Optimal Control and Estimation – Final Exam

Prof. Dr. Moritz Diehl, Greg Horn, IMTEK, Universität Freiburg, and ESAT-STADIUS, KU Leuven

August 14, 2014, 10:15-13:15, Freiburg, HS 026, Gebäude 101

page	0	1	2	3	4	5	6
points on page (max)	6	13	15	15	6	10	15
points obtained							
intermediate sum							

Note: Klausur eingesehen am: Unterschrift des Prüfers:

Last name: First name: Matriculation number:

Field of studies: Targeted degree: PhD ☐ Master ☐ Lehramt ☐ other ☐

Please fill in your name above. For the multiple choice questions, tick **exactly one** box for the right answer (only one answer is right, and a wrong answer gives -1/3 points). For the text questions, give a short formula or text answer just below the question in the space provided, or, if necessary, write on the back page or one of the extra pages at the end, and add a comment “see back page” or similar. The exam is an open book exam, in particular the course script is allowed. Some legal comments are found in a footnote (in German). [1]

1. **Convex Sets**

   Which of the following sets Ω is convex ? Encircle the convex sets. $A \in \mathbb{R}^{m \times n}$ is some fixed matrix and $c \in \mathbb{R}^n$ a fixed vector.

   (a) $\Omega = \{x \in \mathbb{R}^2 \mid \exp(-x^\top x) \leq 0.5\}$

   (b) $\Omega = \{x \in \mathbb{R}^n \mid \|x\|_2^2 \leq c^\top x\}$

   (c) $\Omega = \{x \in \mathbb{R}^n \mid \|Ax\|_2 \geq 10\}$

   (d) $\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|y\|_2^2 + \|x\|_2^2 \leq 2 + 2x^\top y\}$

   (e) $\Omega = \{x \in \mathbb{R}^n \mid c^\top x = x^\top x\}$

   (f) $\Omega = \{x \in \mathbb{R}^n \mid c^\top x \leq x^T A^T A x\}$

6	

points on page: 6	

[1] PRÜFUNGSUNFÄHIGKEIT: Durch den Antritt dieser Prüfung erklaren Sie sich für prüfungsfähig. Sollten Sie sich während der Prüfung nicht prüfungsfahig fühlen, können Sie aus gesundheitlichen Gründen auch während der Prufung von dieser zurücktreten. Gemäß den Prüfungsordnungen sind Sie verpflichtet, die für den Rücktritt oder das Versäumnis geltend gemachten Gründe unverzüglich (innerhalb von 3 Tagen) dem Prüfungsamt durch ein Attest mit der Angabe der Symptome schriftlich anzuzeigen und glaubhaft zu machen. Weitere Informationen: https://www.tf.uni-freiburg.de/studium/pruefungen/pruefungsunfaehigkeit.html.
TÄUSCHUNG/STÖRUNG: Sofern Sie versuchen, während der Prüfung das Ergebnis ihrer Prüfungsleistung durch Täuschung (Abschreiben von Kommilitonen ...) oder Benutzung nicht zugelassener Hilfsmittel (Email, Mobiltelefon, ...) zu beeinflussen, wird die betreffende Prüfungsleistung mit “nicht ausreichend” (5,0) und dem Vermerk Täuschung bewertet. Als Versuch gilt bei schriftlichen Prüfungen und Studienleistungen bereits der Besitz nicht zugelassener Hilfsmittel während und nach der Ausgabe der Prüfungsaufgaben. Sollten Sie den ordnungsgemßen Ablauf der Prüfung stören, werden Sie vom Prüfer/Aufsichtsführenden von der Fortsetzung der Prüfung ausgeschlossen. Die Prüfung wird mit “nicht ausreichend” (5,0) mit dem Vermerk Störung bewertet.

2. **Gauss-Newton**

We want to use a full-step Gauss-Newton method to solve the simple least-squares problem $\min_{x\in\mathbb{R}} \frac{1}{2}\|R(x)\|_2^2$ with $R(x) = (x^3 - 2, x - 1)^\top$. Given the current iterate x_k, which exact formula determines the next iterate x_{k+1} ?

(a) ☐ $x_k - \frac{3x_k^5 - 6x_k^2 + x_k - 1}{9x_k^4 + 1}$	(b) ☐ $x_k - \frac{3x_k^2 + 1}{x_k^6 - 4x_k^3 + x_k^2 - 2x_k + 5}$
(c) ☐ $x_k + \frac{3x_k^5 - 6x_k^2 + x_k - 1}{15x_k^4 - 12x_k + 1}$	(d) ☐ $x_k - \frac{3x_k^5 - 6x_k^2 + x_k - 1}{15x_k^4 - 12x_k + 1}$

3	

3. **Linear Quadratic Dynamic Programming**

Use dynamic programming to solve the following simple discrete time OCP with one state and one control by hand.

$$\underset{x_0, x_1, x_2, u_0, u_1}{\text{minimize}} \quad \sum_{k=0}^{1} u_k^2 \; + \; 10x_2^2$$

subject to

$$\begin{aligned} x_0 &= 5 \\ x_{k+1} &= x_k + u_k, \quad k = 0, 1. \end{aligned}$$

10	

points on page: 13	

4. **Multi-stage OCP with Multiple Shooting**:

You are given a dynamical system with

$$\begin{aligned} \dot{x}(t) &= f_1(x(t), u(t)) \quad t \in [0, T_1] \\ \dot{x}(t) &= f_2(x(t), u(t)) \quad t \in [T_1, T_2] \end{aligned}$$

where times T_1 and T_2 are constant numbers where $0 < T_1 < T_2$, and $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}^{n_u}$.

