Multi-Parametric Toolbox (MPT)

Michal Kvasnica and Colin Jones

Prototype & Deploy Optimization-Bazed Controllers

Motivating Example

JR-Maglev (581 km/h)

Motivating Example

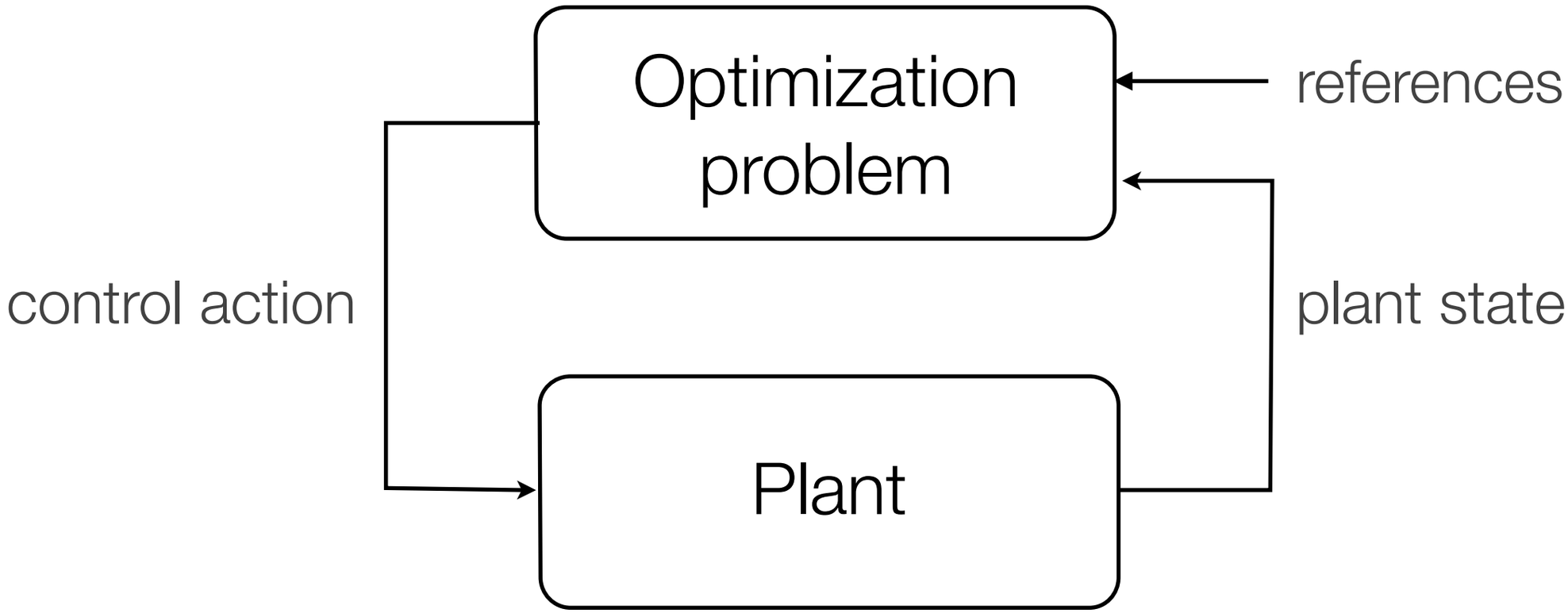
Fast linear dynamics

All states can be measured

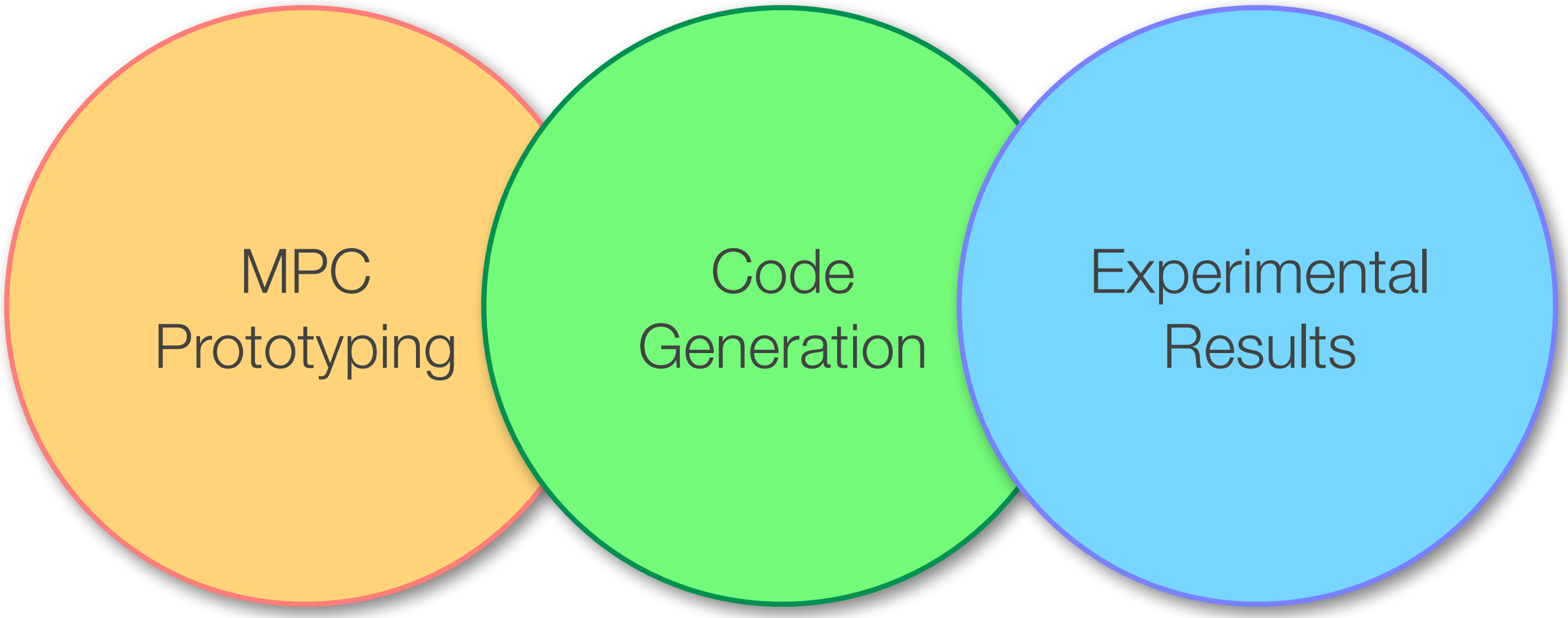
Constraints on inputs & outputs

Simple implementation hardware

Model Predictive Control



Agenda



MPC Prototyping

$$\underbrace{\begin{bmatrix} x_t \\ x_{t+1} \\ x_{t+2} \\ \vdots \\ x_{t+N-1} \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^{N-1} \end{bmatrix}}_{\tilde{A}} x_t + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ B & 0 & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A^{N-2}B & A^{N-3}B & \cdots & B & 0 \end{bmatrix}}_{\tilde{B}} \underbrace{\begin{bmatrix} u_t \\ u_{t+1} \\ u_{t+2} \\ \vdots \\ u_{t+N-1} \end{bmatrix}}_{U}$$

MPC Prototyping

Regulation with disturbance rejection

Reference tracking

Soft constraints

Slew rate constraints

Custom setups

Disturbance Rejection

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} q y_k^2 + r u_k^2$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k$$

$$u_{\min} \leq u_k \leq u_{\max}$$

$$y_{\min} \leq y_k \leq y_{\max}$$

$$x_0 = x(t)$$

```
sys = LTISystem('A', A, 'B', B, 'C', C)
sys.u.min = -2.29, sys.u.max = 2.71
sys.y.min = -1.40, sys.y.max = 3.60
sys.y.penalty = QuadFunction(q)
sys.u.penalty = QuadFunction(r)
N = 5
ctrl = MPCController(sys, N)
ctrl.construct()
```

Command-Line Implementation

```
u0 = ctrl.evaluate(x0)

u0 =

   -2.2900
```

```
[u0, feasible, openloop] = ctrl.evaluate(x0)

openloop =

    cost: 96.5649
       U: [-2.2900 -1.7824 0.1978 0.2407 0]
       X: [2x6 double]
       Y: [2 1.7822 1.1774 0.4492 -0.2010]
```

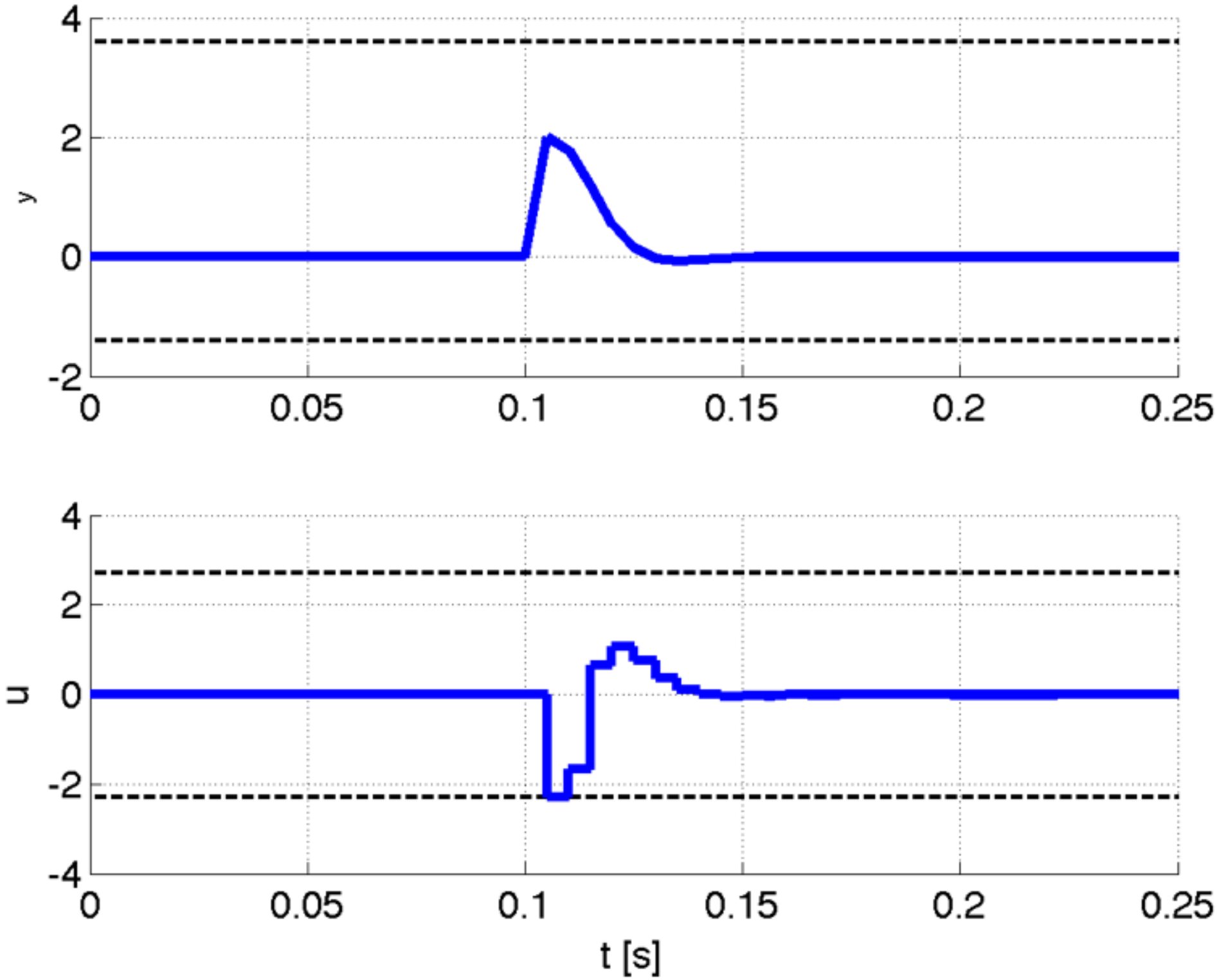
Closed-Loop Implementation (Command Line)

```
loop = ClosedLoop(ctrl, sys)
data = loop.simulate(x0, Nsim)

data =

        X: [2x31 double]
        U: [1x30 double]
        Y: [1x30 double]
     cost: [1x30 double]
```

Disturbance Rejection



MPC Prototyping

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Tracking of a Constant Reference

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} q(y_k - y_{\text{ref}})^2 + r u_k^2$$

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sys.u.min = -2.29, sys.u.max = 2.71
sys.y.min = -1.40, sys.y.max = 3.60
sys.y.penalty = QuadFunction(10)
sys.u.penalty = QuadFunction(1)


