyset$$

$$\mathcal{A}(\bar{x}_1) = \{ 1 \}$$

$$\mathcal{A}(\bar{x}_2) = \{ 1, 2 \}$$

2.1 SPECIAL CLASSES

LP:

$$\boxed{\begin{aligned} &\min_{x\in\mathbb{R}^n} && c^T x \\ &\text{s.t.} && Ax-b=0 \\ & && Cx-d\leq 0 \end{aligned}}$$

WITH c, A, b, C, d FIXED MATRICES OF APPR. DIMENSION

(LINEAR f, g, h)

QP:

$$\boxed{\begin{aligned} &\min_{x} && c^T x + \frac{1}{2} x^T B x \\ &\text{s.t.} && Ax-b=0 \\ & && Cx-d\leq 0 \end{aligned}}$$

$f(x) = c^T x + \frac{1}{2} x^T B x$

$\nabla f(x) = c + Bx$

$\nabla^2 f(x) = B$

B "HESSIAN"

IF $B \succcurlyeq 0$ QP IS "CONVEX QP"

IF $B \succ 0$ "STRICTLY CONVEX QP"

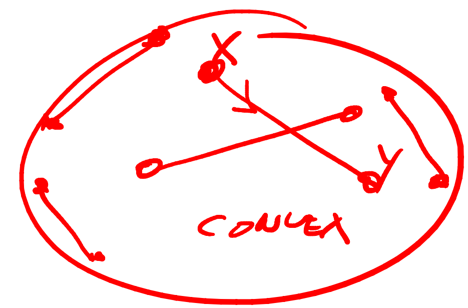
OTHERWISE: NON-CONVEX QP

CONVEX OPTIMIZATION:

DEF: SET $\Omega \subset \mathbb{R}^n$ IS "CONVEX SET" IFF

$$\boxed{\forall x,y \in \Omega:\ x + t(y-x) \in \Omega \quad \forall t \in [0,1]}$$

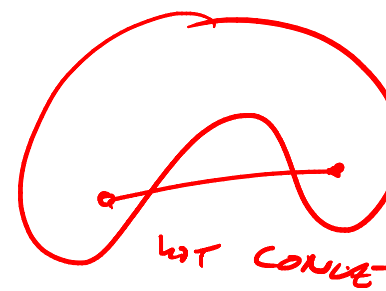
$$(1-t)x + t \cdot y \in \Omega$$

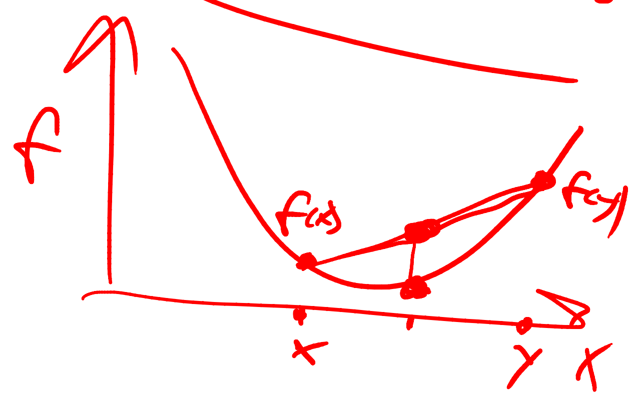


DEF: $f: \Omega \to \mathbb{R}$ IS "CONVEX FCN" IFF Ω IS CONVEX SET AND

$$\boxed{\forall x,y \in \Omega\ \forall t \in [0,1]:\quad f\big(x + t(y-x)\big) \le f(x) + t\big(f(y) - f(x)\big)}$$

$$(1-t)f(x) + t\, f(y)$$





DEF: CONVEX OPTIMIZATION PROBLEM

FEASIBLE SET Ω, OBJECTIVE $f : \Omega \to \mathbb{R}$

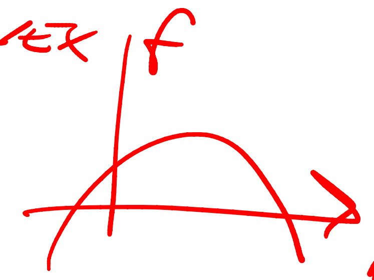
$$\boxed{\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad x \in \Omega}$$

