Dynamic System Models for Optimal Control

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Overview

- Ordinary Differential Equations (ODE)
- Boundary Conditions, Objective
- Differential-Algebraic Equations (DAE)
- Multi Stage Processes
- Partial Differential Equations (PDE)
- From continuous to discrete time
- Linear Quadratic Regulator (LQR)

Dynamic Systems and Optimal Control

- "Optimal control" = **optimal choice of inputs for a dynamic system**

Dynamic Systems and Optimal Control

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- What type of dynamic system?
  - Stochastic or deterministic?
  - Discrete or continuous time?
  - Discrete or continuous states?

Dynamic Systems and Optimal Control

- "Optimal control" = **optimal choice of inputs for a dynamic system**
- What type of dynamic system?
  - Stochastic or deterministic?
  - Discrete or continuous time?
  - Discrete or continuous states?
- In this course, treat deterministic differential equations and discrete time systems

Continous and discrete time deterministic systems

- Continuous time Ordinary Differential Equation (ODE):

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [0, T]$$

states $x \in \mathbb{R}^{n_x}$, control inputs $u \in \mathbb{R}^{n_u}$, $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$,

- Discrete time systems:

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \ldots$$

states $x_k \in X$, control inputs $u_k \in U$. Sets X, U can be continuous or discrete.

(Some other dynamic system classes)

- Games like chess: discrete time and state (chess figure positions), adverse player exists.
- Robust optimal control: like chess, but continuous time and state (adverse player exists in form of worst-case disturbances)
- Control of Markov chains: discrete time, system described by transition probabilities

$$P(x_{k+1}|x_k, u_k), \quad k = 0, 1, \ldots$$

- Stochastic Optimal Control of ODE: like Markov chain, but continuous time and state

Ordinary Differential Equations (ODE)

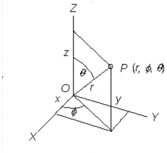
System dynamics can be manipulated by controls and parameters:

$$\boxed{\dot{x}(t) = f(t, x(t), u(t), p)}$$

- simulation interval: $[t_0, t_{\mathrm{end}}]$
- time $t \in [t_0, t_{\mathrm{end}}]$
- state $x(t) \in \mathbb{R}^{n_x}$
- controls $u(t) \in \mathbb{R}^{n_u}$ $\longleftarrow$ manipulated
- design parameters $p \in \mathbb{R}^{n_p}$ $\longleftarrow$ manipulated

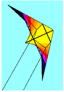
ODE Example: Dual Line Kite Model

- Kite position relative to pilot in spherical polar coordinates r, ϕ, θ. Line length r fixed.
- System states are $x = (\theta, \phi, \dot{\theta}, \dot{\phi})$.
- We can control roll angle $u = \psi$.
- Nonlinear dynamic equations:



$$\ddot{\theta} = \frac{F_\theta(\theta, \phi, \dot{\theta}; \dot{\phi}, \psi)}{rm} + \sin(\theta)\cos(\theta)\dot{\phi}^2$$

$$\ddot{\phi} = \frac{F_\phi(\theta, \phi, \dot{\theta}; \dot{\phi}, \psi)}{rm\sin(\theta)} - 2\cot(\theta)\dot{\phi}\dot{\theta}$$



- Summarize equations as $\dot{x} = f(x, u)$.

Initial Value Problems (IVP)

THEOREM [Picard 1890, Lindelöf 1894]:

Initial value problem in ODE

$$\dot{x}(t) = f(t, x(t), u(t), p), \quad t \in [t_0, t_{\mathrm{end}}],$$
$$\dot{x}(t_0) = x_0$$

- with given initial state x_0, design parameters p, and controls $u(t)$,
- and Lipschitz continuous $f(t, x, u(t), p)$

has **unique** solution

$$x(t), \quad t \in [t_0, t_{\mathrm{end}}]$$

NOTE: Existence but not uniqueness guaranteed if $f(t, x, u(t), p)$ only continuous [G. Peano, 1858-1932]. Non-uniqueness example: $\dot{x} = \sqrt{|x|}$

Boundary Conditions

Constraints on initial or intermediate values are important part of dynamic model.

STANDARD FORM:

$$\boxed{r(x(t_0), x(t_1), \ldots, x(t_{\text{end}}), p) = 0, \quad r \in \mathbb{R}^{n_r}}$$

E.g. fixed or parameter dependent initial value x_0:

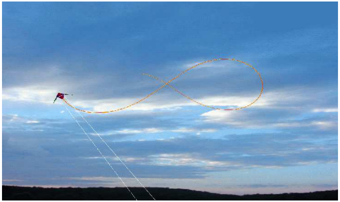
$$x(t_0) - x_0(p) = 0 \qquad (n_r = n_x)$$

or periodicity:

$$x(t_0) - x(t_{\text{end}}) = 0 \qquad (n_r = n_x)$$

NOTE: Initial values $x(t_0)$ need not always be fixed!