N = 5
ctrl = MPCController(sys, N)
ctrl.construct()
```

Tracking of a Constant Reference

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} q(y_k - y_{\text{ref}})^2 + r u_k^2$$

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sys.y.penalty = QuadFunction(10)
sys.u.penalty = QuadFunction(1)
sys.y.with('reference')

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sys.y.penalty = QuadFunction(10)
sys.u.penalty = QuadFunction(1)
sys.y.with('reference')
sys.y.reference = 3
N = 5
ctrl = MPCController(sys, N)
ctrl.construct()
```
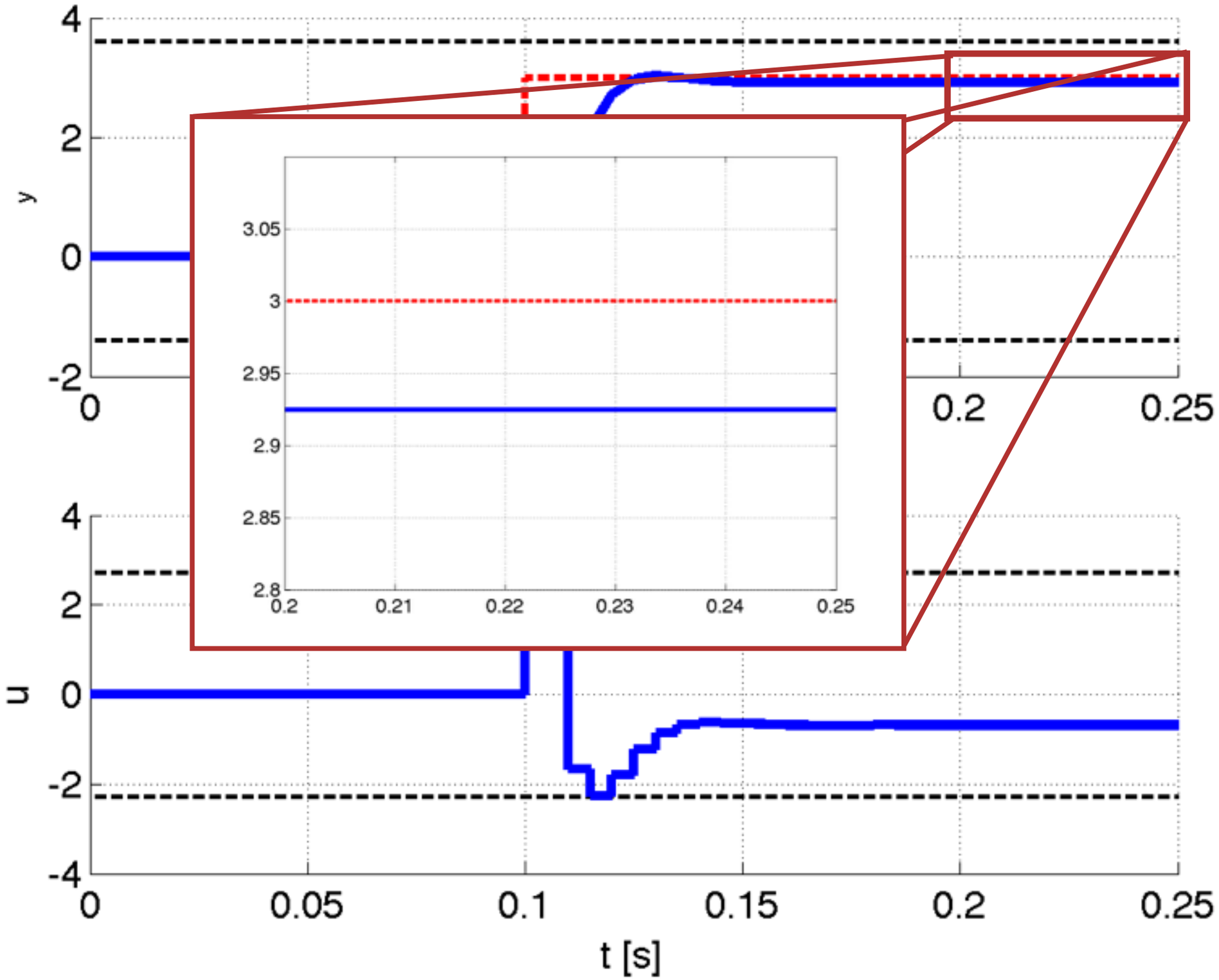

Tracking of a Constant Reference



Tracking of a Constant Reference

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} q(y_k - y_{\text{ref}})^2 + r\Delta u_k^2$$

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$$y_{\min} \leq y_k \leq y_{\max}$$

$$\Delta u_k = u_k - u_{k-1}$$

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sys.y.with('reference')
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sys.y.with('reference')
sys.y.reference = 3
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Tracking of a Constant Reference

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sys.y.min = -1.40, sys.y.max = 3.60
sys.y.penalty = QuadFunction(10)
sys.u.with('deltaPenalty')

sys.y.with('reference')
sys.y.reference = 3
N = 5
ctrl = MPCController(sys, N)
```

Tracking of a Constant Reference

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} q(y_k - y_{\text{ref}})^2 + r\Delta u_k^2$$

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sys.y.min = -1.40, sys.y.max = 3.60
sys.y.penalty = QuadFunction(10)
sys.u.with('deltaPenalty')
sys.u.deltaPenalty = QuadFunction(1)
sys.y.with('reference')
sys.y.reference = 3
N = 5
ctrl = MPCController(sys, N)
```
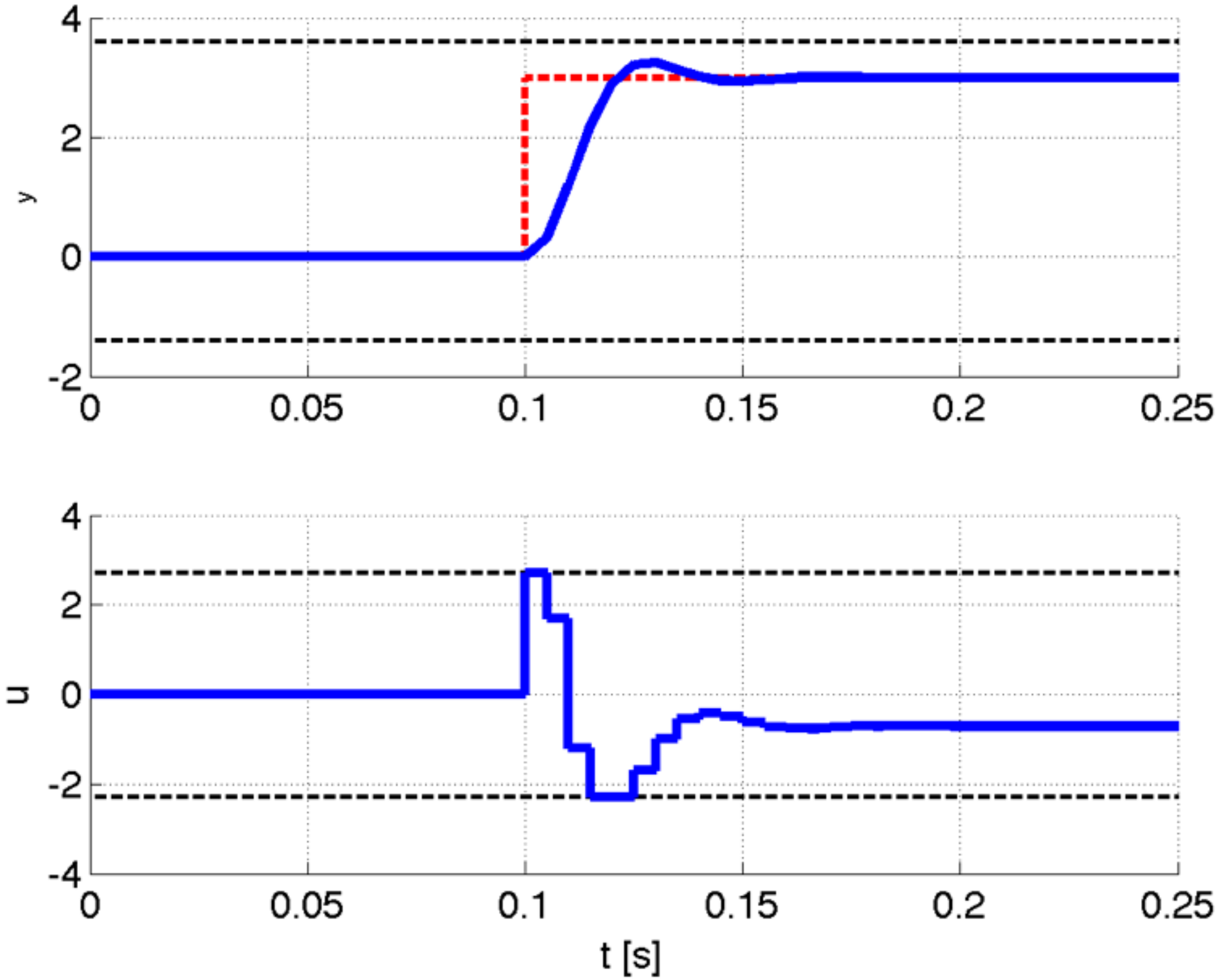

Tracking of a Constant Reference



Tracking of a Time-Varying References

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} q(y_k - y_{\text{ref}})^2 + r\Delta u_k^2$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k$$

$$u_{\min} \leq u_k \leq u_{\max}$$

$$y_{\min} \leq y_k \leq y_{\max}$$

$$\Delta u_k = u_k - u_{k-1}$$

$$x_0 = x(t)$$

```
sys = LTISystem('A', A, 'B', B, 'C', C)
sys.u.min = -2.29, sys.u.max = 2.71
sys.y.min = -1.40, sys.y.max = 3.60
sys.y.penalty = QuadFunction(1)
sys.u.with('deltaPenalty')
sys.u.deltaPenalty = QuadFunction(1)
sys.y.with('reference')
sys.y.reference = 3
N = 5
ctrl = MPCController(sys, N)
```

Tracking of a Time-Varying References

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} q(y_k - y_{\text{ref}})^2 + r\Delta u_k^2$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k$$

$$u_{\min} \leq u_k \leq u_{\max}$$

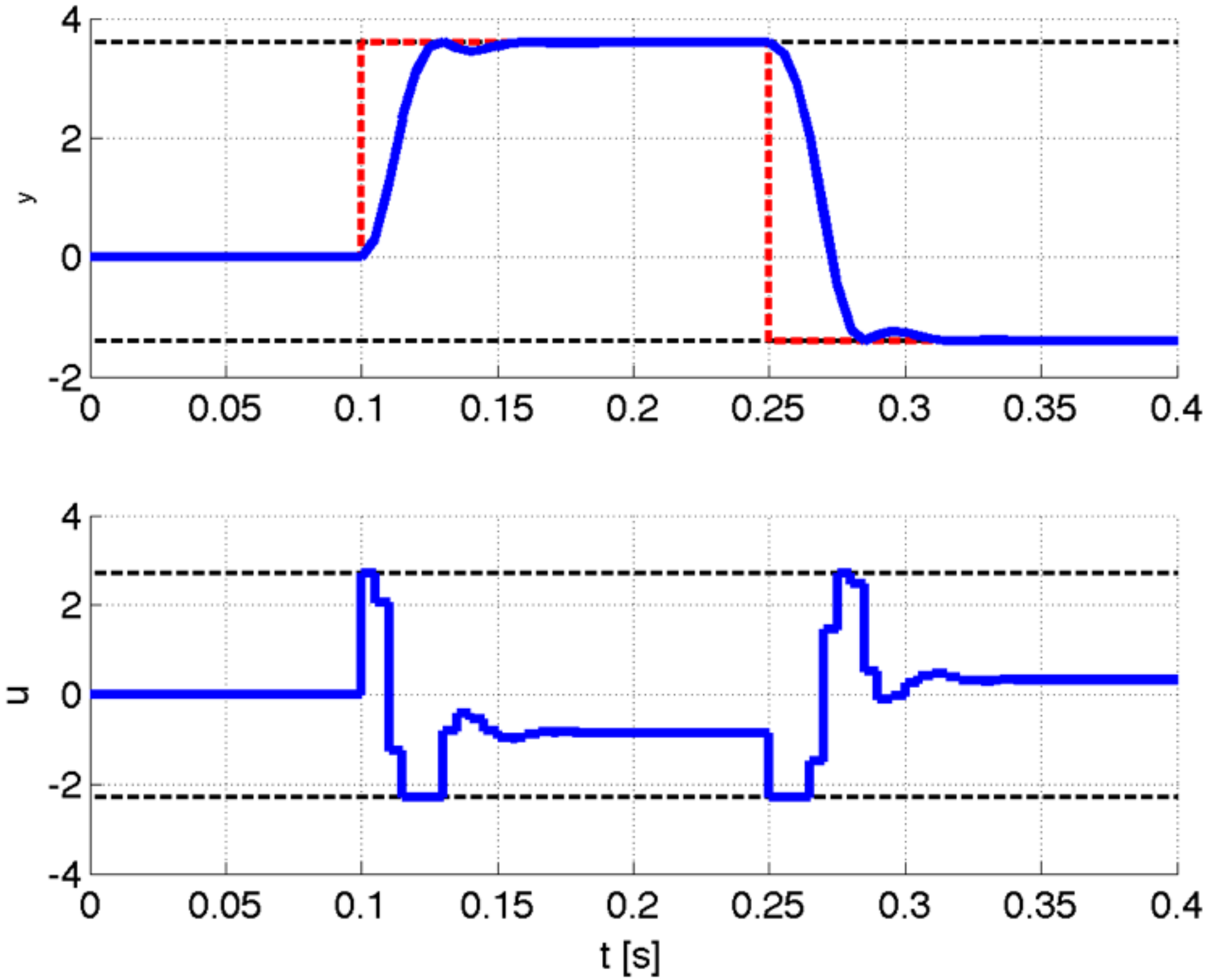
$$y_{\min} \leq y_k \leq y_{\max}$$

$$\Delta u_k = u_k - u_{k-1}$$

$$x_0 = x(t)$$

```
sys = LTISystem('A', A, 'B', B, 'C', C)
sys.u.min = -2.29, sys.u.max = 2.71
sys.y.min = -1.40, sys.y.max = 3.60
sys.y.penalty = QuadFunction(1)
sys.u.with('deltaPenalty')
sys.u.deltaPenalty = QuadFunction(1)
sys.y.with('reference')
sys.y.reference = 'free'
N = 5
ctrl = MPCController(sys, N)
```