IS CONVEX IFF

Ω IS CONVEX &

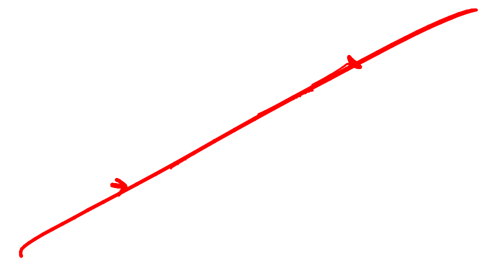
f IS "

DEF: f "CONCAVE" IFF $(-f)$ IS CONVEX
(FOR FUNCTIONS ONLY)



THM 2: SUBLEVEL SET $\{x \in \Omega \mid h(x) \leq 0\}$ OF CONVEX FCN $h : \Omega \to \mathbb{R}$ IS CONVEX SET.

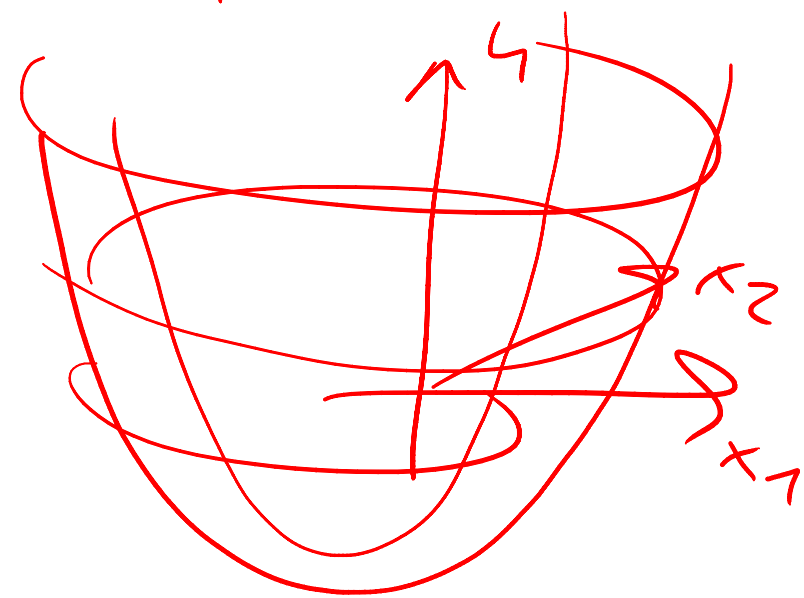
EXAMPLE: A) $h(x) = -1 + x_1 + x_2$



AFFINE: CONVEX (CONCAVE)

$\{x \in \mathbb{R}^2 \mid -1 + x_1 + x_2 \le 0\}$ CONVEX SET

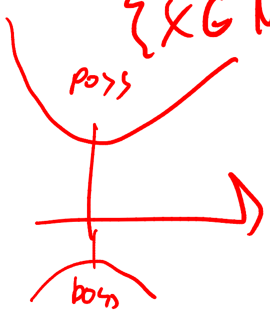
B) $h(x) = x_1^2 + x_2^2 - 1$



$\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 - 1 \le 0\}$ CONVEX

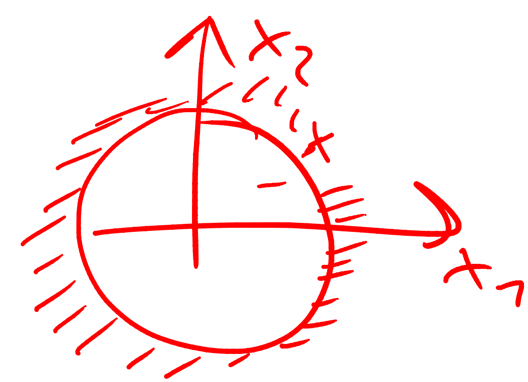
C) $h(x) = x_1^2 - x_2^2 + 1$

$\{x \in \mathbb{R}^2 \mid x_1^2 - x_2^2 + 1 \le 0\}$ CONVEX?



