COST-TO-GO

$$\boxed{J_k(\bar{x}_k) = \underset{\substack{x_k, x_{k+1}, \dots x_N \\ u_k, \dots, u_{N-1}}}{\text{minimize}} \sum_{i=k}^{N-1} L_k(x_k, u_k) + E(x_N) \quad \text{s.t.} \quad x_{i+1} = f_k(x_i, u_i), \quad x_k = \bar{x}_k}$$

$$J_N(\bar{x}_N) = E(\bar{x}_N)$$

DP-BACKWARDS-RECURSION

$$\rightarrow \boxed{J_k(\bar{x}_k) = \min_{u_k} L_k(\bar{x}_k, u_k) + J_{k+1}(f_k(\bar{x}_k, u_k))}$$

$$\boxed{u_k^*(\bar{x}_k) = \arg\min_{u_k} \quad '' \quad ''}$$

7.2 LINEAR QUADRATIC PROBLEMS

$$\boxed{\begin{aligned} &\underset{\substack{x_0,\dots,x_N \\ u_0,\dots,u_{N-1}}}{\text{minimize}} \quad \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q_k & S_k^T \\ S_k & R_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^T P_N x_N \\ &\text{s.t.} \quad x_0 - \bar{x}_0 = 0 \\ &\qquad x_{k+1} - A_k x_k - B_k u_k = 0 \quad , k = 0, \dots, N-1 \end{aligned}}$$

$\begin{bmatrix} Q_k & S_k \\ S_k & R_k \end{bmatrix}$, P_k SYMMETRIC

$$L_k = \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$\boxed{J_k = \min_u L_k + J_{k+1}(f_k(\ ))}$$

$$J_N = E = x^T P_N x$$

$$f = Ax + Bu$$

DP-RECURSION: ASSUME $J_{k+1}(x) = x^T P_{k+1} x$

$$\boxed{J_k(x) = \min_u \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q_k & S_k^T \\ S_k & R_k \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + (A_k x + B_k u)^T P_{k+1} (A_k x + B_k u)}$$

$\boxed{\text{THM}}$

WILL PROVE: (IF $R_k + B_k^T P_{k+1} B_k \succ 0$)

$$\boxed{J_k(x) = x^T P_k x} \quad \text{WITH}$$

$$P_k = \underbrace{Q_k + A_k^T P_{k+1} A_k}_{\bar{Q}} - (\overbrace{S_k^T + A_k^T P_{k+1} B_k}^{\bar{S}^T})(\overbrace{R_k + B_k^T P_{k+1} B_k}^{\bar{R}})^{-1} (\underbrace{S_k + B_k^T P_{k+1} A_k}_{\bar{S}})$$

$$J_k(x) = \min_u \begin{bmatrix} x \\ u \end{bmatrix}^T \left(\underbrace{\begin{bmatrix} Q_k & S_k^T \\ S_k & R_k \end{bmatrix} + \begin{bmatrix} A_k^T P_{k+1} A_k & A_k^T P_{k+1} B_k \\ B_k^T P_{k+1} A_k & B_k^T P_{k+1} B_k \end{bmatrix}}_{=: \begin{bmatrix} \bar{Q} & \bar{S}^T \\ \bar{S} & \bar{R} \end{bmatrix} \text{ (DROP INDEX)}} \right) \begin{bmatrix} x \\ u \end{bmatrix}$$

LEMMA (SCHUR-COMPLEMENT)

$$\boxed{\min_u \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \cancel{= x^T P_{new} x} = x^T (Q - S^T R^{-1} S) x \quad \text{(IF } R \succ 0\text{)}}$$

$$f(u) = \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \boxed{x^T Q x + 2 u^T S x + u^T R u} = f(u)$$

u^* ? $\quad \nabla f(u^*) = 0$

$$\boxed{\nabla f(u) = 0 + 2Sx + 2Ru}$$

$$u^T R u = \sum_{i=1}^{n_u} \sum_{j=1}^{n_u} R_{ij} u_i u_j$$

$$\frac{\partial (u^T R u)}{\partial u_k} = \sum_{j=1}^{n_u} R_{kj} u_j + \sum_{i=1}^{n_u} R_{ik} u_i$$

$$= [Ru + R^T u]_k$$

$\nabla f(u^*) = 0$

$\Updownarrow$

$Ru^* = -Sx \iff \boxed{u^* = -R^{-1} S x}$

$f(u^*)$?