Tracking of a Time-Varying References



MPC Prototyping

Regulation with disturbance rejection

Reference tracking

Soft constraints

Slew rate constraints

Custom setups

Soft Constraints

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} q(y_k - y_{\text{ref}})^2 + r\Delta u_k^2$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k$$

$$u_{\min} \leq u_k \leq u_{\max}$$

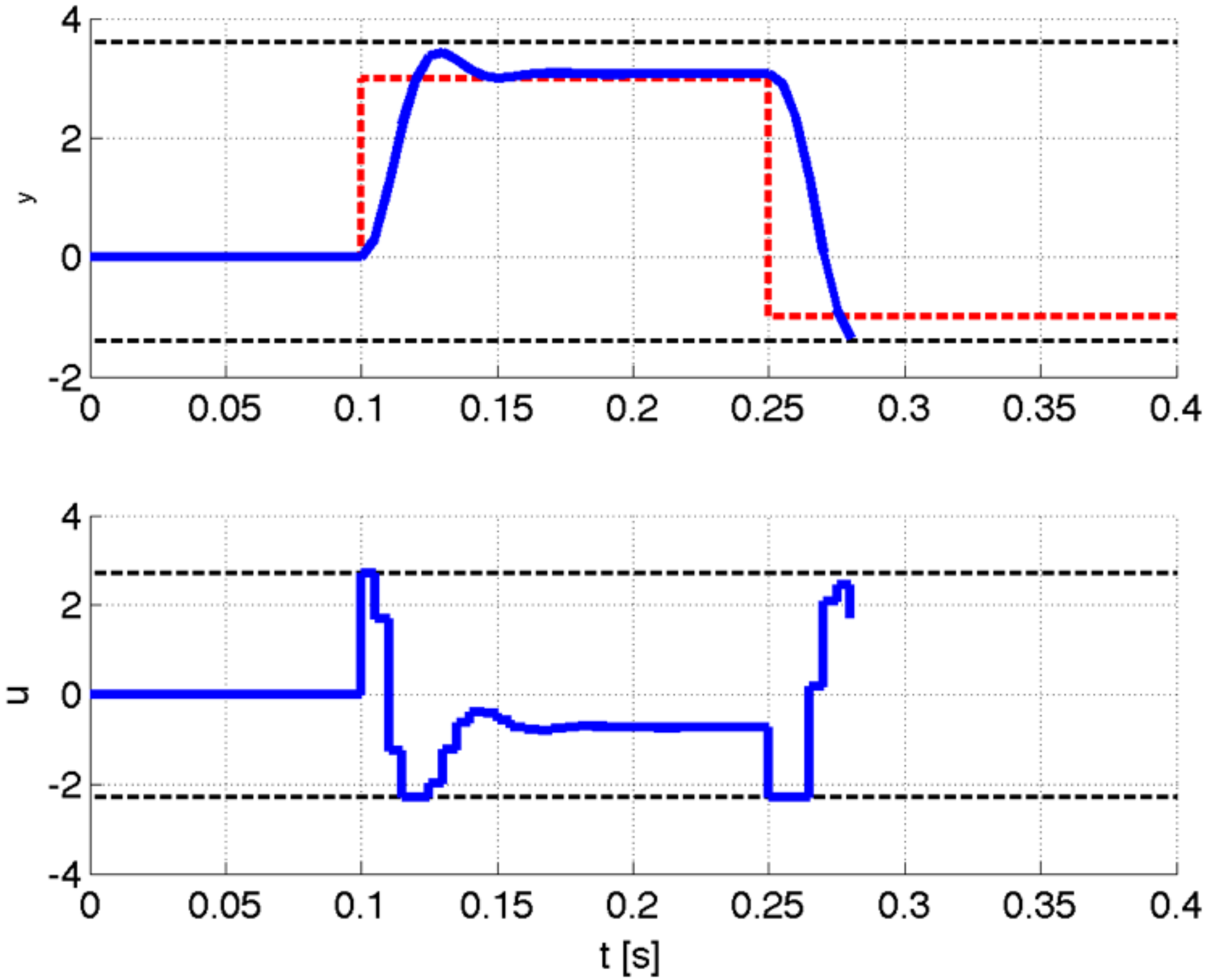
$$y_{\min} \leq y_k \leq y_{\max}$$

$$\Delta u_k = u_k - u_{k-1}$$

$$x_0 = x(t)$$

```
simsys = LTISystem('A', A+0.01*A, 'B', B, 'C', C)
loop = ClosedLoop(ctrl, simsys)
```

Tracking of a Time-Varying References



Soft Constraints

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} q(y_k - y_{\text{ref}})^2 + r\Delta u_k^2 + sz_k^2$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k$$

$$u_{\min} \leq u_k \leq u_{\max}$$

$$y_{\min} - z_k \leq y_k \leq y_{\max} + z_k$$

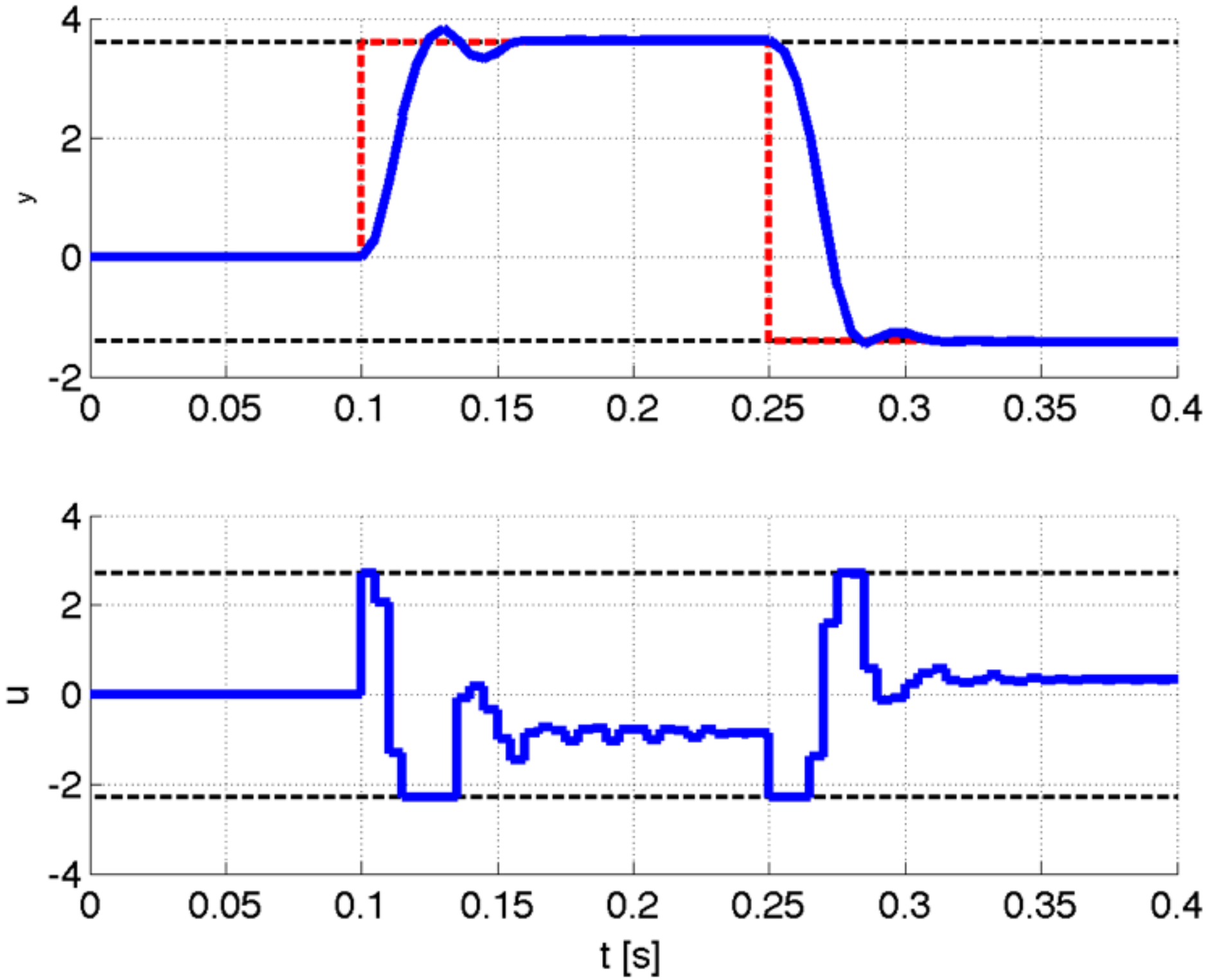
$$0 \leq z_k \leq z_{\max}$$

$$\Delta u_k = u_k - u_{k-1}$$

$$x_0 = x(t)$$

```
sys.y.with('softMin')
sys.y.softMin.maximalViolation = zmax
sys.y.softMin.penalty = QuadFunction(s)

sys.y.with('softMax')
sys.y.softMax.maximalViolation = zmax
sys.y.softMax.penalty = QuadFunction(s)
```

Soft Constraints



MPC Prototyping

Regulation with disturbance rejection

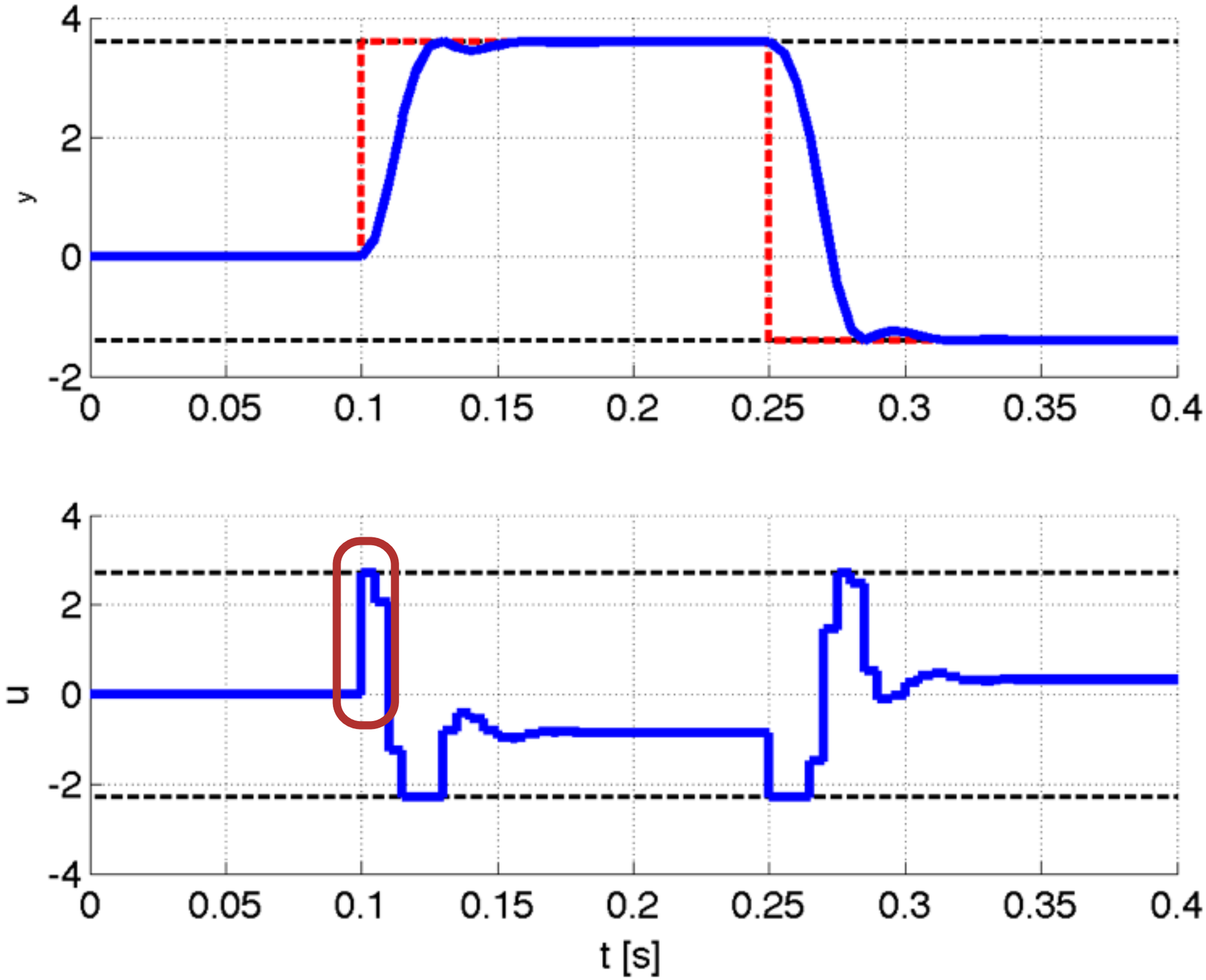
Reference tracking

Soft constraints

Slew rate constraints

Custom setups

Slew Rate Constraints



Slew Rate Constraints

$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} q(y_k - y_{\text{ref}})^2 + r\Delta u_k^2$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k$$