$x_2^2 \ge x_1^2 + 1$

$x_2 \ge \pm\sqrt{x_1^2 + 1}$



THM 4: C^2 FUNCTION $f: \Omega \to \mathbb{R}$, Ω OPEN AND CONVEX

f CONVEX IFF $\nabla^2 f(x) \succeq 0 \quad \forall x \in \Omega$

EX: A) $f(x) = c^T x + a$ $\qquad \nabla f(x) = c \qquad \nabla^2 f(x) = 0$

$\succeq 0$

B) $f(x) = c^T x + \frac{1}{2} x^T B x$ $\qquad \nabla^2 f(x) = B$

CONVEX $\iff$

$B \succeq 0$

C) $f(x) = e^x$ $\qquad \nabla^2 f(x) = e^x \qquad e^x > 0 \Rightarrow$ CONVEX

$\cancel{D) f(c^T x + a)}$ $\quad$ D) $\boxed{f(x) = \exp(c^T x + a)}$

$$\boxed{e^{(c^T x + a)} + \|x\|_2^2 + c^T x \leq 0}$$

$\nabla f(x) = c \cdot f(x)$

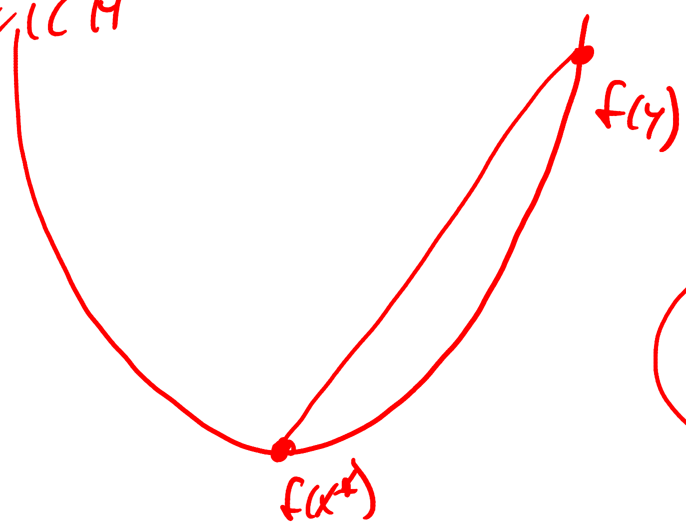
$\nabla^2 f(x) = \underbrace{f(x)}_{>0} \cdot c \cdot c^T \overset{?}{\succeq} 0$ $\quad$ YES: $f(x)$ CONVEX

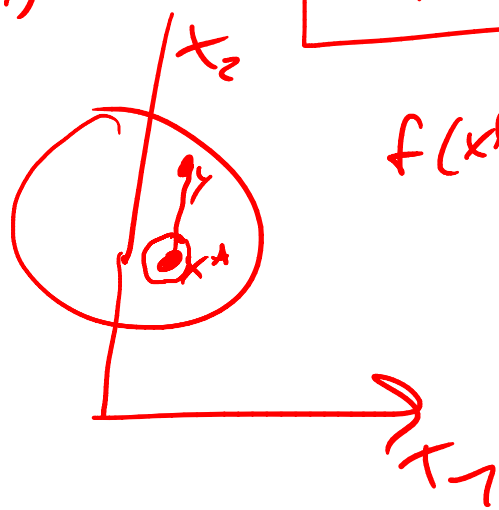
THM 3: LOCAL IMPLIES GLOBAL CONVEXITY FOR CONVEX OPT. PROBLEMS:

IF
$$\boxed{\begin{array}{ll} \min_{x \in \mathbb{R}^n} & f(x) \\ \text{s.t.} & x \in \Omega \end{array}}$$
IS CONVEX AND x^* A LOCAL MINIMIZER