Kite Example: Periodic Solution Desired



- Formulate periodicity as constraint.
- Leave $x(0)$ free.
- Minimize integrated power per cycle

$$\min_{x(\cdot),u(\cdot)} \int_0^T L(x(t), u(t))dt$$

subject to

$$\begin{aligned} x(0) - x(T) &= 0 \\ \dot{x}(t) - f(x(t), u(t)) &= 0, \quad t \in [0, T]. \end{aligned}$$

Objective Function Types

Typically, distinguish between

- *Lagrange term* (cost integral, e.g. integrated deviation):

$$\int_0^T L(t, x(t), u(t), p)dt$$

- *Mayer term* (at end of horizon, e.g. maximum amount of product):

$$E(T, x(T), p)$$

- Combination of both is called *Bolza objective*.

Differential-Algebraic Equations (DAE) - Semi-Explicit

Augment ODE by **algebraic equations** g and **algebraic states** z

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), z(t), u(t), p) \\ 0 &= g(t, x(t), z(t), u(t), p) \end{aligned}$$

- differential states $x(t) \in \mathbb{R}^{n_x}$
- algebraic states $z(t) \in \mathbb{R}^{n_z}$
- algebraic equations $g(\cdot) \in \mathbb{R}^{n_z}$

Standard case: index one $\Leftrightarrow$ matrix $\frac{\partial g}{\partial z} \in \mathbb{R}^{n_z \times n_z}$ invertible.

Existence and uniqueness of initial value problems similar as for ODE.

Tutorial DAE Example

Regard $x \in \mathbb{R}$ and $z \in \mathbb{R}$, described by the DAE

$$\begin{aligned} \dot{x}(t) &= x(t) + z(t) \\ 0 &= \exp(z) - x \end{aligned}$$

- Here, one could easily eliminate $z(t)$ by $z = \log x$, to get the ODE

$$\dot{x}(t) = x(t) + \log(x(t))$$

Tutorial DAE Example

Regard $x \in \mathbb{R}$ and $z \in \mathbb{R}$, described by the DAE

$$\begin{aligned} \dot{x}(t) &= x(t) + z(t) \\ 0 &= \exp(z) - x + z \end{aligned}$$

- Now, z cannot be eliminated as easily as before, but still, the DAE is well defined because $\frac{\partial g}{\partial z}(x, z) = \exp(z) + 1$ is always positive and thus invertible.

Fully Implicit DAE

A fully implicit DAE is just a set of equations:

$$0 = f(t, x(t), \dot{x}(t), z(t), u(t), p)$$

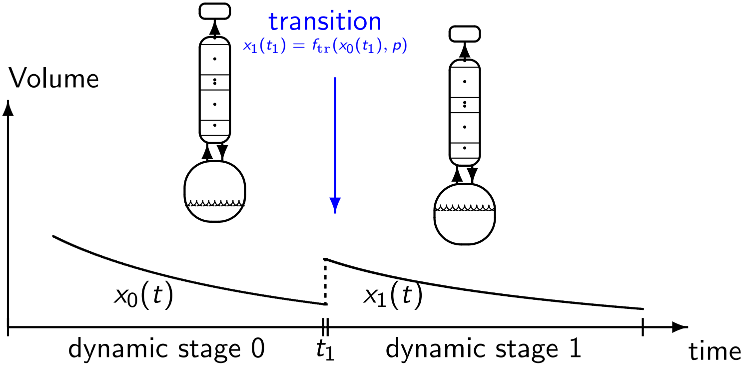
- derivative of differential states $\dot{x}(t) \in \mathbb{R}^{n_x}$
- algebraic states $z(t) \in \mathbb{R}^{n_z}$

Standard case: fully implicit DAE of index one $\Leftrightarrow$ matrix $\frac{\partial f}{\partial(\dot{x},z)} \in \mathbb{R}^{(n_x+n_z)\times(n_x+n_z)}$ invertible.

Again, existence and uniqueness similar as for ODE.