$$u_{\min} \leq u_k \leq u_{\max}$$

$$y_{\min} \leq y_k \leq y_{\max}$$

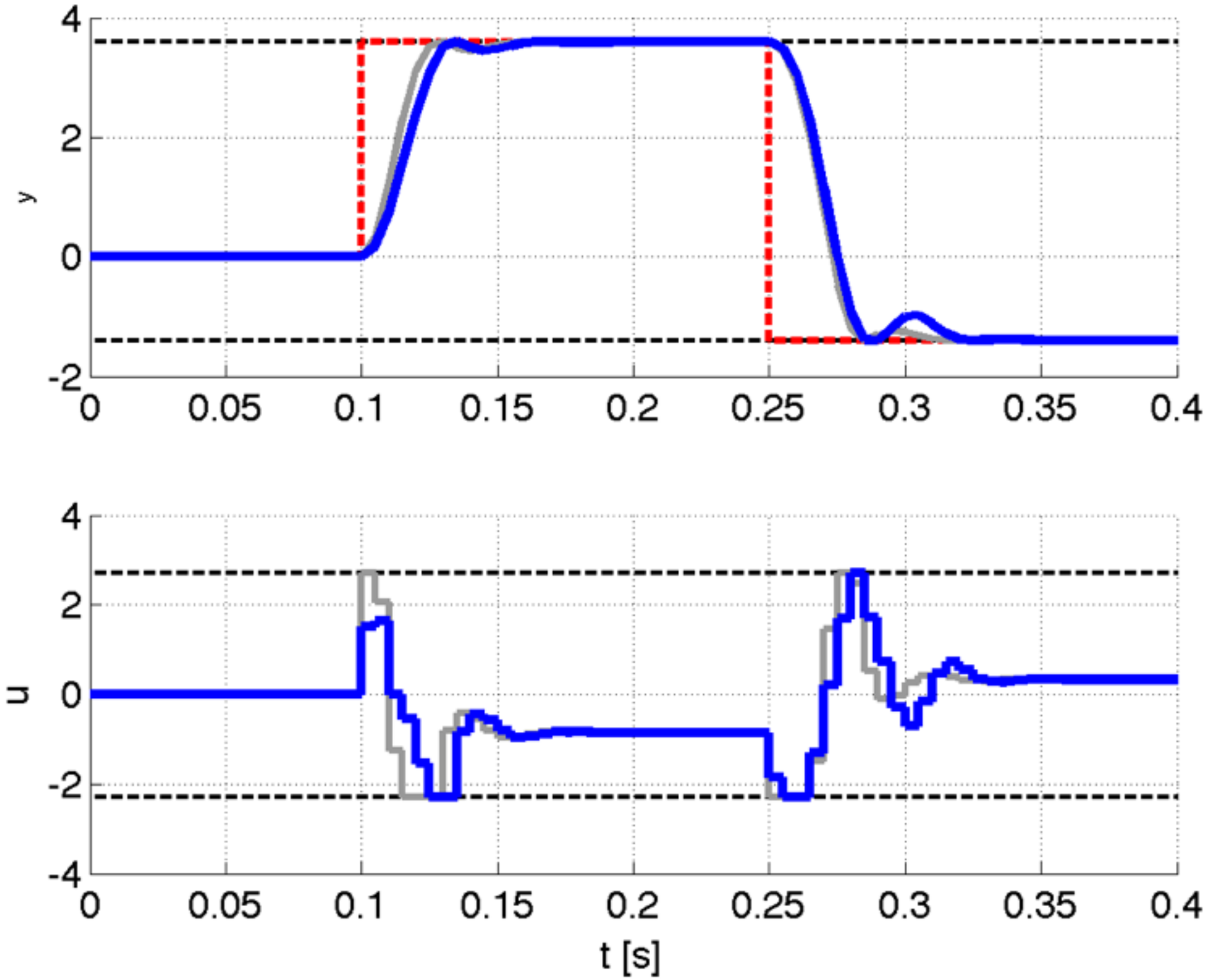
$$\Delta u_k = u_k - u_{k-1}$$

$$\Delta u_{\min} \leq \Delta u_k \leq \Delta u_{\max}$$

$$x_0 = x(t)$$

```
sys.u.with('deltaMin')
sys.u.deltaMin = -1
sys.u.with('deltaMax')
sys.u.deltaMax = 1.5
```

Slew Rate Constraints



MPC Prototyping

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MPT is based on YALMIP


Berthelemy inv. et P.
anno 1802.

Johan Löfberg

YALMIP

Language for rapid prototyping of optimization problems

Matlab-based, textbook-like coding

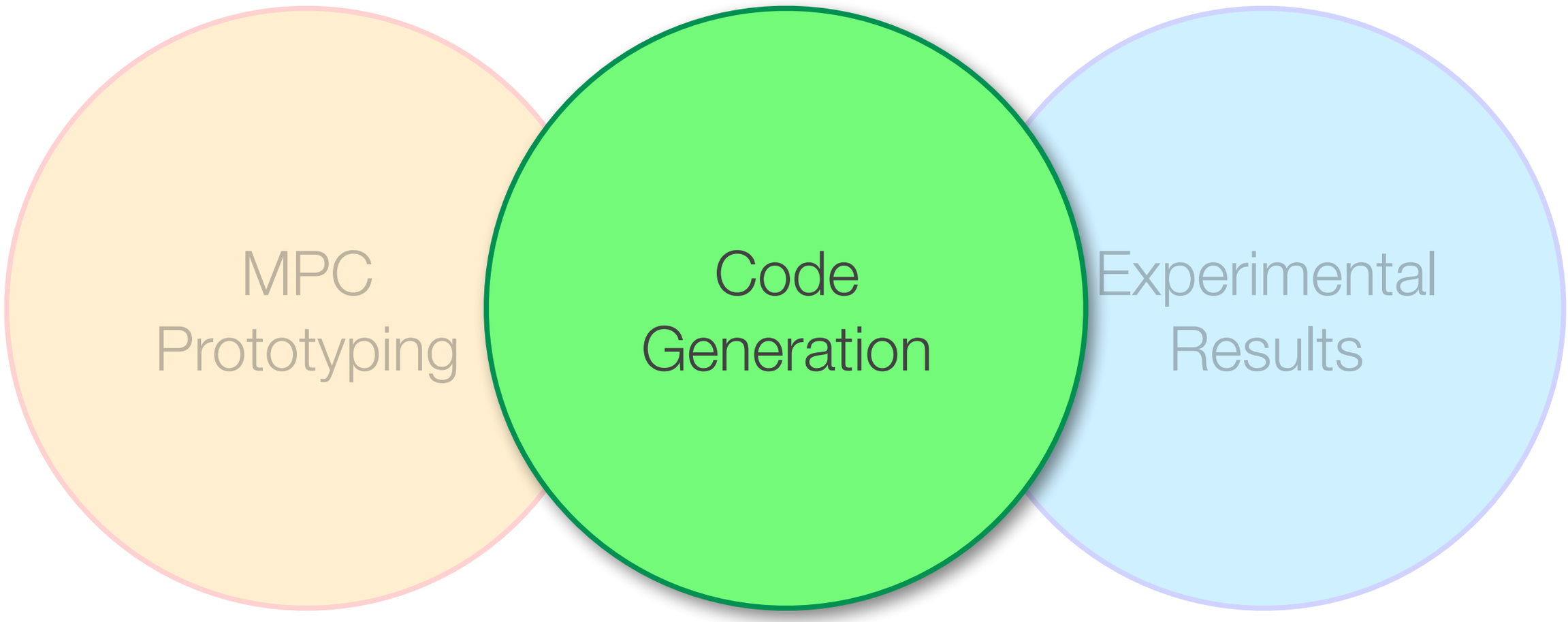
$$\min\ |x-y| + (y-3)^2$$
$$\text{s.t. } x^2 + y^2 \leq 1$$

$$\min\ z^T Q z + c^T z$$
$$\text{s.t. } z^T H_i z + A_i z \leq b_i$$

```
sdpvar x y
objective = abs(x-y) + (y-3)^2
constraints = [ x^2 + y^2 <= 1 ]
optimize(constraints, objective)
value([x y])
ans =

    0.3074    0.9516
```