THEN x^* IS GLOBAL MINIMIZER

SKETCH





$$\boxed{f(y) \geq f(x^*)} \quad \forall y \in \Omega$$

$$f(x^* + t(y - x^*)) \geq f(x^*)$$

$\forall t$ SMALL ENOUGH

$t > 0$

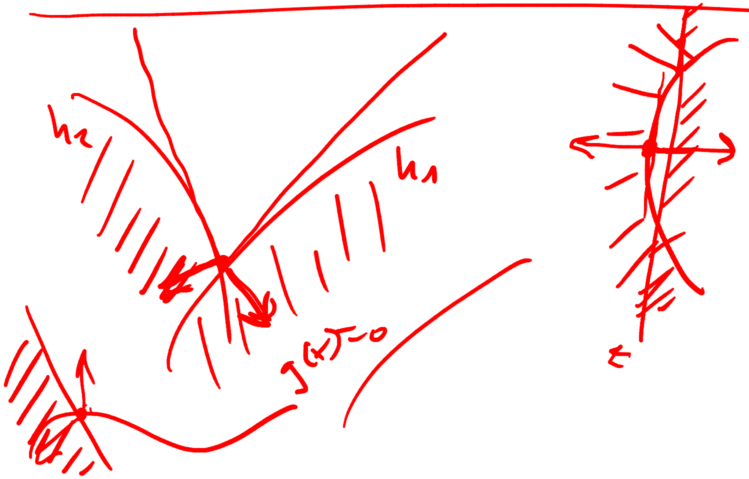
$$\cancel{f(x^*)} + t(\underbrace{f(y) - f(x^*)}_{\geq \cancel{f(x^*)}})$$

2.2 FIRST ORDER OPTIMALITY COND.

x^* MIN. $\boxed{\min_x \; f(x)}$ $\Longrightarrow$ $\nabla f(x^*) = 0$

x^* MIN. $\boxed{\begin{array}{ll} \min & f(x) \\ \text{st} & g(x) = 0 \\ & h(x) \leq 0 \end{array}}$ $\Longrightarrow$ $\exists\, \lambda^*, \mu^*$

$$\left\{ \begin{array}{l} \nabla \mathcal{L}(x^*, \lambda^*, \mu^*) = 0 \\ \\ g(x^*) = 0 \\ h(x^*) \leq 0 \\ \mu^* \geq 0 \\ \mu^{*T} h = 0 \end{array} \right\}$$



LINEAR INDEPENDENCE CONSTRAINT QUALIFICATION (LICQ)

$\bar{x}$ IS FEASIBLE POINT, LICQ HOLDS AT $\bar{x}$ IFF $\nabla h_i(\bar{x})$ FOR $i \in \mathcal{A}(\bar{x})$ AND ALL $\nabla g_i(\bar{x})$ ARE LINEARLY INDEPENDENT.

$$\tilde{g}(x) = \begin{bmatrix} g(x) \\ h_i(x), \; i \in \mathcal{A} \end{bmatrix}$$

LICQ $\Longleftrightarrow$ RANK $\frac{\partial \tilde{g}}{\partial x}$ IS FULL ROW RANK

$$\frac{\partial \tilde{g}}{\partial x}(\bar{x}) = \left.\begin{bmatrix} \nabla g_1^T(\bar{x}) \\ \vdots \\ \nabla g_m^T(\bar{x}) \\ \hline \nabla h_i^T \\ \vdots \end{bmatrix}\right\} n \text{ ROWS}$$

$$\frac{\partial g}{\partial x}(x) =: \nabla g(x)^T$$

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^T \cdot g(x) + \mu^T \cdot h(x)$$ "LAGRANGIAN FUNCTION"

$$\nabla_x \mathcal{L}(x, \lambda, \mu) = \nabla f(x) + \nabla g(x) \cdot \lambda + \underline{\underline{\nabla h(x) \cdot \mu}} = \nabla f(x) + \nabla \tilde{g}(x) \cdot \tilde{\lambda}$$

$$\tilde{\lambda} = \begin{bmatrix} \lambda \\ \mu_i \; i \in \mathcal{A} \end{bmatrix}$$

