$$\boxed{x_{k+1} = f_k(x_k, u_k)} \qquad x^+ = f(x,u) \qquad \begin{array}{l} x \in \mathbb{R}^{n_x} \\ u \in \mathbb{R}^{n_u} \end{array}$$

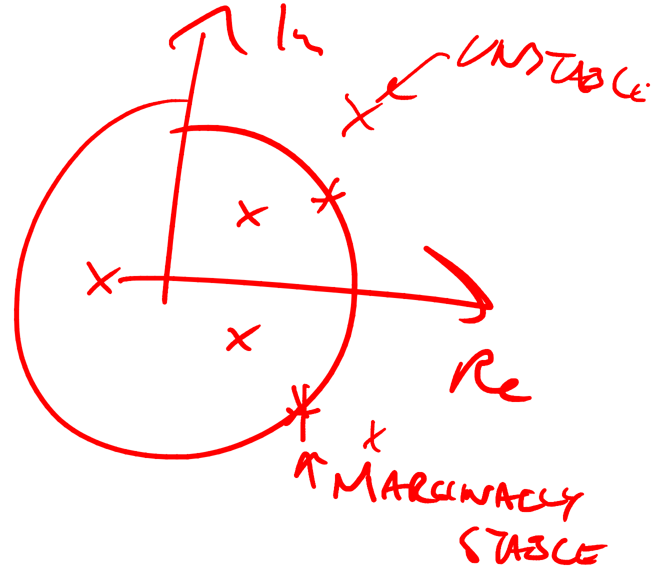
LTI

$$x^+ = Ax + Bu$$

$\mathrm{eig}(A) \subset$ unit disc

$\Downarrow$

stable



$$\mathcal{C} = [B \,|\, AB \,|\, A^2B \,|\, \dots \,|\, A^{n_x-1} \cdot B]$$

if $\cancel{\mathcal{C}}$ $\mathrm{rnk}(\mathcal{C}) = n_x \iff$ CONTROLLABLE

AFFINE SYSTEMS (1.3 CONT.)
(TIME VARYING)

$$\boxed{x_{k+1} = A_k \cdot x_k + B_k u_k + c_k}$$

FROM $\boxed{x_{k+1} = f_k(x_k, u_k)}$ $\to$ $\boxed{x_{k+1} = A_k x_k + B_k u_k + c_k}$

BY LINEARIZATION ALONG TRAJECTORIES

$\boxed{(\bar{x}_0, \ldots, \bar{x}_{N-1}), (\bar{u}_0, \ldots, \bar{u}_{N-1})}$

(TAYLOR)

$$f_k(x_k, u_k) \doteq f_k(\bar{x}_k, \bar{u}_k) + \frac{\partial f_k}{\partial x}(\bar{x}_k, \bar{u}_k)(x_k - \bar{x}_k) + \frac{\partial f_k}{\partial u}(\bar{x}_k, \bar{u}_k) \cdot (u_k - \bar{u}_k)$$

$$x_{k+1} \approx A_k x_k + B_k u_k + c_k$$

$\Delta x_k := x_k - \bar{x}_k$, $\Delta u_k = u_k - \bar{u}_k$

$A_k = \frac{\partial f_k}{\partial x}(\quad)$, $B_k = \frac{\partial f_k}{\partial u}$

$$\boxed{c_k = f_k(\bar{x}_k, \bar{u}_k) - \frac{\partial f}{\partial x}(\;)\,\bar{x}_k - \frac{\partial f}{\partial u}(\;)\cdot \bar{u}_k}$$

$$\underbrace{(x_{k+1} - \bar{x}_{k+1})}_{\Delta x_{k+1}} = A_k \cdot \Delta x_k + B_k \Delta u_k + \underbrace{\left(f_k(\bar{x}_k, \bar{u}_k) - \bar{x}_{k+1}\right)}_{\substack{=0 \text{ FOR} \\ \text{FEASIBLE} \\ \text{TRAJECTORIES}}}$$

$$x_N = (A_{N-1} \cdot \ldots \cdot A_0)\, x_0 + \sum_{k=0}^{N-1} (A_{N-1} \cdot \ldots \cdot A_{k+1})(B_k u_k + c_k)$$