$$f(u^*) = x^T Q x + 2(-R^{-1} S x)^T S x + (-R^{-1} S x)^T R (-R^{-1} S x)$$

$$= \text{''} \quad -2 x^T S^T R^{-1} S x + x^T S R^{-1} \underbrace{R R^{-1}}_{=I} S x$$

$$= x^T Q x - 1 \cdot x^T S^T R^{-1} S x \quad \square$$

$$= x^T \underbrace{\left(Q - S^T R^{-1} S\right)}_{P_{new} = P_k} x \quad \square$$

RESULT: GIVEN P_N, Q_k, S_k, R_k, A_k, B_k $\quad k=0,\dots,N-1$

CAN COMPUTE $P_{N-1}, P_{N-2}, \dots, P_0$

$$u_k^*(x_k) = -\underline{\bar{R}^{-1}\bar{S}} \cdot x_k = -\left(R_k + B_k^T P_{k+1} B_k\right)^{-1}\left(S_k + B_k^T P_{k+1} A_k\right) x_k$$

$$= -K_k \cdot x_k$$

OPTIMAL FEEDBACK CONTROL

FORWARD RECURSION

GIVEN $\bar{x}_0 =: x_0$

COMPUTE $u_0, x_1, \dots, u_{N-1}, x_N$

BY $\quad x_{k+1} = A_k \cdot x_k + B_k \cdot u_k^*(x_k)$

7.4 INFINITE HORIZON PROBLEMS (LQ)

$$\min_{x,u} \sum_{k=0}^{\infty} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$\text{s.t.} \quad x_{k+1} = A x_k + B u_k \quad , k = 0, 1, 2, \ldots$$

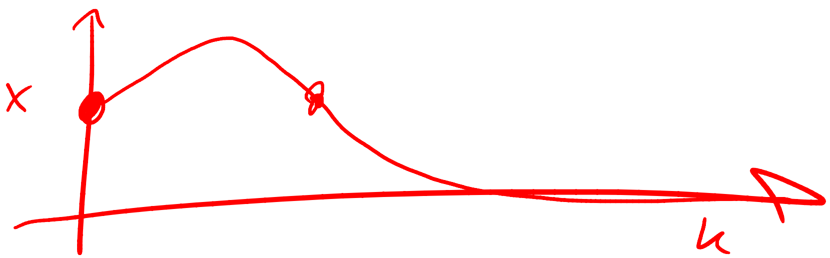
$$\boxed{u^*(x) = -K \cdot x}$$

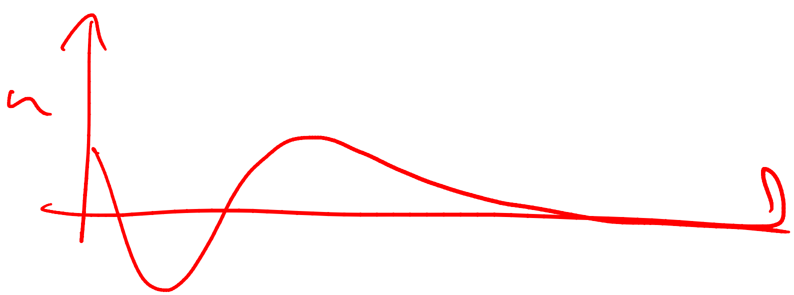
$$\cancel{K = R^{-1} S}$$

$$\boxed{K = (R + B^T P B)^{-1} (S + B^T P A)}$$

LINEAR QUADRATIC REGULATOR (LQR)

SOLUTION FOR FINITE, BUT LONG HORIZONS





$$J_{k+1} = J_k \quad \text{if } k \gg 1$$

$$\boxed{J(x) = \min_u L(x,u) + J(f(x,u))}$$

$$P = \bar{Q} - \bar{S}^T \bar{R}^{-1} \bar{S}$$

$$= Q + A^T P A - (S + B^T P A)^T (R + B^T P B)^{-1} (S + B^T P A)$$

ALGEBRAIC RICCATI EQUATION