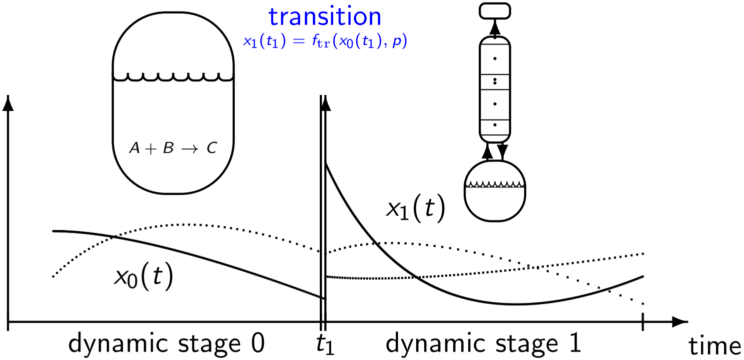
Multi Stage Processes

Two dynamic stages can be connected by a discontinuous "transition". **E.g. Intermediate Fill Up in Batch Distillation**



Multi Stage Processes II

Also **different** dynamic systems can be coupled. **E.g. batch reactor followed by distillation (different state dimensions)**



Partial Differential Equations

- Instationary partial differential equations (PDE) arise e.g in transport processes, wave propagation, ...
- Also called "distributed parameter systems"
- Often PDE of subsystems are coupled with each other (e.g. flow connections)
- Method of Lines (MOL): discretize PDE in space to yield ODE or DAE system.
- Often MOL can be interpreted in terms of compartment models.

From continous ODE to discrete time systems

- Solution $x(t)$ of ODE $\dot{x} = f(x, u)$ can be computed by **numerical integration** (details in talk by Rien)
- if control is kept constant $u(t) \equiv q$ and initial value $x(0) = s$ specified, integrator delivers solution trajectory

$$x(t; s, q)$$

- for sampling time Δt, can use $f_{\mathrm{d}}(s, q) := x(\Delta t; s, q)$ to obtain discrete time system

$$s_{k+1} = f_{\mathrm{d}}(s_k, q_k)$$

- In case of linear ODE $\dot{x} = Ax + Bu$, discrete linear system $s_{k+1} = A_{\mathrm{d}} s_k + B_{\mathrm{d}} q$ can be obtained by matrix exponentials:

$$A_{\mathrm{d}} := e^{A\Delta t}, \quad B_{\mathrm{d}} := \int_0^{\Delta t} e^{At} B \mathrm{d}t$$

Linearization of Nonlinear Systems

- Nonlinear discrete time system $f_\mathrm{d}(s, q)$ can be linearized at any point $(\bar{s}, \bar{q})$ to obtain first order Taylor expansion:

$$f_\mathrm{d}(s, q) \approx f_\mathrm{d}(\bar{s}, \bar{q}) + \underbrace{\frac{\partial f_\mathrm{d}}{\partial s}(\bar{s}, \bar{q})}_{=:A_\mathrm{d}}(s - \bar{s}) + \underbrace{\frac{\partial f_\mathrm{d}}{\partial q}(\bar{s}, \bar{q})}_{=:B_\mathrm{d}}(q - \bar{q})$$

- If evaluated at steady state, derivatives are identical to matrix exponentials of linearized continous time system (matter of convenience which way to go).

Linear Quadratic Regulator (LQR)

Simplest optimal control problem: linear system $x_{k+1} = Ax_k + Bu_k$ with quadratic cost on infinite horizon:

$$\min_{u_0, x_1, u_1, \ldots} \sum_{k=0}^{\infty} x_k^\top Q x_k + u_k^\top R u_k$$

- Solved with help of **discrete time Riccati equation**

  $$P = Q + A^T PA - (A^T PB)(R + B^T PB)^{-1}(B^T PA)$$

  to determine matrix P yielding **optimal feedback control**

  $$u^*(x) = -\underbrace{(R + B^T PB)^{-1}(B^T PA)}_{=K} x$$

- Implemented e.g. in MATLAB's dlqr command.

Summary

Dynamic models for optimal control consist of

- differential equations (ODE/DAE/PDE)
- boundary conditions, e.g. initial/final values, periodicity
- objective in Lagrange and/or Mayer form
- transition stages in case of multi stage processes

PDE can be transformed to DAE by Method of Lines (MOL)

ODE standard form for this course:

$$\boxed{\dot{x}(t) \quad = \quad f(x(t), u(t))}$$

- Discrete time models can be obtained by numerical integration
- Linear quadratic regulator (LQR) can easily be computed for linearized systems

References

- K.E. Brenan, S.L. Campbell, and L.R. Petzold: The Numerical Solution of Initial Value Problems in Differential-Algebraic Equations, SIAM Classics Series, 1996.
- U.M. Ascher and L.R. Petzold: Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations. SIAM, 1998.

Exercise on Linear Quadratic Regulator (LQR)

Tasks:

- Learn how to use integrators and get derivatives from them
- integrate and linearize ODE of test problem (inverted pendulum) to get linear system $x_{k+1} = Ax_k + Bu_k$
- Get LQR by dlqr command
- Simulate nonlinear **closed-loop** system

$$x_{k+1} = f_{\mathrm{d}}(x_k, \bar{u} - K(x_k - \bar{x}))$$

- Outlook to rest of the course:
  1001 sophisticated ways to replace LQR feedback
  $\bar{u} - K(x_k - \bar{x})$ by **embedded optimization**