AFFINE MAP IN $(u_0, \ldots, u_{N-1};\ x_0)$

(SOMETIMES USEFUL TO ELIMINATE STATES)

1.4 OPTIMIZATION PROBLEMS

- DECISION VARS
- OBJECTIVE
- CONSTRAINTS

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x)$$
$$\text{subject to} \quad g(x) = 0$$
$$h(x) \leq 0$$

EQUALITY

INEQUALITY

DIVISIONS:

$\boxed{\text{• FINITE}}$ VS. INFINITE DIMENSIONAL DECISION VARS

DISCRETE TIME, FINITE HORIZON OPT. CONTROL.

CONT. TIME OPT. CONT

CONT. VS. INTEGER

$\mathbb{R}^n$ $\mathbb{Z}^n$ $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$$\boxed{\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0 \\ & h(x) \leq 0 \end{aligned}}$$

CONT.

f, g, h : NONLINEAR, BUT DIFFABLE :

NONLINEAR PROGRAM (NLP)

f, g, h AFFINE :

LINEAR PROGRAM (LP) "CONVEX"

g, h AFFINE, f QUADRATIC (+LINEAR)

QUADRATIC PROGRAM (QP) "

MIXED-INTEGER PROGRAMMING (MIP)

$$\boxed{\begin{aligned} \min_{\substack{x \in \mathbb{R}^n \\ z \in \mathbb{Z}^m}} \quad & f(x, z) \\ \text{s.t.} \quad & g(x, z) = 0 \\ & h(x, z) \leq 0 \end{aligned}}$$

INTEGER PROGRAMMING (IP)

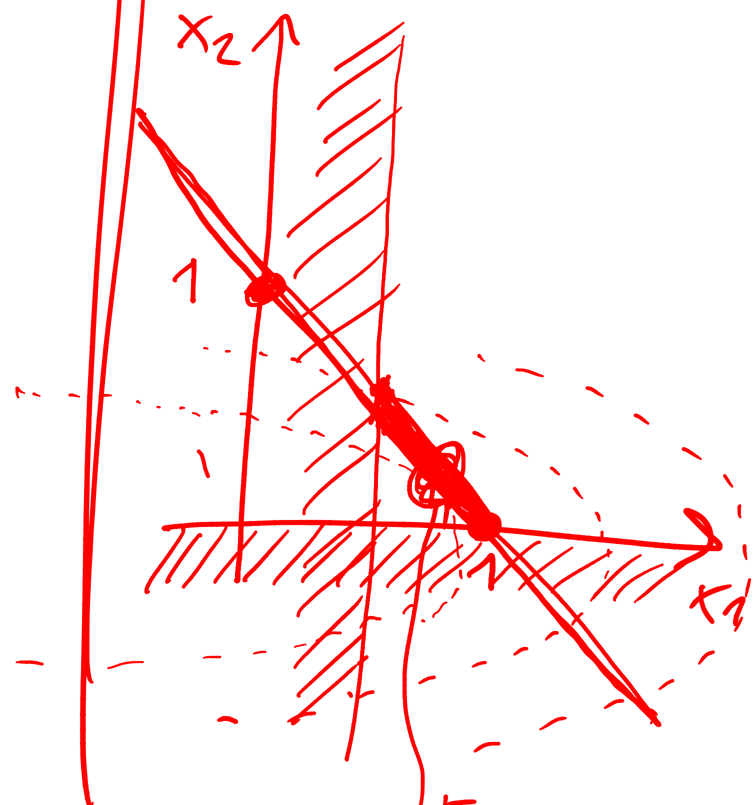
$$\boxed{\begin{aligned} \min_{z \in \mathbb{Z}^m} \quad & f(z) \\ \text{s.t.} \quad & g(z) = 0 \\ & h(z) \leq 0 \end{aligned}}$$

TWO DIM. EXAMPLE

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & x_1 + (x_2)^2 \\ \text{s.t.} \quad & -1 + x_1 + x_2 = 0 \\ & \begin{bmatrix} -x_2 \\ \frac{1}{2} - x_1 \end{bmatrix} \le 0 \end{aligned}$$

QP!



