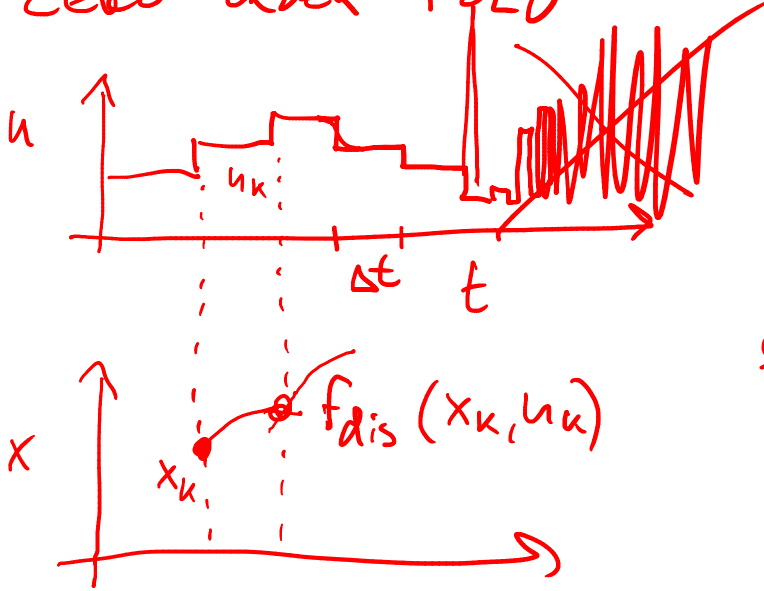
ZERO ORDER HOLD



$$\boxed{\dot x = f(x,u)}$$

sol. map

u piecewise constant

(Picard-Lindelöf)

$$\boxed{x_{k+1} = f_{dis}(x_k, u_k)}$$

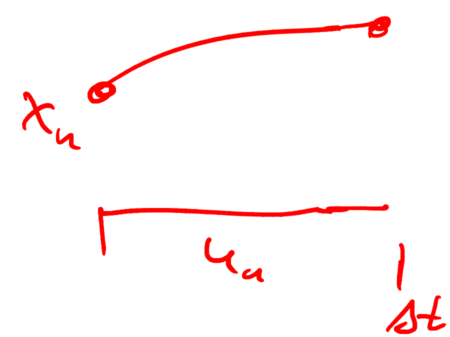
SOLUTION MAP FOR LTI SYSTEMS (p. 12)

$\dot x = A \cdot x + B \cdot u$ A, B fixed

f_{dis} ? $\boxed{\dot x = A \cdot x + B u_k \quad , \quad x(0) = x_k}$

$$\boxed{x(t) = e^{At} \cdot x_k + \int_0^t e^{A(t-\tau)} \cdot B \cdot \underbrace{\cancel{u(\tau)}}_{u_k} \, d\tau}$$

$$\boxed{\begin{aligned} \dot x &= A \cdot e^{At} \cdot x_k + \dots \\ &= A[\ x\ ] + B \cdot u_k \end{aligned}}$$



$$x(\Delta t; x_k, u_k) = \underbrace{\left[e^{A\Delta t}\right]}_{=:} \cdot x_k + \underbrace{\left[\int_0^{\Delta t} e^{A(\Delta t - \tau)} \cdot B \cdot d\tau\right]}_{=:} \cdot u_k$$

$$= A_{dis} \cdot x_k + B_{dis} \cdot u_k$$

LINEAR !

