Modelling and System Identification – Microexam 2

Prof. Dr. Moritz Diehl, IMTEK, Universität Freiburg, and ESAT-STADIUS, KU Leuven

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Nachname: Vorname: Matrikelnummer:

Fach: Studiengang: Bachelor ☐ Master ☐ Lehramt ☐ Sonstiges ☐

Please fill in your name above and tick exactly one box for the right answer of each question below.

1. What is the covariance matrix of $Z = 3X + Y$ if the random variables $Y, X \in \mathbb{R}^n$ are independent and have covariance matrices Σ_x, Σ_y?

(a) ☐ $9\Sigma_x^{-1} + \Sigma_y^{-1}$	(b) ☒ $9\Sigma_x + \Sigma_y$	(c) ☐ $(3\Sigma_x^{-1} + \Sigma_y^{-1})^{-1}$	(d) ☐ $3\Sigma_x + \Sigma_y$

2. What is the covariance matrix of the random variable Y if $Y = BX$ with $B \in \mathbb{R}^{n\times m}$ fixed and Σ_x the covariance matrix of $X \in \mathbb{R}^m$?

(a) ☐ $B^\top \Sigma_x B$	(b) ☐ $B\Sigma_x^{-1}B^\top$	(c) ☒ $B\Sigma_x B^\top$	(d) ☐ $(B\Sigma_x^{-1}B^\top)^{-1}$

3. Which of the following dynamic models with inputs $u(t)$ and outputs $y(t)$ is **not** time varying ?

(a) ☐ $\dot{y}(t) = u(t) + \cos(t)$	(b) ☒ $\ddot{y}(t) = u(t)^3$	(c) ☐ $t^3\ddot{y}(t) = u(t)$	(d) ☐ $\dot{y}(t)^3 = t^2 u(t)$

4. Which of the following dynamic models with inputs $u(t)$ and outputs $y(t)$ is **neither** linear **nor** affine.

(a) ☐ $t^3\ddot{y}(t) = u(t)$	(b) ☐ $\ddot{y}(t) = t^3 u(t)$	(c) ☐ $\dot{y}(t) = u(t) + \cos(t)$	(d) ☒ $\dot{y}(t)^3 = u(t)$

5. Which of the following dynamic models with inputs $u(t)$ and outputs $y(t)$ is a linear time invariant (LTI) system ?

(a) ☐ $\ddot{y}(t) = t \cdot u(t)$	(b) ☐ $\dot{y}(t) = u(t) + \sin(t)$	(c) ☒ $\ddot{y}(t) = \frac{1}{3}u(t)$	(d) ☐ $\dot{y}(t)^3 = u(t)$

6. Which of the following models with input $u(k)$ and output $y(k)$ is **not** linear-in-the-parameters w.r.t. $\theta \in \mathbb{R}^2$?

(a) ☐ $y(k) = \theta_1 u(k)^2 + \theta_2 \sin(u(k))$	(b) ☐ $y(k) = \theta_1 y(k-1) + \theta_2 u(k)$
(c) ☒ $y(k) = \theta_1 u(k) + \sin(\theta_2 u(k))$	(d) ☐ $y(k) = \sin(y(k-1)) \cdot (\theta_1 + \theta_2 u(k))$

7. Which transfer function $G(s)$ describes the system $\dot{x}(t) = x(t) + u(t),\ y(t) = x(t) + u(t)$?

(a) ☐ $\frac{s}{s+1}$	(b) ☒ $\frac{s}{s-1}$	(c) ☐ $\frac{1}{s+1}$	(d) ☐ $\frac{s+1}{s+1}$

8. Which system is described by the transfer function $G(s) = \frac{3}{s^2-1}$?

(a) ☐ $\ddot{y} - 3\dot{y} = u$	(b) ☒ $\ddot{y} - y = 3u$	(c) ☐ $\ddot{y} + 3\dot{y} = u$	(d) ☐ $\ddot{y} - \dot{y} = 3u$

9. What solution $y(t)$ has the system $T\dot{y}(t) + y(t) = u(t)$ with initial value $y(0) = -1$ for constant input $u(t) = 0$?