λ, μ : LAGRANGE MULTIPLIERS

THM 5: KARUSH-KUHN-TUCKER (KKT) CONDITIONS

IF x^* IS LOC. MINIMIZER ~~THEN~~ AND LICQ HOLDS AT x^* THEN:

$\exists \lambda^*, \mu^*$:

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = 0$$

$$g(x^*) = 0$$

$$\left.\begin{array}{ll} \left.\begin{array}{ll} h_i(x^*) & \leq 0 \\ \mu_i^* & \geq 0 \\ h_i(x^*) \cdot \mu_i^* & = 0 \end{array}\right\} & i = 1, \ldots, n_h \end{array}\right\} \text{"COMPLEMENTARY CONDITIONS"}$$

DEF: "KKT-POINT" (x^*, λ^*, μ^*) IF KKT-COND. ARE SATISFIED

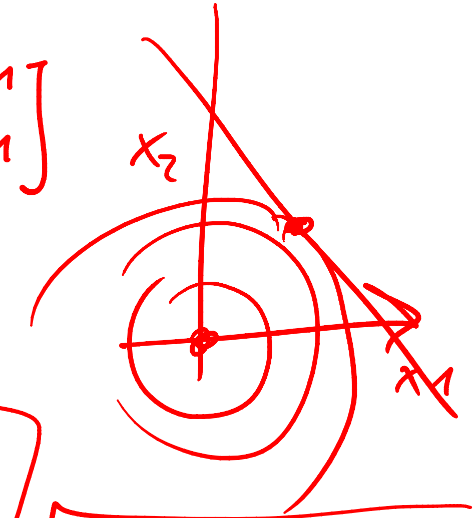
"STATIONARY POINT"

EX (*): $\boxed{\begin{aligned} f(x) &= \tfrac{1}{2}x^T x \\ g(x) &= -1 + x_1 + x_2 \end{aligned}}$ $\quad x \in \mathbb{R}^2$

h IS EMPTY

$\nabla f(x) = x$

$\nabla g(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



x^* IS LOC. MIN $\Longrightarrow$ $\exists\, \lambda^*$ WITH

$$\boxed{\begin{aligned} \nabla f + \nabla g \cdot \lambda^* &= 0 \\ g(x^*) &= 0 \end{aligned}}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x^* + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \lambda^* = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \cdot x^* = 1$$

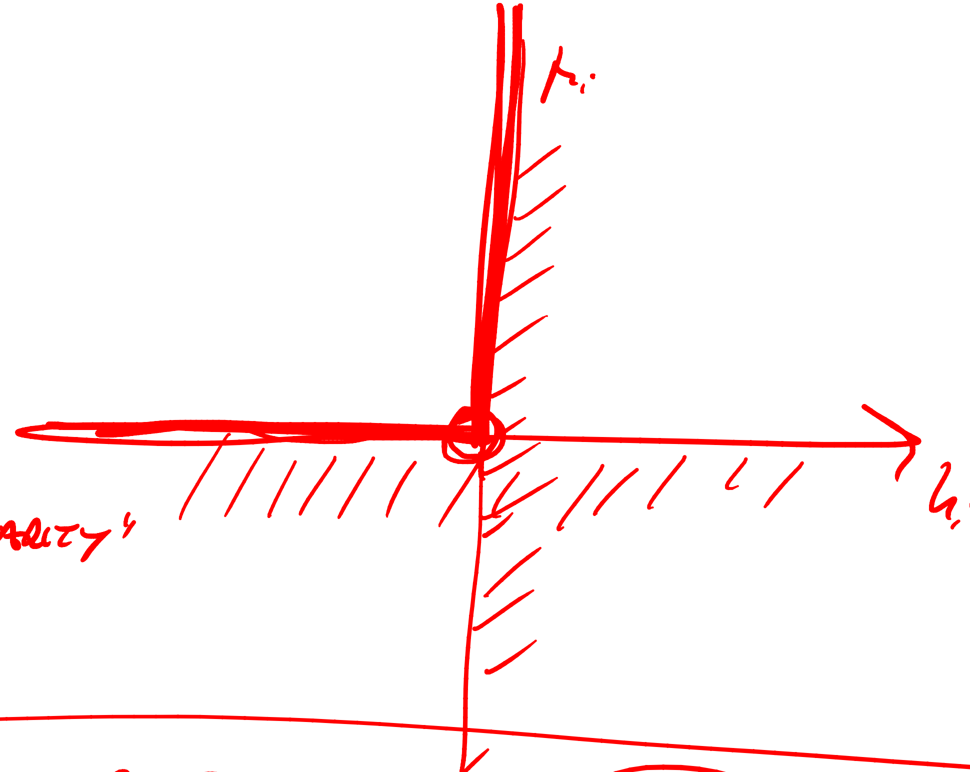
$x^* = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ $\quad \lambda^* = \left[-\tfrac{1}{2}\right]$