$$A_{dis} = e^{A \cdot \Delta t},$$ ALWAYS INVERTIBLE,

$$A_{dis}^{-1} = e^{-A\Delta t}$$ NO ZERO EIGENVALUES

$A \in \mathbb{R}$ $\quad A = \lambda \quad$ (EIGENVALUE)

$$A_{dis} = e^{\lambda \cdot \Delta t}$$

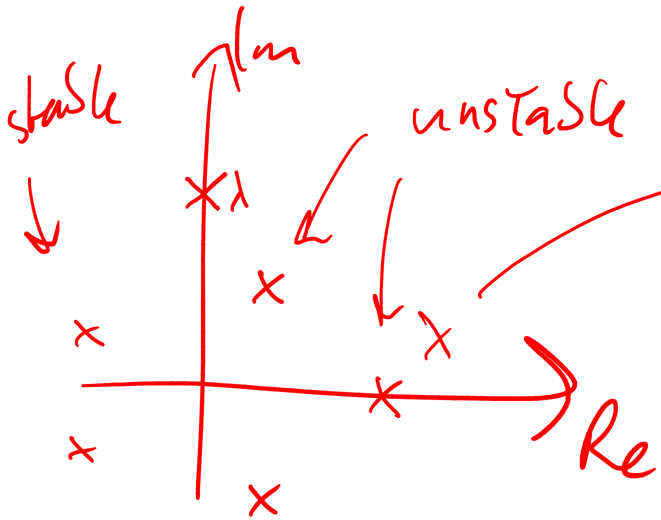
$\lambda > 0 \Rightarrow e^{\lambda \Delta t} > 1$

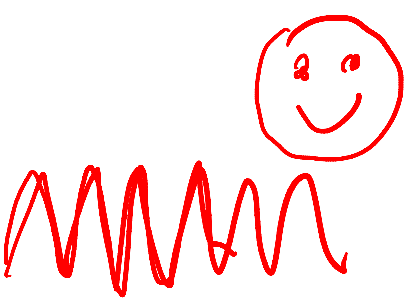
$\operatorname{Re} \lambda > 0 \Rightarrow |e^{\lambda \Delta t}| > 1$

CONT. TIME LTI

$$\boxed{\dot{x} = Ax + Bu}$$

eig(A)





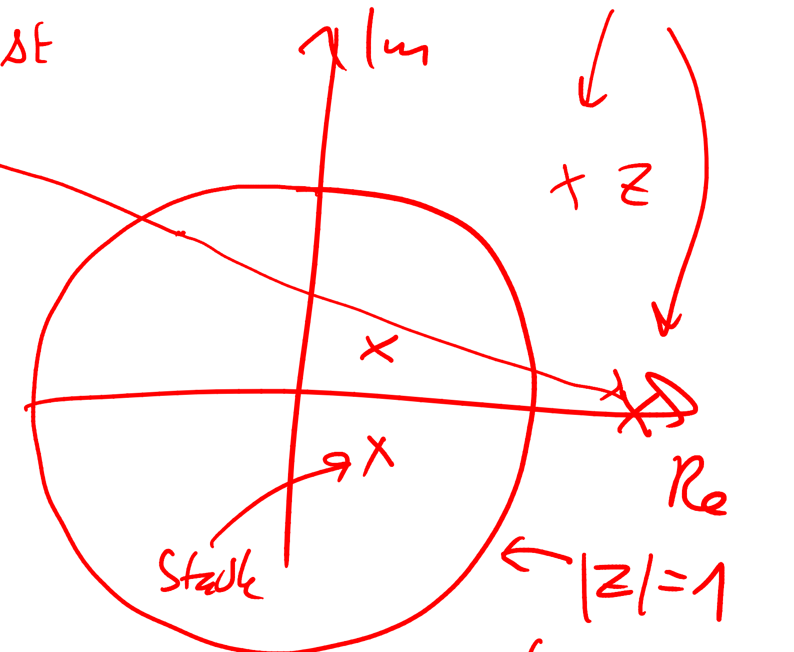
$$\boxed{x_{k+1} = -x_k}$$

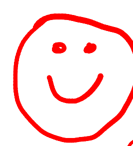
DISCRETE TIME LTI

$$\boxed{x_{k+1} = A_{dis}\, x_k + B_{dis}\, u_k}$$

eig(A_{dis})