(a) ☒ $y(t) = -e^{-t/T}$	(b) ☐ $y(t) = e^{-t/T}$	(c) ☐ $y(t) = -e^{-tT}$	(d) ☐ $y(t) = e^{-tT}$

10. What is the discrete time equivalent for the system $\dot{y}(t) = u(t)$ with sampling time $\Delta T = 2$ (time is unitless for simplicity) under the assumption of zero-order hold for the inputs?

(a) ☐ $y(k+1) = \frac{1}{2}y(k) + u(k)$	(b) ☐ $y(k+1) = y(k) + u(k)$
(c) ☒ $y(k+1) = y(k) + 2u(k)$	(d) ☐ $y(k+1) = 2y(k) + 2u(k)$

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11. Maximum Likelihood Estimator (MLE): Assume a nominal model $h_i(\theta)$ and given measurements y_i, $i = 1, \dots, N$. The measurement noises are i.i.d. and Gaussian. What function of θ does the MLE minimize in this case?

(a) ☐ $\sum_{i=1}^N \lvert y_i - h_i(\theta)\rvert$	(b) ☐ $\lvert \sum_{i=1}^N y_i - \sum_{i=1}^N h_i(\theta)\rvert$
(c) ☒ $\sum_{i=1}^N (y_i - h_i(\theta))^2$	(d) ☐ $\sum_{i=1}^N \frac{1}{\sigma_i}\lvert y_i - h_i(\theta)\rvert$

12. The PDF of a random variable Y is given by $p(y) = \frac{1}{2}\exp(-|y-\theta|)$, with unknown $\theta \in \mathbb{R}$. We obtained three measurements, $y(1) = 1$, $y(2) = 2$, and $y(3) = 27$. What is the minimizer θ^* of the negative log-likelihood function ?

(a) ☐ 1	(b) ☒ 2	(c) ☐ 10	(d) ☐ 27

13. Bayesian estimation: we want to estimate the resistivity ρ of a new material and found in the only existing previous article that an estimate of ρ is given by 5Ωm with standard deviation 2Ωm. Our own measurement apparatus has Gaussian errors with standard deviation 4Ωm, and we obtained N measurements, $y(1), \dots, y(N)$ of ρ. What function is minimized by the Bayesian Maximum-A-Posteriori (MAP) estimator in this context?

(a) ☐ $\frac{(\rho-5\Omega\mathrm{m})^2}{2\Omega\mathrm{m}} + \sum_{i=1}^N \frac{(y(i)-\rho)^2}{4\Omega\mathrm{m}}$	(b) ☒ $\frac{(\rho-5\Omega\mathrm{m})^2}{(2\Omega\mathrm{m})^2} + \sum_{i=1}^N \frac{(y(i)-\rho)^2}{(4\Omega\mathrm{m})^2}$
(c) ☐ $-\frac{(\rho-5\Omega\mathrm{m})^2}{2\Omega\mathrm{m}} + \sum_{i=1}^N \frac{(y(i)-\rho)^2}{4\Omega\mathrm{m}}$	(d) ☐ $\frac{(\rho-5\Omega\mathrm{m})}{2\Omega\mathrm{m}} + \sum_{i=1}^N \frac{(y(i)-\rho)^2}{4\Omega\mathrm{m}}$

14. Modelling a hot iron (Bügeleisen): regard an electrically heated iron with heat capacity C and temperature $T(t)$. Heat losses to the outside result in an energy flow $\lambda(T(t) - T_0)$ (where λ is a constant and T_0 the outside temperature). The electrical coil provides a heating power $Q(t)$. Regard $Q(t)$ as input and $T(t)$ as output. Which differential equation models this system?

(a) ☐ $\dot{T} = -\frac{C}{\lambda}(T - T_0) + \frac{Q}{\lambda}$	(b) ☐ $\dot{T} = -\frac{\lambda}{C}(T - T_0) - \frac{Q}{C}$
(c) ☐ $\dot{T} = \lambda(T - T_0) - \frac{Q}{C}$	(d) ☒ $\dot{T} = -\frac{\lambda}{C}(T - T_0) + \frac{Q}{C}$

15. Given a one step ahead prediction model $\boxed{y(k) = \theta_1 y(k-1) + \theta_2 u(k)^2 + \epsilon(k)}$ with unknown parameter vector $\theta = (\theta_1, \theta_2)^T$, and assuming i.i.d. noise $\epsilon(k)$ with zero mean, and given a sequence of N scalar input and output measurements $u(1), \dots, u(N)$ and $y(1), \dots, y(N)$, we want to compute the linear least squares (LLS) estimate $\hat{\theta}$ by minimizing a function $f(\theta) = \|y_N - \Phi_N\theta\|_2^2$. How do we need to choose the matrix Φ_N and vector y_N?