The initial and final states must be constrained:

$$\begin{aligned} x(0) &= \bar{x}_0 \\ x(T_2) &= \bar{x}_\mathrm{F} \end{aligned}$$

The objective function is

$$\int_0^{T_1} g_1(x(t), u(t))\,\mathrm{d}t + \int_{T_1}^{T_2} g_2(x(t), u(t))\,\mathrm{d}t$$

This problem will be discretized using direct multiple shooting, using piecewise constant controls with N_1 and N_2 equidistant intervals on each stage. You don't have to explicitly write out an integration scheme, you can just write something like:

$$\begin{aligned} s_{k+1} &= F_1(s_k, u_k) \\ s_{k+1} &= F_2(s_k, u_k) \end{aligned}$$

but you must specify which differential equations are being integrated with each F. You may not assume that the integration scheme is forward Euler.

Write down the NLP for this problem, specifying the optimization variables, the objective function and the constraints. Be sure to write down the dimensions of all optimization variables and to give correct ranges for all indices.

15	

points on page: 15	

5. **Trajectory estimation**

You are given a discrete-time dynamical system with

$$x_{k+1} = f(x_k, u_k, w_k)$$

where u_k are the known inputs while w_k are unknown disturbances to the system. The system has sensors which are modeled as

$$y_i = h(x_i) + v_i$$

where y_i are some recorded sensor data, and v_i are unknown sensor noises. Variables w_i and v_i are independent identically distributed Gaussian variables with zero mean and respective covariances Σ_w and Σ_v.

Your goal is to take $N+1$ sensor data $y_0, y_1 \dots y_N$ recorded at equally spaced time instants $t_k = k\Delta t$ (where Δt is the sampling time of the discrete time system) and recreate the state trajectory. Besides this, there is no a-priori knowledge given. Using a simultaneous approach, formulate a discrete-time OCP which would accomplish this task.

(a) Write down this NLP including the optimization variables and their dimensions, the objective function, and the constraints, using correct ranges for all occuring indices.

(b) Write down the Lagrangian for this NLP.

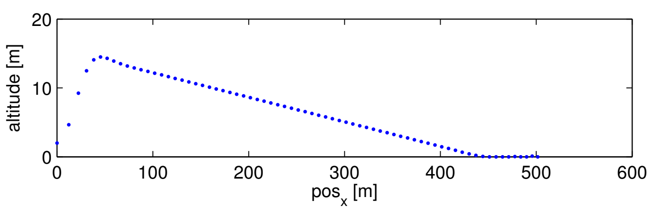
(c) Decide on an ordering of the optimization variables and write the Gauss-Newton Hessian approximation.

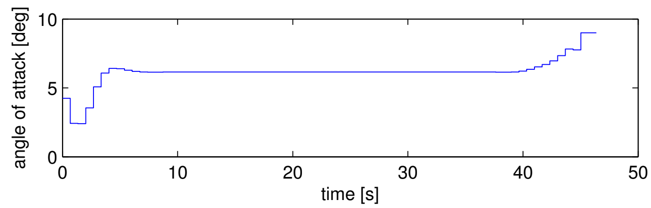
15	

points on page: 15	

6. **Improving accuracy in an OCP discretization**

Below is a plot showing a distance-optimized paper airplane trajectory whose integrator is a Runge-Kutta 4th order (RK4) using 10 RK4 steps per control interval.





As you can see, there is a lot going on at the beginning (takeoff and ascent), and the end (landing flare), but not much going on in the middle (efficient cruise). Describe one way in which this problem could be reformulated, increasing accuracy in the integration without taking more time to solve.

5	

Bonus point: describe a different way

1	

points on page: 6	

7. **Minimum-time OCP with direct single shooting**

   Given a dynamic system

$$\dot{x}(t) = f(x(t), u(t)) \quad t \in [0, T]$$

   and objective

$$T + \int_0^T g(x(t), u(t))\,\mathrm{d}t$$

   Write down the NLP resulting from the direct single shooting formulation of this problem.

   As before, you don't have to explicitly write out an integration scheme, but you must specify which differential equations are being integrated and on which of the decision variables the solution depends (again, you may not assume that the integration scheme is forward Euler.)

10	

points on page: 10	

8. **Direct Collocation with Midpoint Rule for Simple Pendulum**

Regard the following continuous time optimal control problem for a simple pendulum system with states $x = [\theta, \omega]^\top$ and control u on the fixed time horizon $[0, T]$:

$$\underset{x(\cdot),\, u(\cdot)}{\text{minimize}} \qquad \int_0^T u(t)^2 \, dt$$

subject to

$$\begin{aligned} x(0) &= [\pi, 0]^\top \\ \dot{x}(t) &= f(x(t), u(t)), \qquad t \in [0, T] \\ x(T) &= [0, 0]^\top \end{aligned}$$

with dynamics given by

$$\begin{aligned} \dot{\theta}(t) &= \omega(t) \\ \dot{\omega}(t) &= \sin(\theta(t)) + u(t) \end{aligned}$$

Use the direct collocation method to transform the problem into a nonlinear programming problem (NLP). Use piecewise constant controls on N equal intervals and the implicit midpoint rule for discretization on each interval (this results in a first order polynomial to represent the states on the interval). If you like, eliminate the collocation node value at the midpoint by $\frac{(x_k + x_{k+1})}{2}$ such that the NLP variables are just given by $(x_0, u_0, \ldots x_{N-1}, u_{N-1}, x_N)$. Formulate the NLP and be careful to give correct ranges for the indices.

15	

points on page: 15	