$e^{\lambda \cdot \Delta t}$

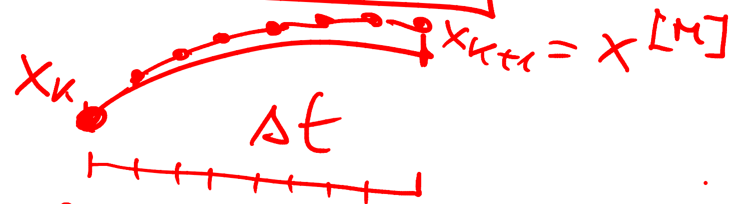




$e^{\lambda \Delta t} = -1$ $\qquad \lambda \Delta t = j\pi$

NUMERICAL INTEGRATION METHODS

$$\boxed{\dot x = f(x,u)}, \quad \boxed{\Delta t} \longrightarrow \boxed{x_{k+1} = f_{d_3}(x_k, u_k)}$$



$$h = \frac{\Delta t}{M}$$

EULER INTEGRATOR

$$\dot x = f(x)$$

$$x\left(t + \frac{\Delta t}{M}\right) \doteq x(t) + \frac{\Delta t}{M} \cdot \frac{\partial x}{\partial t}(t) \qquad \text{(TAYLOR)}$$

$$= x(t) + \frac{\Delta t}{M} \cdot f(x)$$

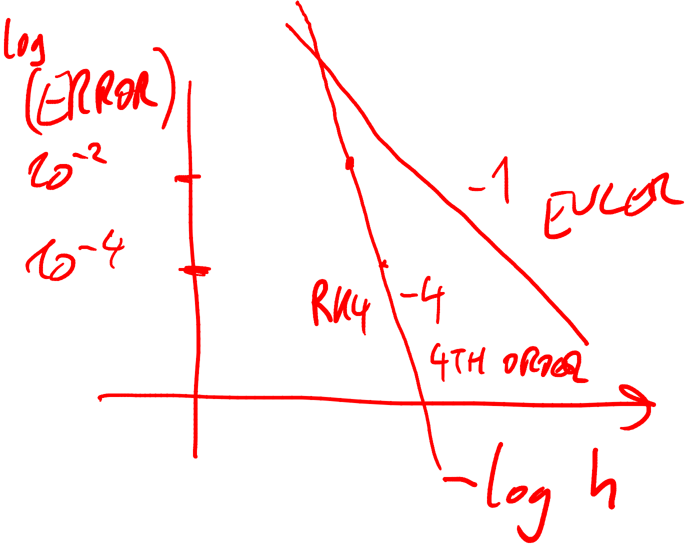
$x_k \rightarrow$, $u_k \rightarrow$

$$\boxed{\begin{aligned} x^{[0]} &= x_k \\ x^{[i+1]} &= x^{[i]} + \frac{\Delta t}{M} \cdot f(x^{[i]}, u_k), \quad i = 0, \dots, M-1 \\ x_{k+1} &= x^{[M]} \end{aligned}} \rightarrow x_{k+1} := f_{d_3}(x_k, u_k)$$

$$\text{ERROR} = O\left(\frac{\Delta t}{M}\right)$$

$$= O(h)$$

FIRST-ORDER METHOD



RK4 (p.14)