(a) ☐ $\Phi_N = \begin{bmatrix} y(1) & u(1)^2 \\ \vdots & \vdots \\ y(N) & u(N)^2 \end{bmatrix}, \quad y_N = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}$	(b) ☒ $\Phi_N = \begin{bmatrix} y(1) & u(2)^2 \\ \vdots & \vdots \\ y(N-1) & u(N)^2 \end{bmatrix}, \quad y_N = \begin{bmatrix} y(2) \\ \vdots \\ y(N) \end{bmatrix}$
(c) ☐ $\Phi_N = \begin{bmatrix} y(2) & u(1) \\ \vdots & \vdots \\ y(N) & u(N-1) \end{bmatrix}, \quad y_N = \begin{bmatrix} y(1) \\ \vdots \\ y(N-1) \end{bmatrix}$	(d) ☐ $\Phi_N = \begin{bmatrix} y(2) & u(2)^2 \\ \vdots & \vdots \\ y(N) & u(N)^2 \end{bmatrix}, \quad y_N = \begin{bmatrix} y(2) \\ \vdots \\ y(N) \end{bmatrix}$

16. Regard the unweighted linear least squares estimate $\hat{\theta}$ minimizing $f(\theta) = \|y_N - \Phi_N\theta\|_2^2$, where the measurements are generated by $y_N = \Phi_N\theta_0 + \epsilon_N$ with θ_0 the unknown true value, and $\epsilon_N = (\epsilon(1), \dots, \epsilon(N))^\top$ the measurement errors, which are assumed i.i.d., zero mean, with variance σ^2 (but not necessarily Gaussian). What would be the covariance matrix $\Sigma_{\hat{\theta}}$ of $\hat{\theta}$?

(a) ☐ $(\Phi_N^\top \sigma^2 \Phi_N)^{-1}$	(b) ☐ not computable	(c) ☐ $\sigma(\Phi_N^+)(\Phi_N^+)^\top$	(d) ☒ $\sigma^2(\Phi_N^\top \Phi_N)^{-1}$

17. As above, regard the LLS estimate $\hat{\theta}$ minimizing $f(\theta) = \|y_N - \Phi_N\theta\|_2^2$. But now there might be some correlations between the measurement errors, and we only know that the vector $\epsilon_N = (\epsilon(1), \dots, \epsilon(N))^\top$ has zero mean and the following covariance matrix Σ_{ϵ_N} (not necessarily diagonal). What would be the covariance matrix $\Sigma_{\hat{\theta}}$ of the unweighted LLS estimate $\hat{\theta}$?

(a) ☐ not computable	(b) ☐ $(\Phi_N^\top \Sigma_{\epsilon_N}^{-1} \Phi_N)^{-1}$	(c) ☒ $(\Phi_N^+)\Sigma_{\epsilon_N}(\Phi_N^+)^\top$	(d) ☐ $\Sigma_{\epsilon_N}(\Phi_N^\top \Phi_N)^{-1}$

18. Maximum Likelihood Estimator (MLE) for a coin-toss: we regard a coin thrown into the air that shows either "heads" or "tails" after landing. We know it is a fraudulent coin, and the unknown probability to get "heads" is θ. In an experiment, we have thrown the coin 100 times, and obtained 40 times "heads". What is the negative log likelihood function $f(\theta)$ that we need to minimize in order to obtain the MLE estimate of θ ?

(a) ☐ $40\log\theta + 60\log(1-\theta)$	(b) ☐ $40\theta + 60(1-\theta)$
(c) ☒ $-40\log\theta - 60\log(1-\theta)$	(d) ☐ $-40\theta - 60(1-\theta)$

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