Endless possibilities…

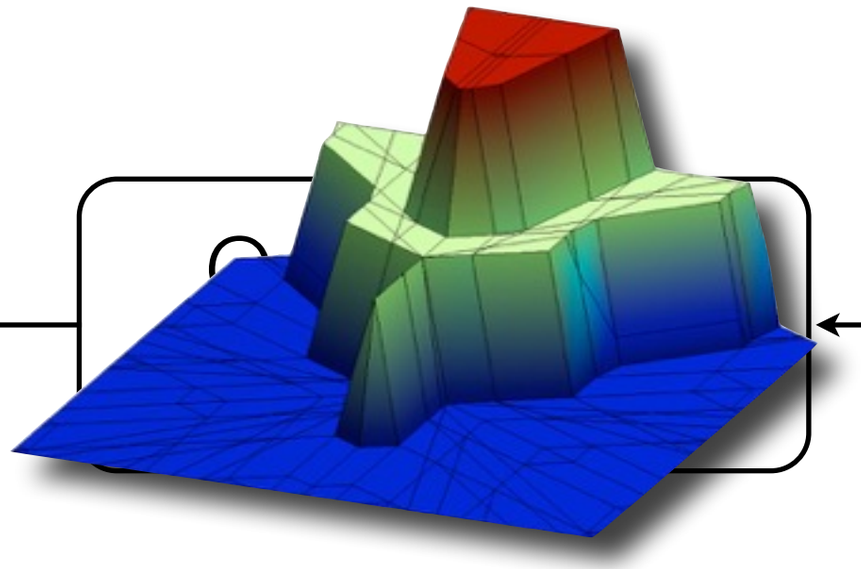
Agenda



Explicit MPC

Off-line

On-line

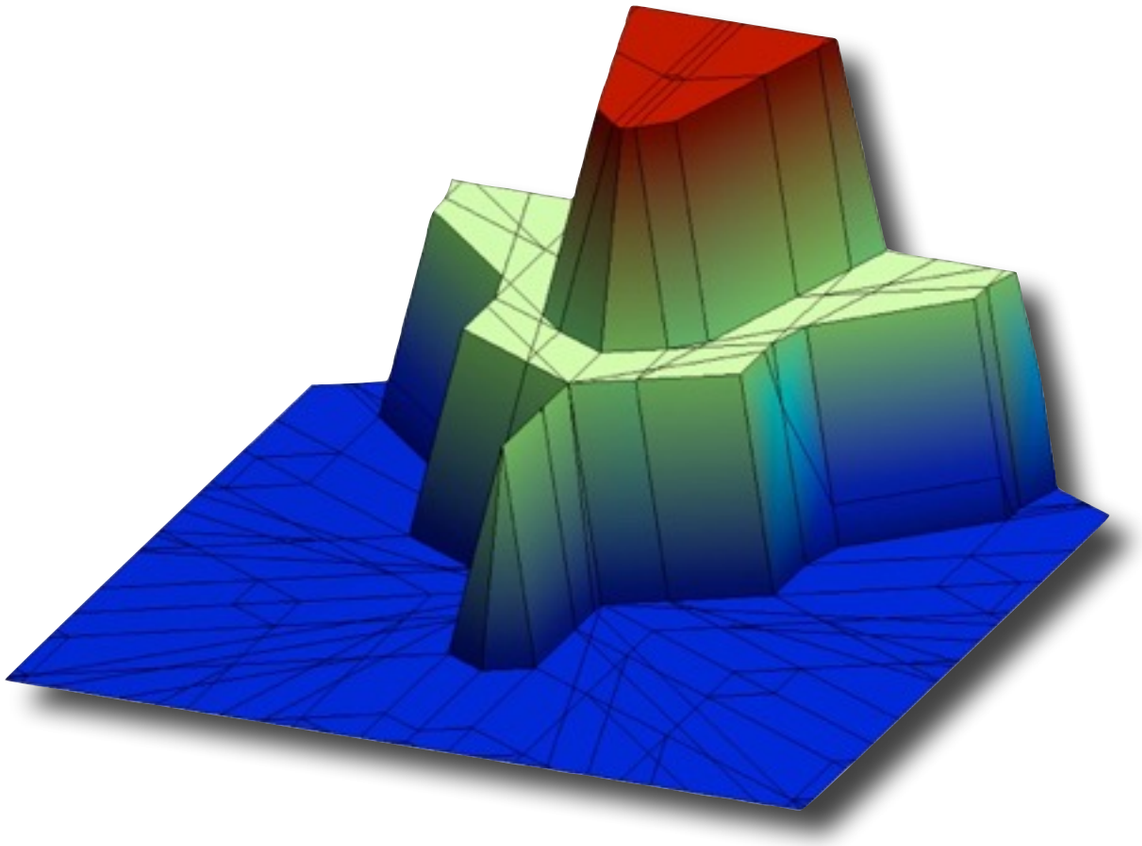


control action

plant state

Plant

Explicit MPC



Simple implementation

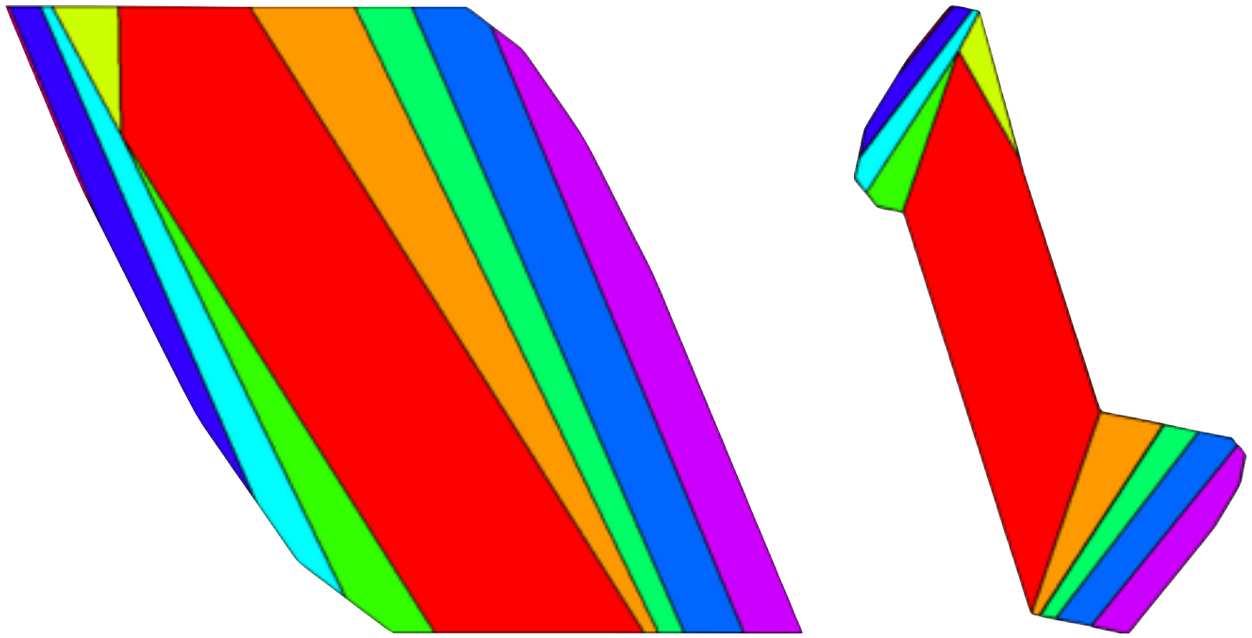
Verifiable performance

Predictable execution

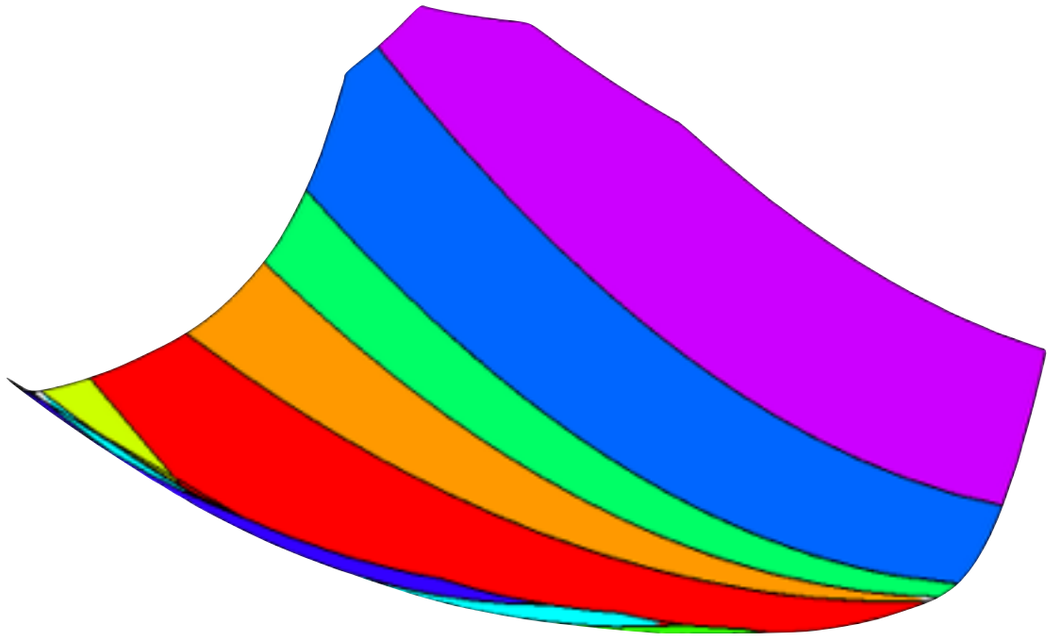
```
explicit = ctrl.toExplicit()
```

Command-Line Implementation

```
u0 = explicit.evaluate(x0)
data = ClosedLoop(explicit, sys).simulate(x0, Nsim)
```



`explicit.partition.plot()`



`explicit.cost.fplot()`

`explicit.feedback.fplot()`

Rigorous Analysis

Invariance computation

- is the MPC problem recursively feasible?
- if not, what is the largest set of initial conditions with such a property?

Stability analysis

- is the closed-loop system stable?
- what is a stabilizing terminal set/penalty?

Implementation analysis

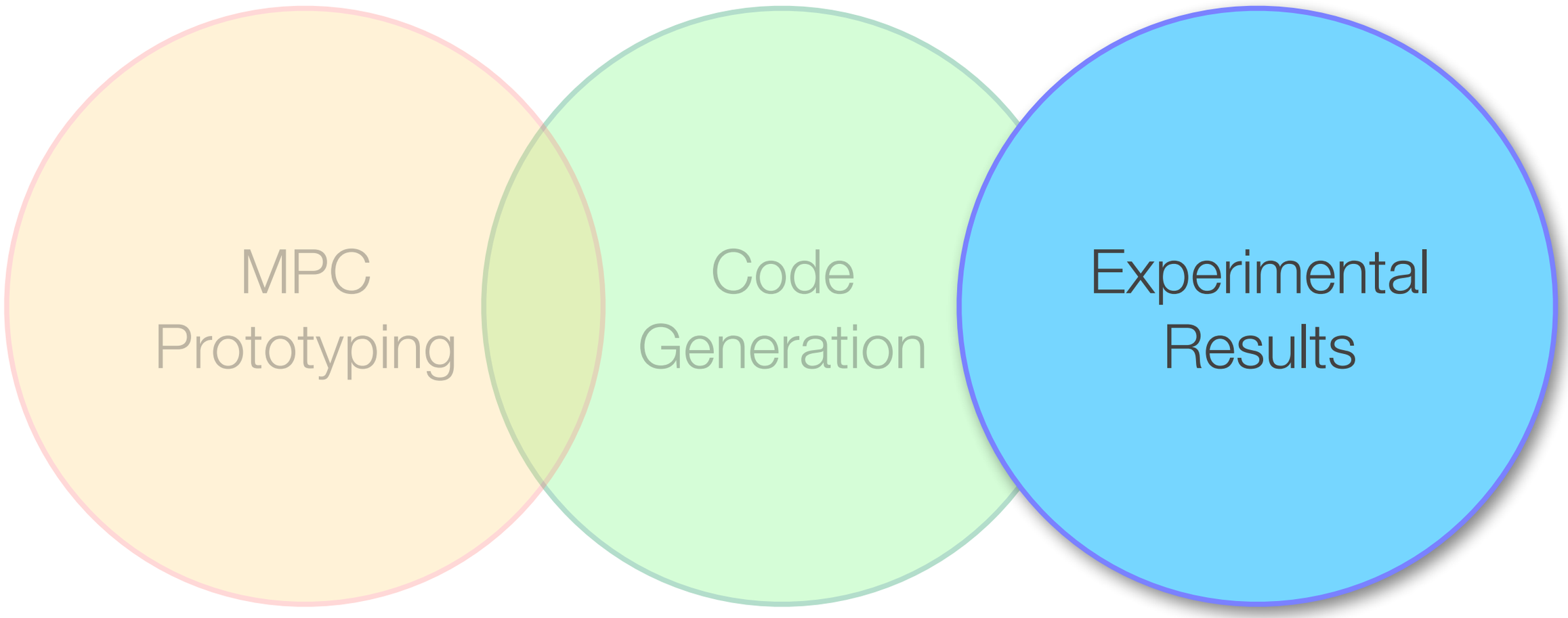
- how much memory and CPU power will I need?