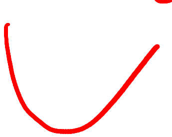
FEASIBLE SET

2 DEC. VARS, CONT.

1 EQUALITY CONSTR

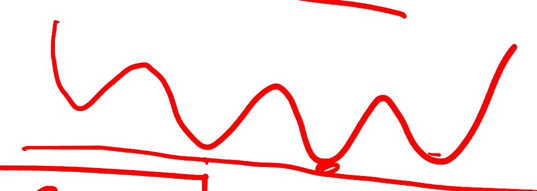
2 INEQUALITY CONSTR

CONVEX vs. NON-CONVEX OPTIMIZATION



EASY, GLOBALLY SOLVABLE
(NO LOCAL MINIMA, THAT ARE NOT GLOBAL)

DIFFICULT.



$\boxed{\text{NOW : CH. 7}}$

CH 7 DYNAMIC PROGRAMMING

$$\underset{\substack{x_0,\dots,x_N\\ u_0,\dots,u_{N-1}}}{\text{minimize}} \quad \sum_{k=0}^{N-1} L_k(x_k,u_k) + E(x_N)$$

$$\text{s.t.} \quad x_{k+1} = f_k(x_k,u_k), \quad k=0,\dots,N-1$$

$$x_0 = \bar{x}_0$$

x_k, u_k SO FAR UNSPECIFIED. TODAY, TWO CASES:

→ a) DISCRETE STATE SPACE $x_k \in \{1,2,\dots,n_{\mathbb{X}}\} =: \mathbb{X}$

ARBITRARY $f: \mathbb{X}\times\mathbb{U} \to \mathbb{X}$ $\quad u_k \in \{1,2,\dots,n_{\mathbb{U}}\} =: \mathbb{U}$

b) CONT. SPACES, LINEAR T.V. SYSTEM, QUADRATIC COST.

a) $n_x = 5, \quad n_u = 2, \quad N = 10$

$k=N$

$$L_0(x_0,u_0) + \underbrace{L_1(x_1,u_1) + \dots + L_{N-1}(x_{N-1},u_{N-1}) + E(x_N)}_{\text{"OPTIMAL COST-TO-GO" } J_1(x_1)}$$

$k=0$

$$\boxed{u_0^* = \arg\min_{u_0} L_0(x_0,u_0) + J_1(f_0(x_0,u_0))}$$

JUST TEST $u_0 = 1$, $u_0 = 2$, ..., $u_0 = n_u$ (n_u EVALS)

$J_N = E$

DP-ALGORITHM (BELLMAN, 1950s)

START: $J_N(x) \equiv E(x) \quad x \in \mathbb{X}$

LOOP FOR $k = N-1, \dots, 0$

$$\boxed{J_k(x) = \min_{u \in \mathbb{U}} L_k(x,u) + J_{k+1}(f_k(x,u))} \quad \left.\right\} n_{\mathbb{X}} \text{ TIMES } n_{\mathbb{U}}$$

(DP-RECURSION, BACKWARDS) $\quad x \in \mathbb{X}$

END-LOOP (TOTAL NO OF L&F EVALUATIONS: $(N \times n_{\mathbb{X}} \times n_{\mathbb{U}})$)

$x_0 \equiv \bar{x}_0$

LOOP FOR $k = 0, \dots, N-1$

$$\boxed{u_k^* = \arg \min_{u \in \mathbb{U}} L_k(x_k,u) + J_{k+1}(f_k(x_k,u))}$$

$x_{k+1} = f_k(x_k, u_k^*)$

ENDLOOP (TOTAL COST: $(N \times n_{\mathbb{U}})$)

(7.1 FINISHED)