$Z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \nabla g^T Z = 0$

$\nabla^2 f = I$ $\quad Z^T \nabla^2 f Z$

FOR CONVEX NLPS, KKT COND ARE <u>SUFFICIENT</u> FOR OPTIMALITY

$h_i \le 0$, $\mu_i \ge 0$ $\quad h_i \cdot \mu_i = 0$



IF $\mu_i - h_i > 0 \quad \forall i$

THEN WE SAY "STRICT COMPLEMENTARITY" HOLDS

2.3 SECOND ORDER CONDITIONS

$$\boxed{\min_x f(x)}$$

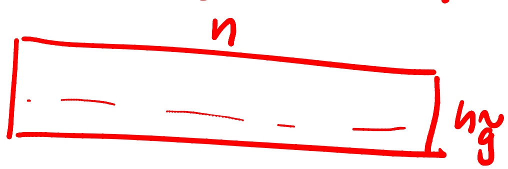
x^* LOC. MIN $\Rightarrow$ $\nabla^2 f(x^*) \succeq 0$

$\Leftarrow$ $\nabla f(x^*) = 0$, $\nabla^2 f(x^*) \succ 0$

THM 7 ~~(SOSC)~~ (SONC / SOSC)

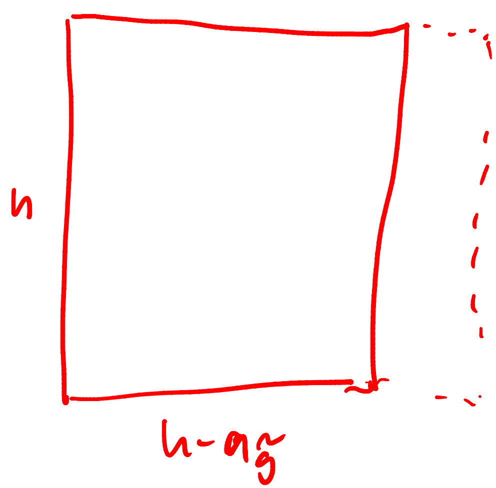
x^* ~~AT~~ WITH LICQ AND λ^*, μ^* SO THAT (x^*, λ^*, μ^*) IS KKT-POINT WITH STRICT COMPL. REGARD Z SPANS NULLSPACE OF $\nabla \tilde{g}(x^*)^T$, I.E. $\nabla \tilde{g}(x^*)^T \cdot Z = 0$