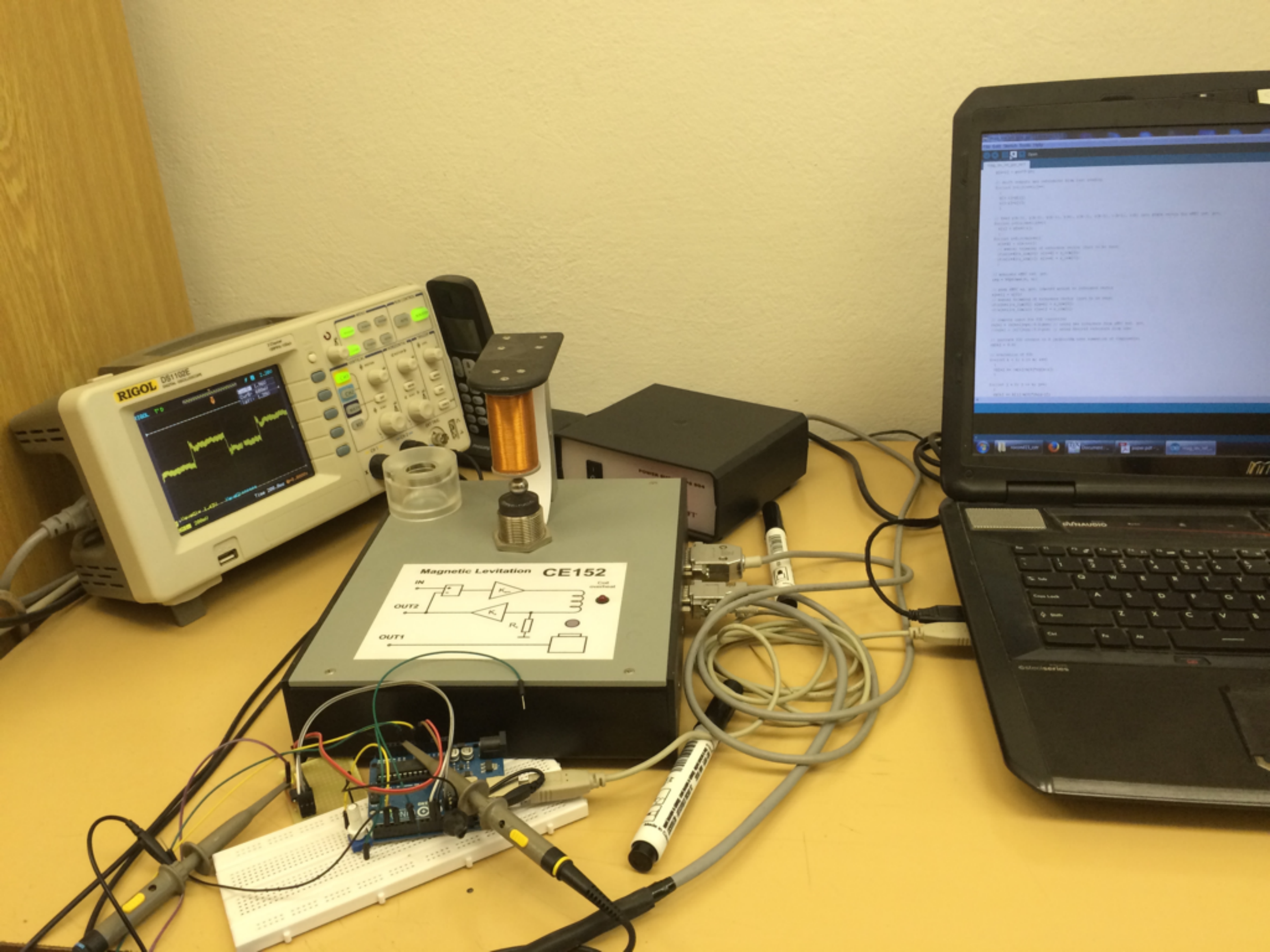
Code Generation

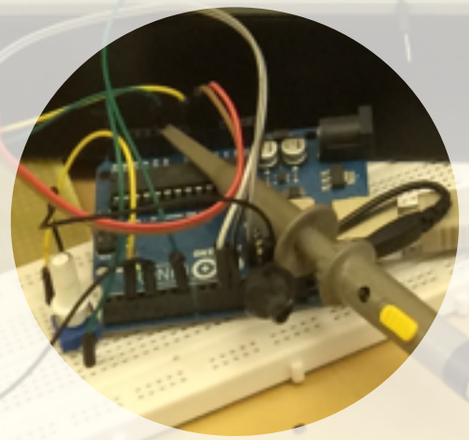
```
>> explicit.feedback.toC('my_controller')

Output written to "mycontroller.c".
C-mex function written to "mycontroller_mex.c".
```

```
static unsigned long mycontroller( double *X, double *U) {
    int ix, iu, ic, nc, isinside; unsigned long ireg, abspos, region; double hx;
    abspos = 0; region = 0;
    for (iu=0; iu<MPT_RANGE; iu++) { U[iu] = 0; }
    for (ireg=0; ireg<MPT_NR; ireg++) {
        isinside = 1; nc = MPT_NC[ireg];
        for (ic=0; ic<nc; ic++) {
            hx = 0; for (ix=0; ix<MPT_DOMAIN; ix++) {
           hx = hx + MPT_A[abspos*MPT_DOMAIN+ic*MPT_DOMAIN+ix]*X[ix]; }
            if ((hx - MPT_B[abspos+ic]) > MPT_ABSTOL) { isinside = 0; break; }
        }
        if (isinside==1) {
            region = ireg + 1; for (iu=0; iu<MPT_RANGE; iu++) {
                for (ix=0; ix<MPT_DOMAIN; ix++) {
             U[iu]=U[iu]+MPT_F[ireg*MPT_DOMAIN*MPT_RANGE+iu*MPT_DOMAIN+ix]*X[ix];
            }
                U[iu] = U[iu] + MPT_G[ireg*MPT_RANGE + iu];
            }
            return region;
        } abspos = abspos + MPT_NC[ireg];
    }
    return region;
}
```