$$\boxed{x^{i+1} = x^i + \Delta t \cdot f(x^i)}$$ EULER

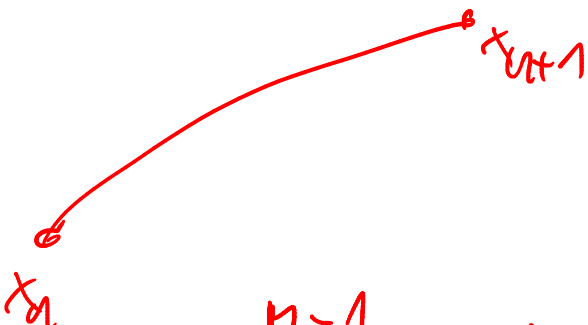
$$k_1 = f(x^i)$$

$$k_2 = f\left(x^i + \frac{h}{2} \cdot k_1\right)$$

$$k_3 = f\left(x^i + \frac{h}{2} k_2\right)$$

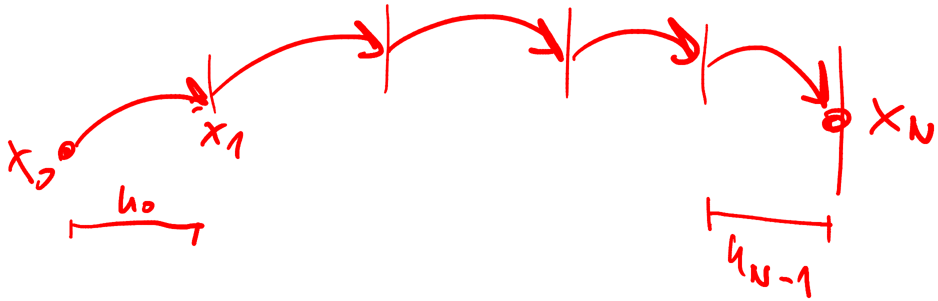
$$k_4 = f(x^i + h \cdot k_3)$$

$$x^{i+1} = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \cdot h + x^i$$



$n = 1$

1.3 DISCRETE TIME SYSTEMS



N - INTERVALS

$$\boxed{x_{k+1} = f_k(x_k, u_k) \quad , \quad k = 0, \dots, N-1}$$

DEF (FORWARD SIMULATION)

MAP FROM $(x_0, u_0, u_1, \dots, u_{N-1})$ TO $(x_0, x_1, x_2, \dots, x_N)$

LTV-SYSTEMS

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} x_0 \\ A_0 \cdot x_0 + B_0 \cdot u_0 \\ A_1 \cdot A_0 \cdot x_0 + A_1 \cdot B_0 \cdot u_0 + B_1 \cdot u_1 \\ \vdots \\ \dots \end{bmatrix}$$

LTI:

$$\underline{x_N} = A^N \cdot x_0 + A^{N-1} \cdot B u_0 + A^{N-2} \cdot B \cdot u_1 + \dots + A^0 \cdot B \cdot u_{N-1}$$

$$= A^N \cdot \underline{x_0} + \left[A^{N-1} B \mid A^{N-2} B \mid \dots \mid B \right] \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

CONTROLLABILITY (FOR LTI SYSTEMS):

CAN REACH ANY x_N FROM ANY x_0 FOR SUFF. BIG N

$$x_N - A^N \cdot x_0 = \underbrace{\left[B \mid \quad \mid \dots \mid A^{N-1} B \right]}_{C_N} \begin{bmatrix} u_{N-1} \\ \vdots \\ u_0 \end{bmatrix}$$

C_N MUST HAVE FULL RANK (FOR N BIG ENOUGH)
($N = n_x$ IS ENOUGH CAYLEY HAMILTON)

$$\boxed{\begin{matrix} x_n \in \mathbb{R}^{n_x} \\ u_n \in \mathbb{R}^{n_u} \end{matrix}}$$

DEF: CONTROLLABILITY MATRIX FOR (A, B)

$$\mathcal{C} := C_{n_x} = [\underbrace{B}_{n_u} | \underbrace{AB}_{n_u} | \dots | \underbrace{A^{(n_x-1)} \cdot B}_{n_u}] \in \mathbb{R}^{n_x \times (n_u \cdot n_x)}$$

DEF: SYSTEM $x_{k+1} = Ax_k + Bu_k$ IS "CONTROLLABLE" $\iff$

$$\boxed{\text{RANK } \mathcal{C} = n_x}$$

(HIGHEST POSSIBLE RANK)

EIGENVECTORS & -VALUES (FOR LTI SYSTEMS)

JORDAN-DECOMPOSITION (ALWAYS POSSIBLE)