$\nabla \tilde{g}(x^*)^T =$



$Z \in \mathbb{R}^{n \times (n - n_{\tilde{g}})}$

$Z =$



($x = Z \cdot v + x^*$ ARE LINEARLY FEASIBLE POINTS)

a) x^* LOC. MIN. $\Longrightarrow Z^T \nabla_x^2 \mathcal{L}(x^*, \lambda^*, \mu^*) Z \succeq 0$

b) " $\Longleftarrow Z^T$ " $Z \succ 0$

$\nabla_x^2 \mathcal{L}$: HESSIAN OF LAGRANGIAN

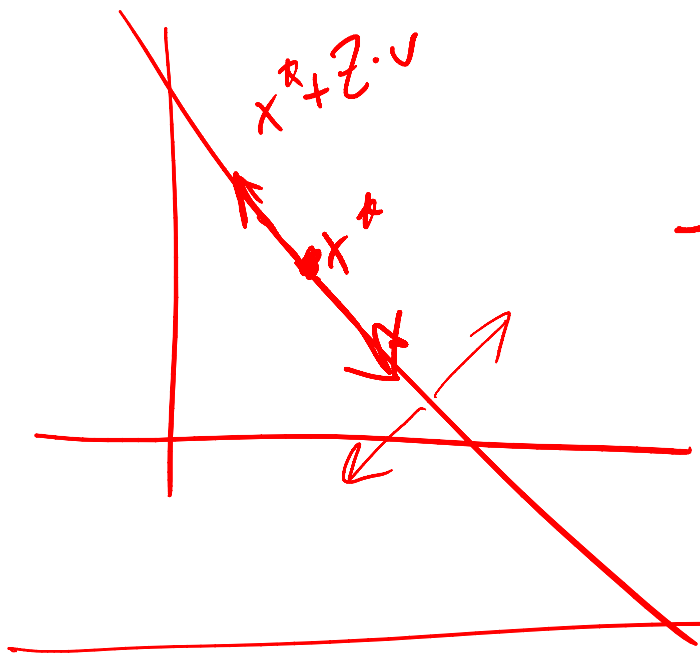
$Z^T \nabla_x^2 \mathcal{L} Z$: REDUCED HESSIAN

EX(*) CONT:

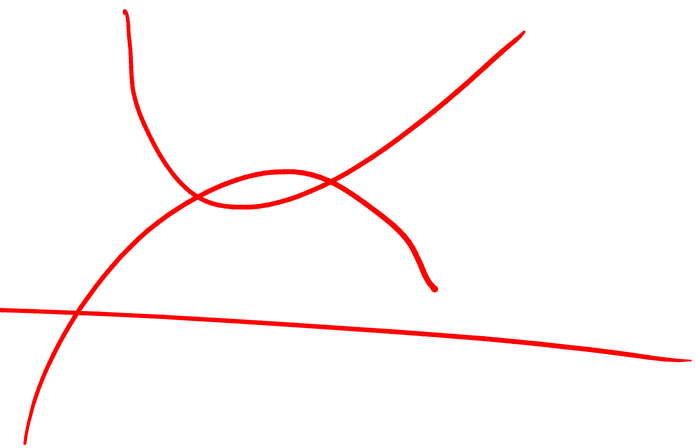
$$\nabla^2 \mathcal{L} = I \quad , \quad z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$z^T \nabla^2 \mathcal{L} z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 1^2 + (-1)^2 = 2 > 0$$



$$-(x_1 + x_2)^2 + (x_1 - x_2)^2 = f(x)$$



QP WITH EQUALITY:

$$\boxed{\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & c^T x + \frac{1}{2} x^T B x \\ \text{s.t.} & Ax + b = 0 \end{array}}$$

$A \in \mathbb{R}^{m \times n}$

LICQ : $\operatorname{rank}(A) = m$, $(m \leq n)$

$\nabla \mathcal{L} = c + Bx + A^T \cdot \lambda$

$\nabla^2 \mathcal{L} = B$

IF:

$z^T B z \succ 0$

KKT POINT IS (LOC.) MINIMIZER
↑ GLOB.

KKT:

$c + Bx + A^T \lambda = 0$

$b + Ax = 0$

$$\begin{bmatrix} B & A^T \\ A & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -c \\ -b \end{bmatrix}$$