Agenda




RIGOL
DS1102E
Magnetic Levitation
CE152
IN
OUT2
OUT1



Embedded Hardware

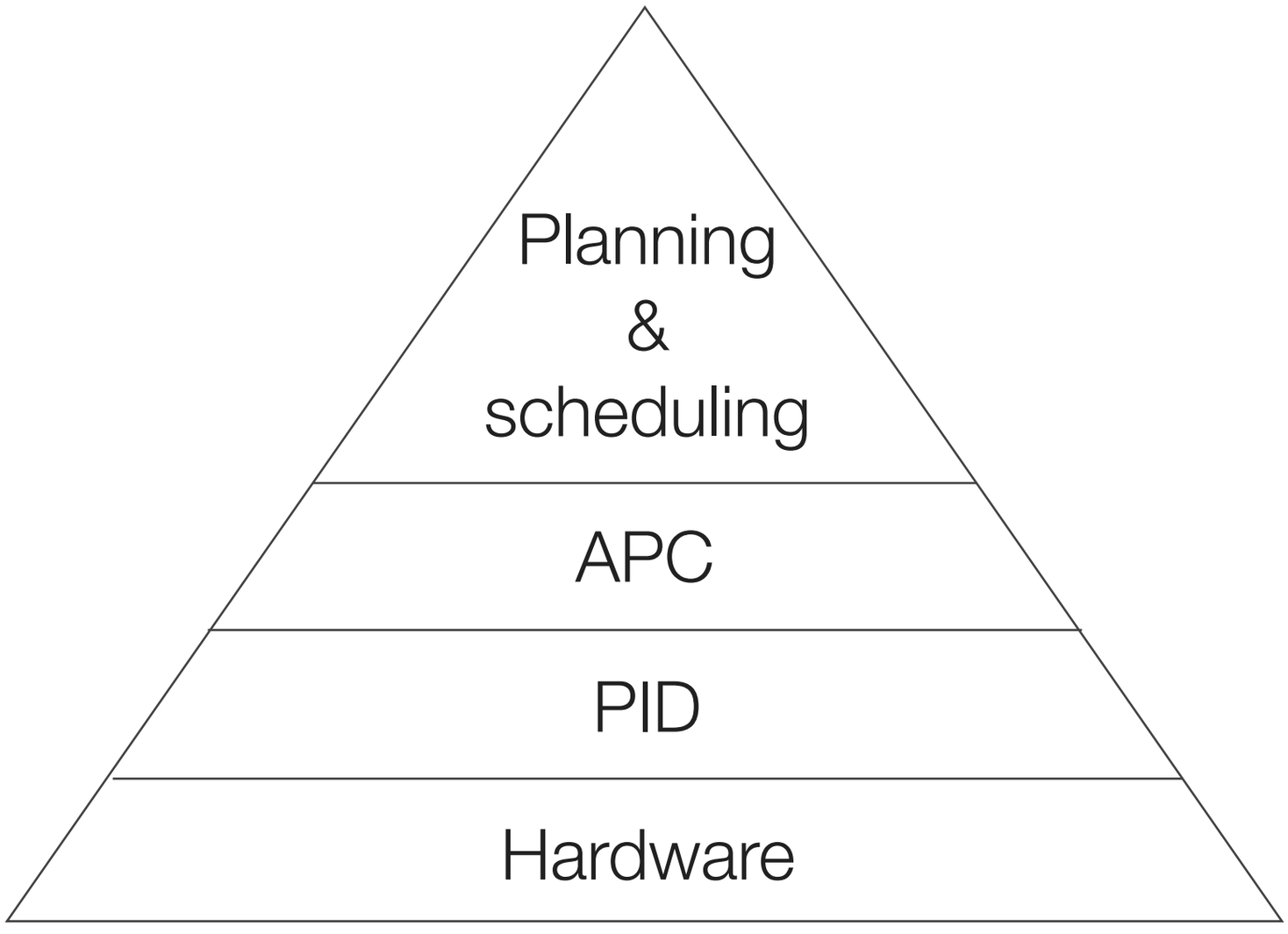
Atmel ATMega328p

8-bit, 16 MHz CPU

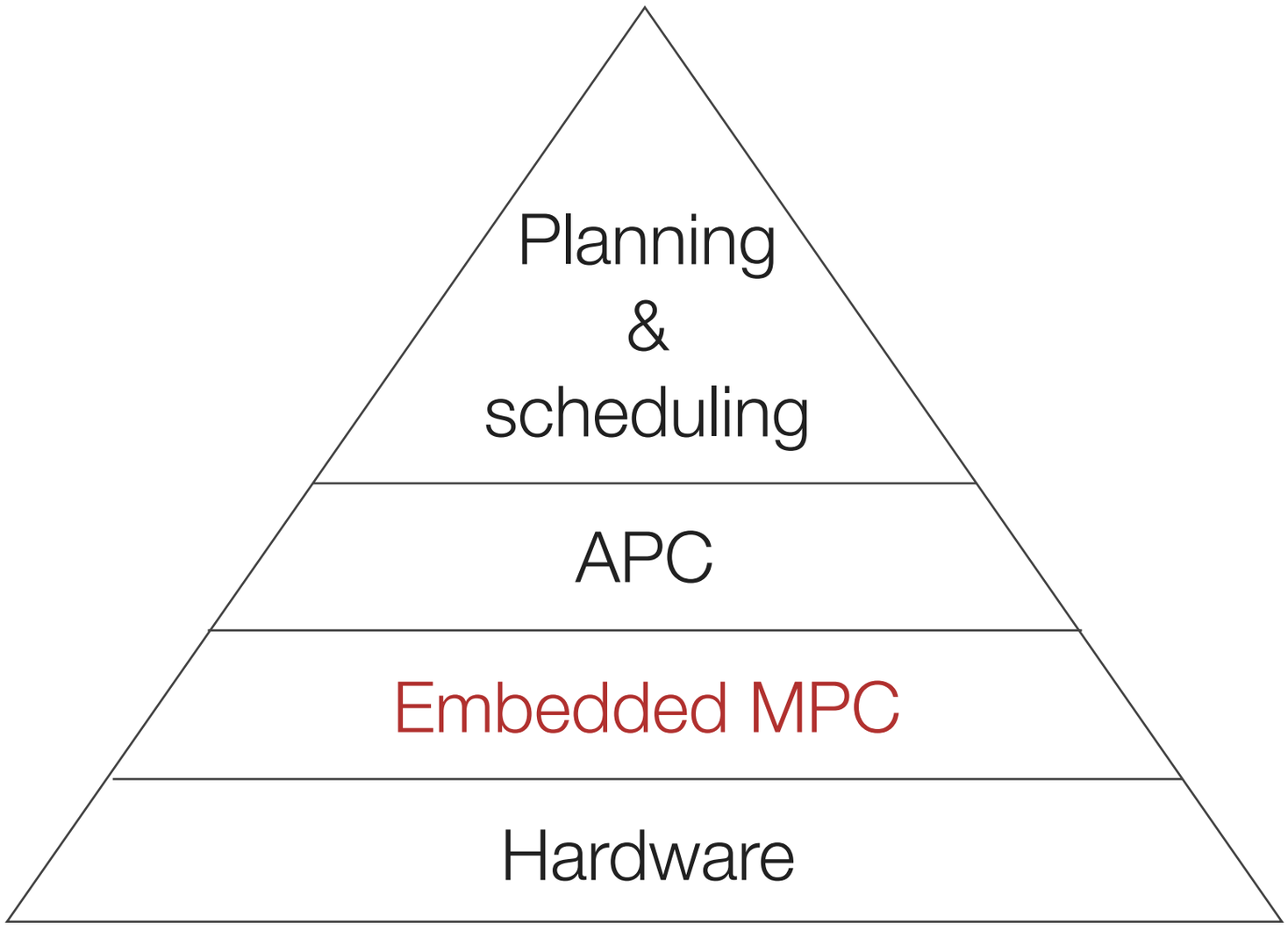
60 kFLOPs/sec

2 kB of SRAM for data

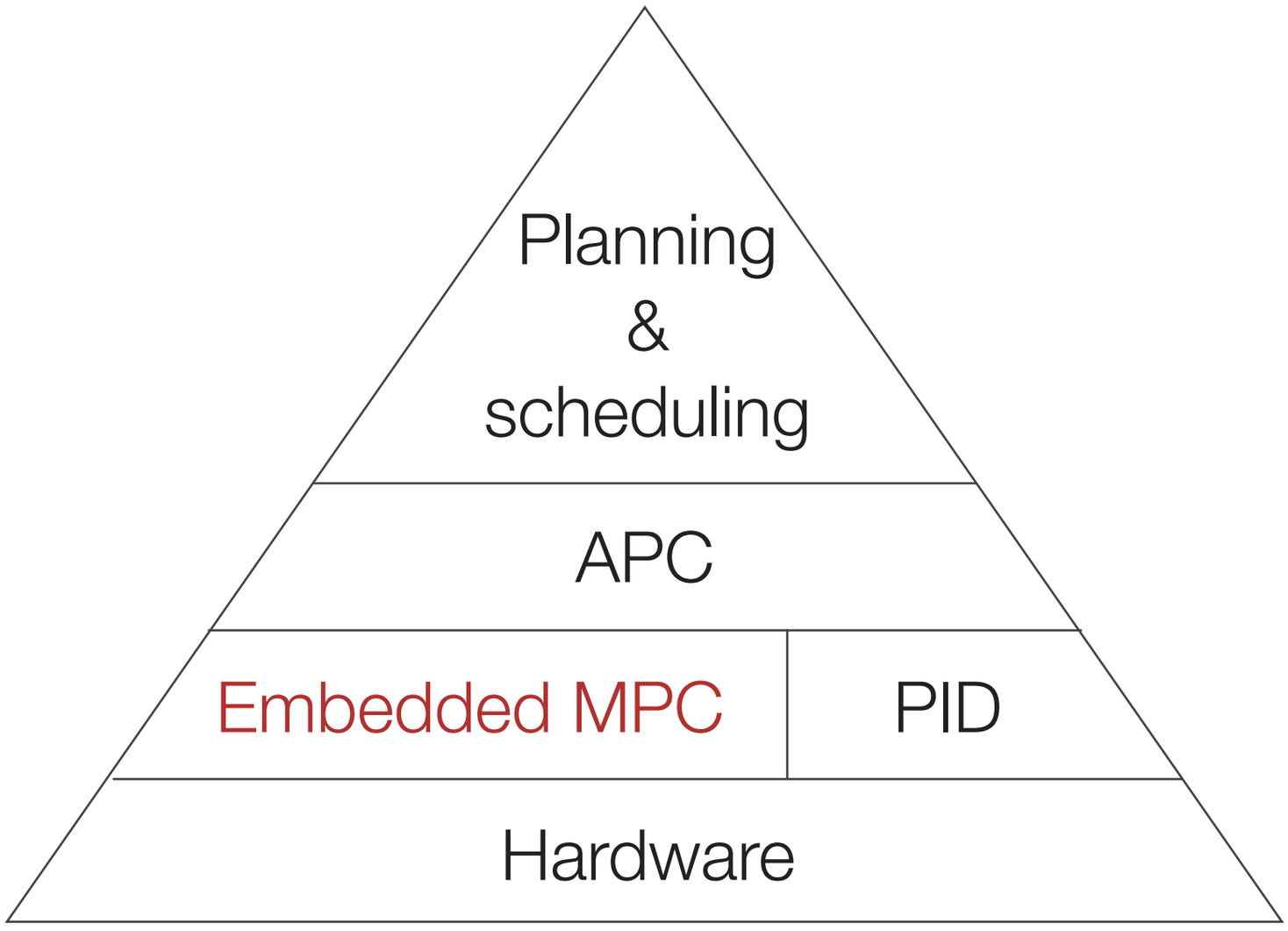
MPC in Process Industries



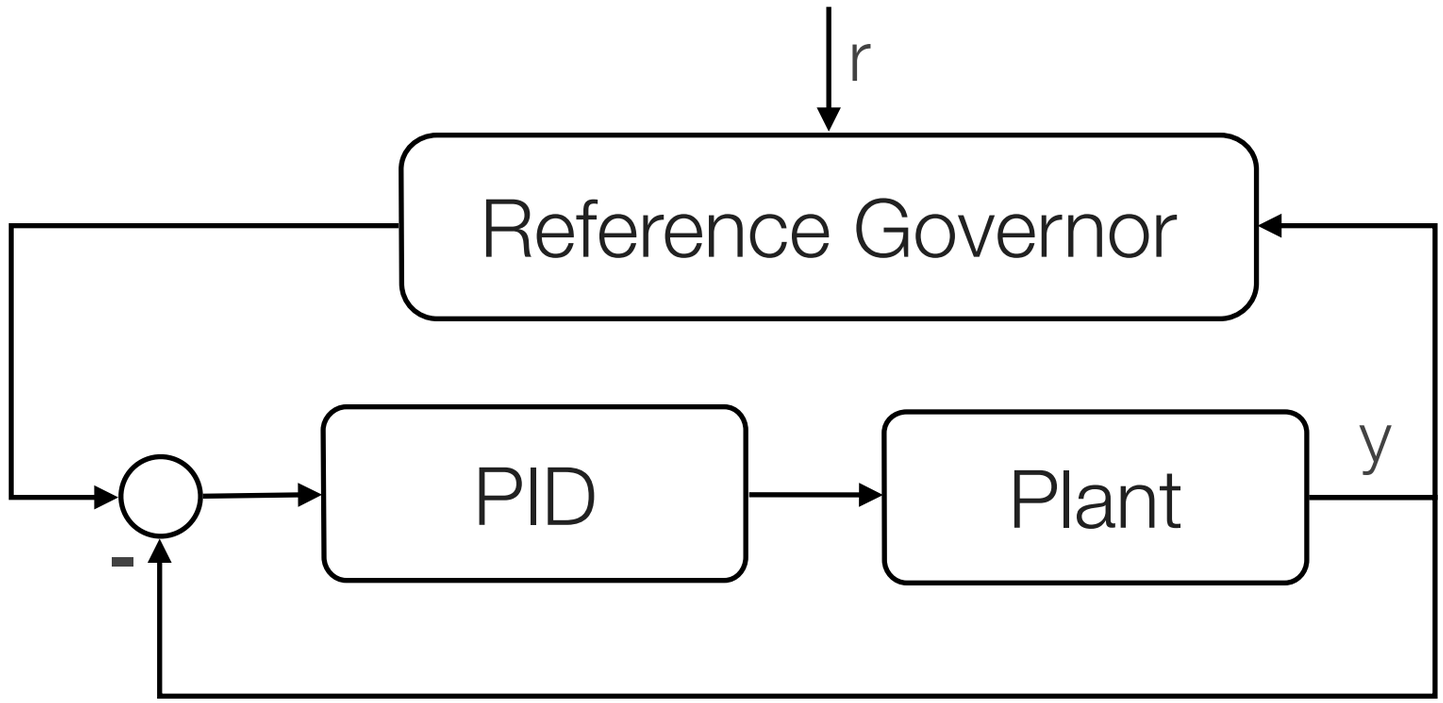
MPC in Process Industries



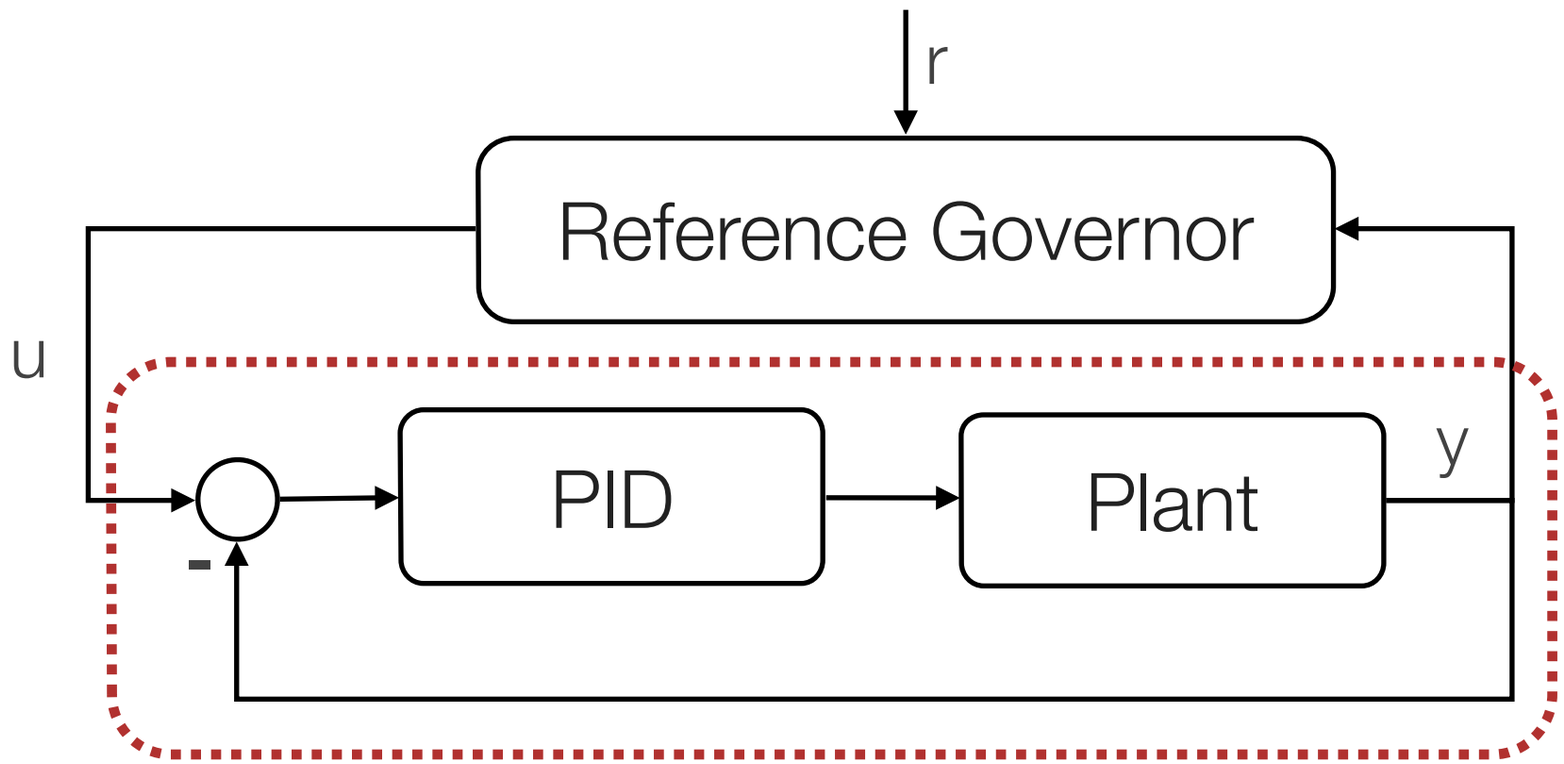
MPC in Process Industries



Reference Governor



Reference Governor



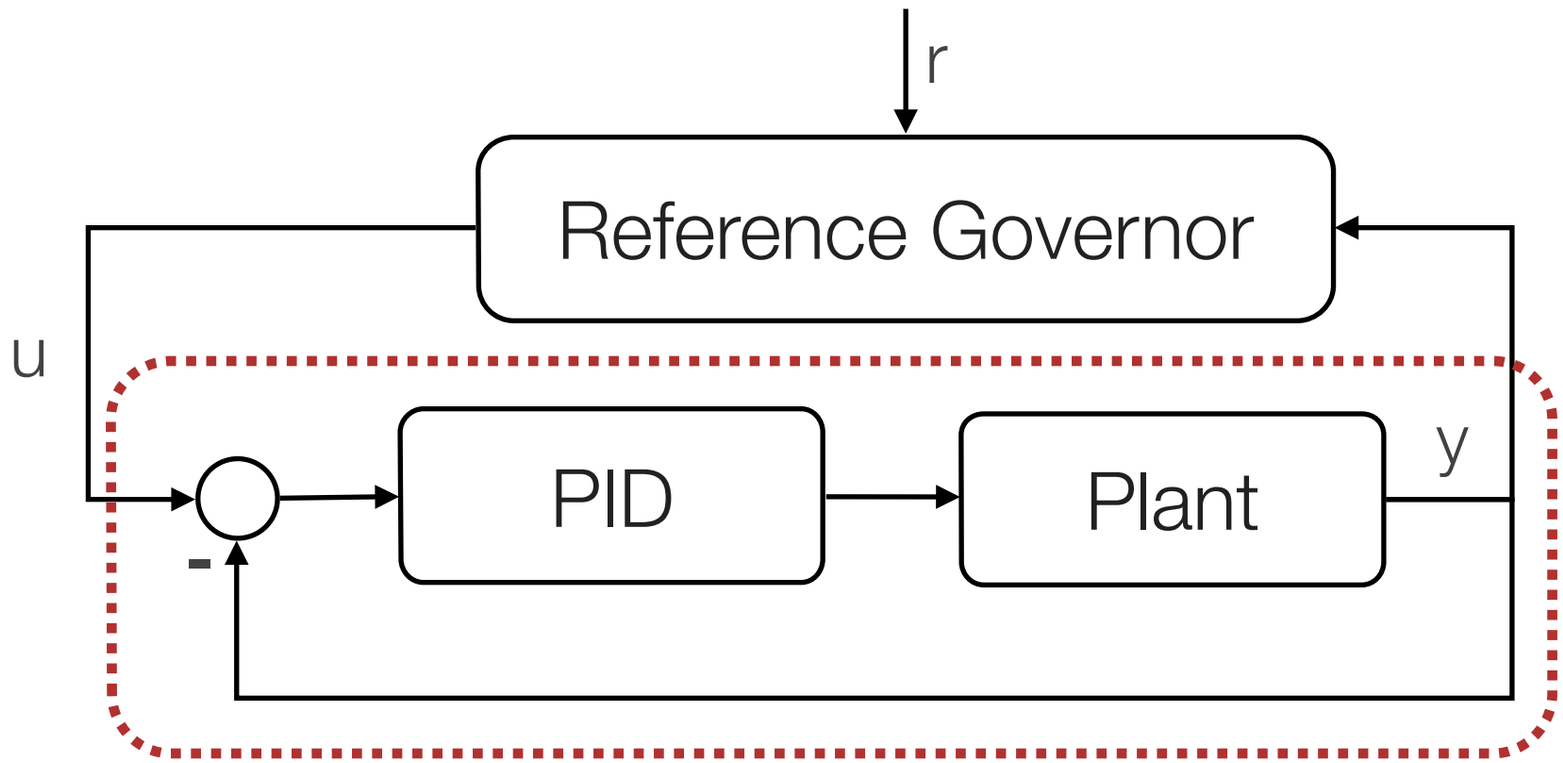
$$\min_{u_0,\ldots,u_{N-1}} \sum_{k=0}^{N-1} Q_{\mathrm{d}} \Delta u_k^2 + Q_{\mathrm{u}}(u_k - r)^2 + Q_{\mathrm{y}}(y_k - u_k)^2$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

$$y_{\min} \leq y_k \leq y_{\max}$$

Reference Governor



MPC setup:

- sampling time 2ms
- prediction horizon 3
- control horizon 1

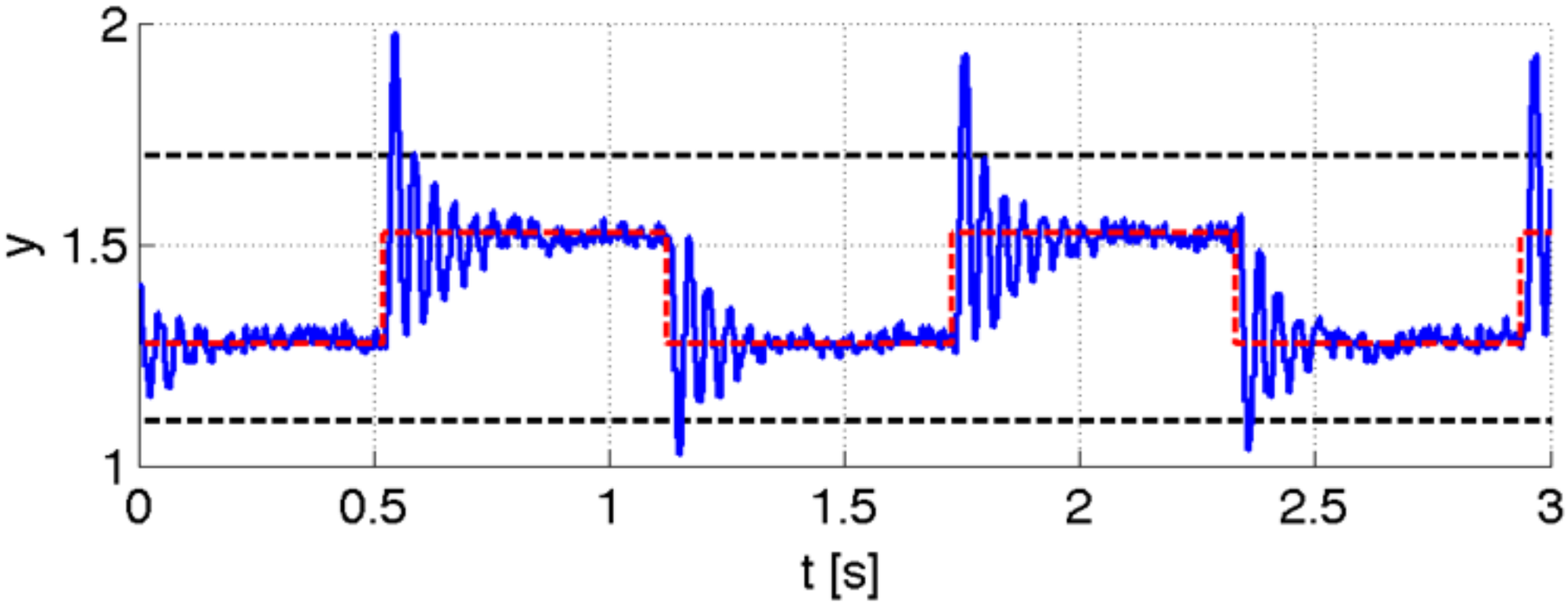
Explicit solution:

- 5 regions in 8D
- binary search tree
- 10 lines of C code

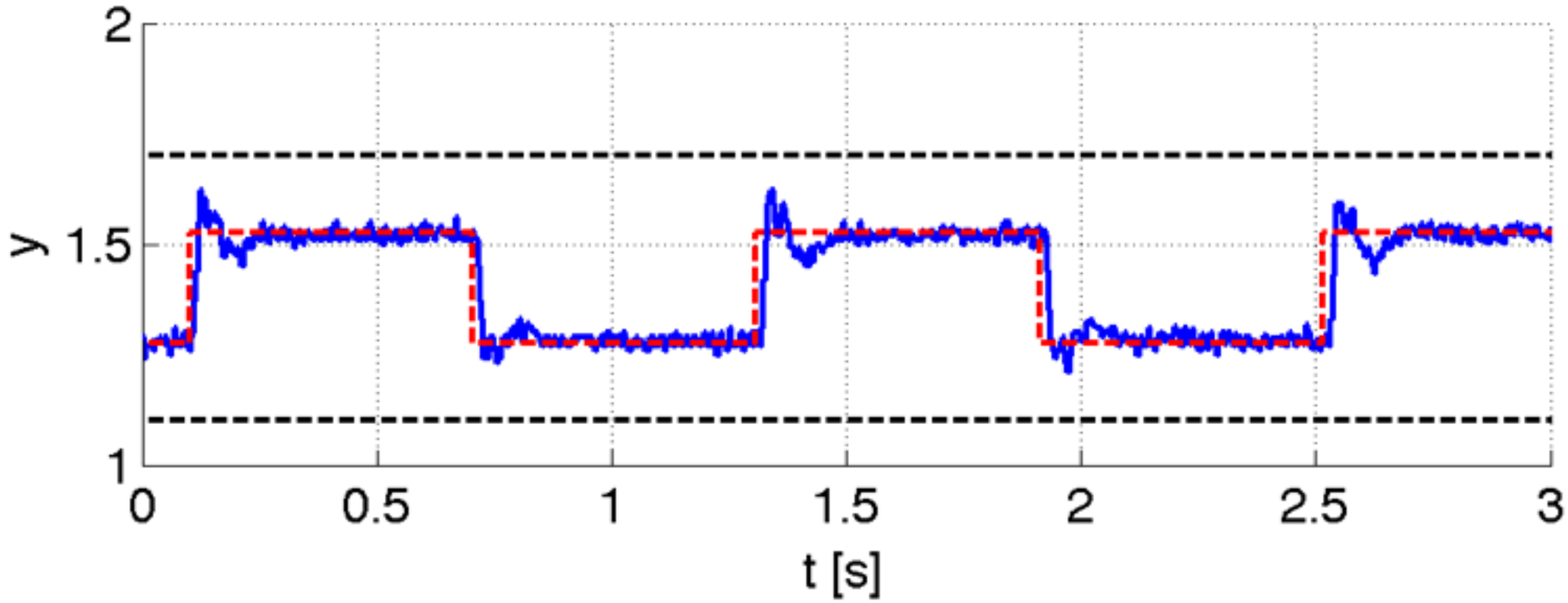
Implementation:

- 621 bytes
- 0.64 ms execution
- 0.50 ms for PID

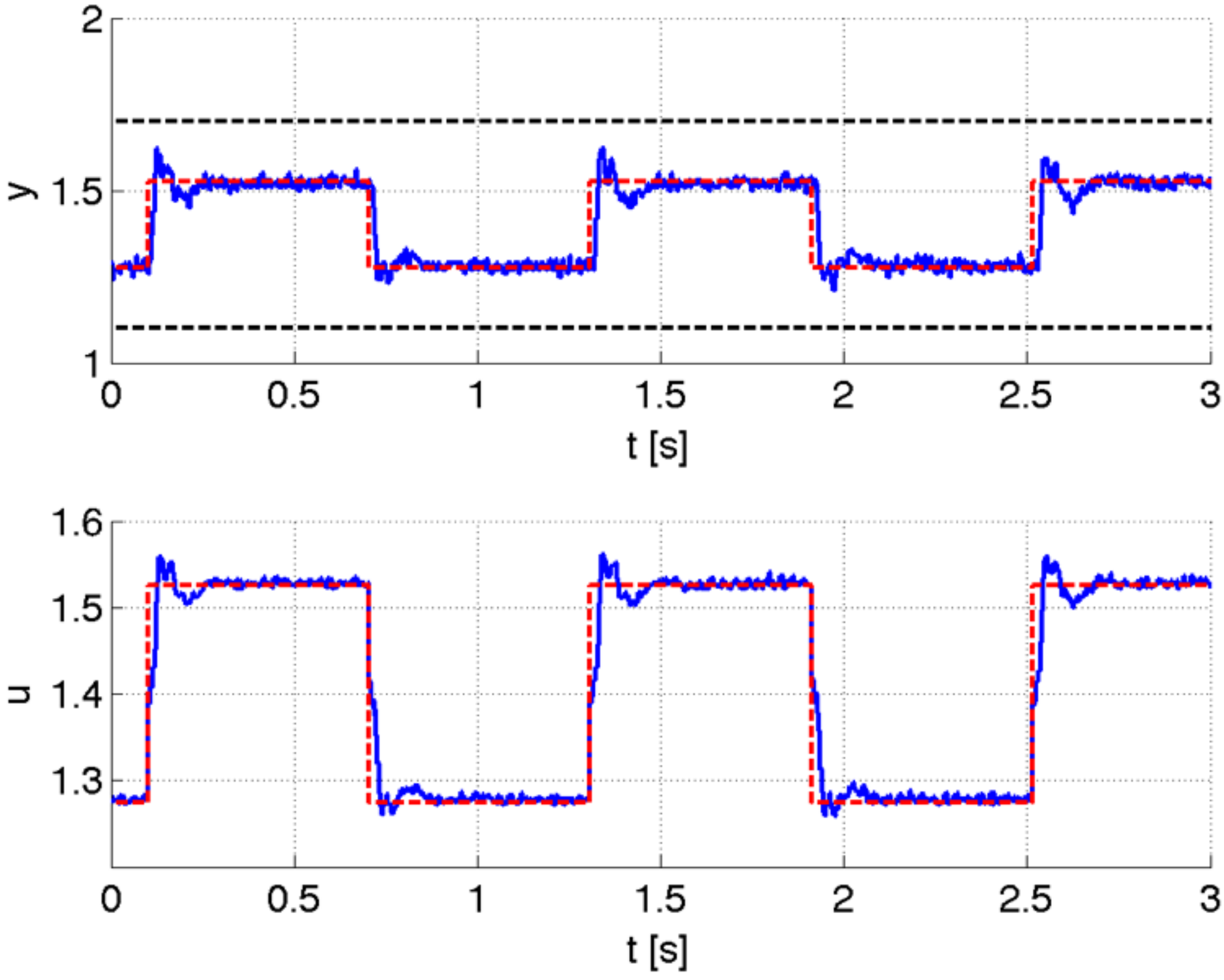
Experimental Results (PID)



Experimental Results (Reference Governor)

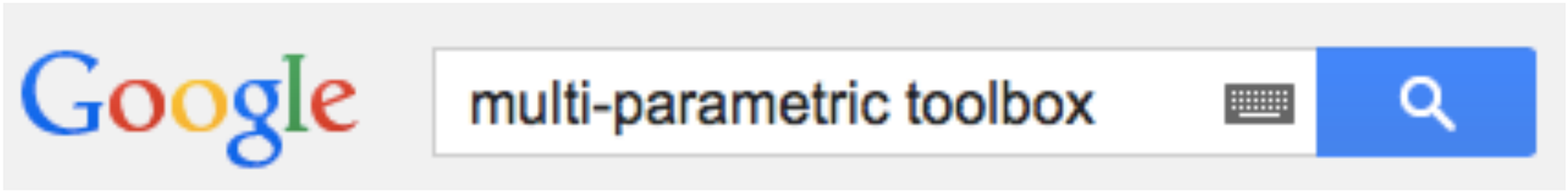


Experimental Results (Reference Governor)



Prototype & Deploy Optimization-Bazed Controllers


Whosoever holds
this hammer, if
he be worthy,
shall possess the
power of...
THOR


Google
multi-parametric toolbox