EVERY $A \in \mathbb{R}^{n_x \times n_x}$ CAN BE WRITTEN $A = Q J Q^{-1}$

WITH Q INVERTIBLE $\left(Q \in \mathbb{R}^{n_x \times n_x} \right)$ AND

$$J = \begin{bmatrix} \boxed{J_1} & & \\ & \ddots & \\ & & \boxed{J_m} \end{bmatrix}$$

WITH m "JORDAN-BLOCKS" J_i, $i = 1, \dots, m$

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}.$$

IF ALL JORDAN BLOCKS HAVE SIZE 1, OUR MATRIX IS "DIAGONALIZABLE"

$$x_N = A^N \cdot x_0 = (Q \cdot J \cdot Q^{-1})^N \cdot x_0 = Q \cdot J^N \cdot (Q^{-1} \cdot x_0)$$

$$J^N = \begin{bmatrix} J_1^N & & 0 \\ & \ddots & \\ 0 & & J_m^N \end{bmatrix} \qquad J_i^N = \begin{bmatrix} \lambda_i^N & N \cdot \lambda_i^{N-1} & \binom{N-1}{1} \cdots & 0 \\ & \ddots & \ddots & 1 \\ & & \ddots & N \lambda_i^{N-1} \\ 0 & & & \lambda_i^N \end{bmatrix} \quad (N < n_k^i)$$

IF $|\lambda_i|$ IS BIGGER THAN ALL OTHER ~~JORDAN~~-EIGENVALUES

FOR LARGE N x_N WILL MOSTLY CONSIST OF THIS MODE (EIGENVECTOR)

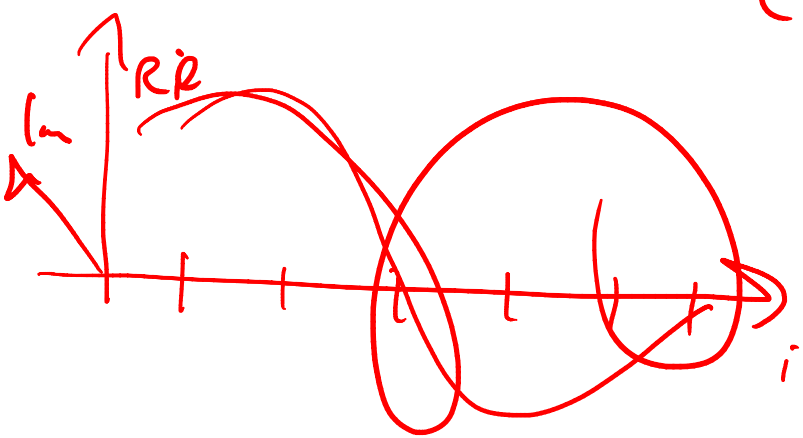
$$\left[\begin{bmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ 0 & & \lambda \end{bmatrix}^3 = \begin{bmatrix} \lambda^2 & 2\lambda & 0 \\ & \lambda^2 & \ddots \\ & & \lambda^2 \end{bmatrix} \cdot \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^3 & 3\lambda & 2\lambda \\ & \ddots & 3\lambda \\ & & \lambda^3 \end{bmatrix}\right]$$

$A \cdot v_i = \lambda_i \cdot v_i$

[ONE v_i PER JORDAN BLOCK]

IF $\lambda_i \in \mathbb{C}$
$v_i \in \mathbb{C}^{n_x}$
$\rightarrow$ $\bar{\lambda}_i$ IS EV
$\bar{v}_i$ IS EV
(BECAUSE $A \in \mathbb{R}^{n_x \times n_x}$)

$$A \cdot \operatorname{Re} v_i = A\left(\frac{v_i + \bar{v}_i}{2}\right)$$
$$= \cancel{A\,\mathrm{Re}} = \frac{1}{2}\left(\lambda_i v_i + \bar{\lambda}_i \cdot \bar{v}_i\right)$$
$$= \operatorname{Re}\left(\lambda_i v_i\right)$$



$i = 1, \ldots, n_x$