

# Constrained Optimization

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*(some slide material was provided by W. Bangerth, K. Mombaur)*

# Nonlinear Programming (Problem Class 3)

- General problem formulation:

$$\begin{array}{ll} \min f(x) & f: D \subset \mathbb{R}^n \rightarrow \mathbb{R} \\ \text{s.t. } g(x) = 0 & g: D \subset \mathbb{R}^n \rightarrow \mathbb{R}^l \\ h(x) \geq 0 & h: D \subset \mathbb{R}^n \rightarrow \mathbb{R}^k \end{array}$$

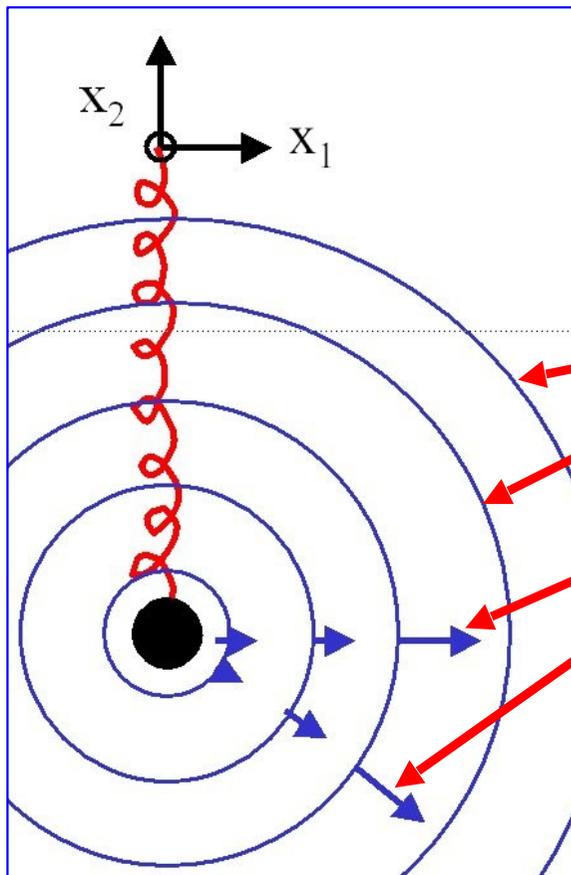
$f$  objective function / cost function

$g$  equality constraints

$h$  inequality constraints

$f, g, h$  shall be smooth (twice differentiable) functions

# Recall: ball on a spring without constraints



$$\min_{x \in \mathbb{R}^2} x_1^2 + x_2^2 + mx_2$$

contour lines of  $f(x)$

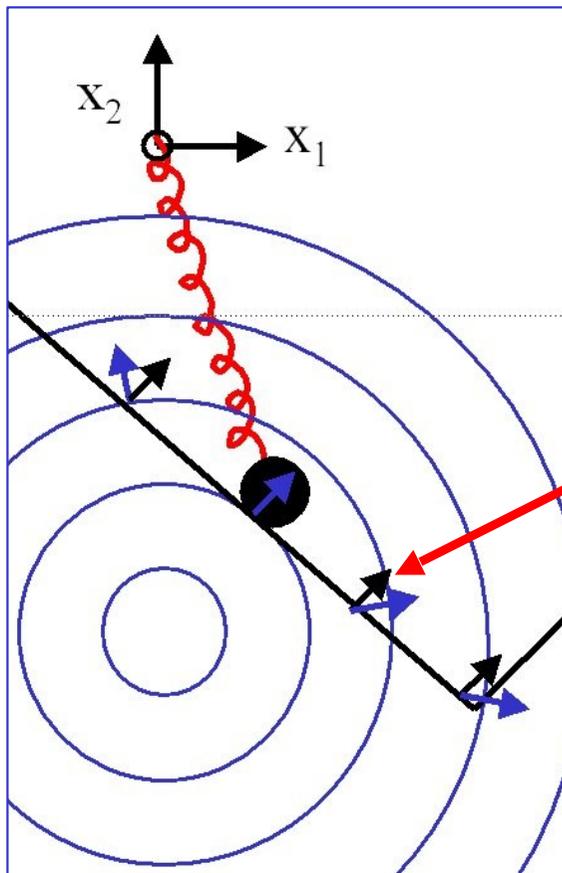
gradient vector

$$\nabla f(x) = (2x_1, 2x_2 + m)$$

unconstrained minimum:

$$0 = \nabla f(x^*) \Leftrightarrow (x_1^*, x_2^*) = \left(0, -\frac{m}{2}\right)$$

# Now: ball on a spring with constraints



gradient  $\nabla h_1$  of active constraint

inactive constraint  $h_2$

$$\min f(x)$$

$$h_1(x) := 1 + x_1 + x_2 \geq 0$$

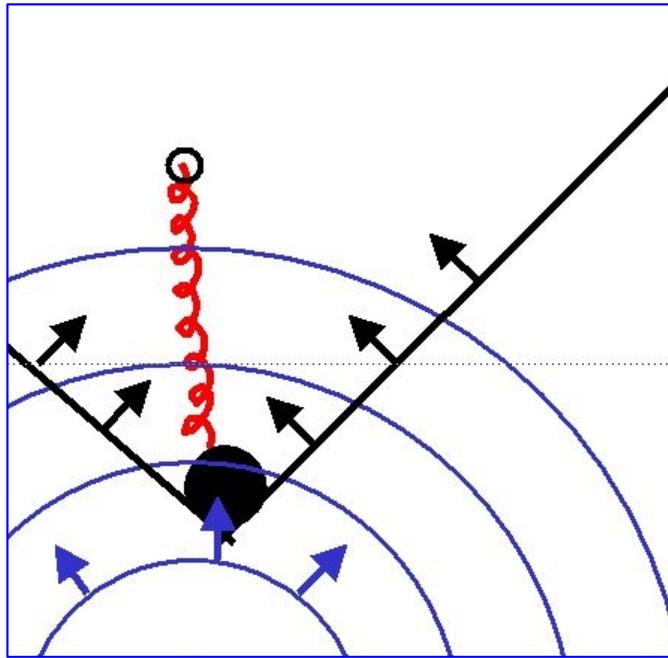
$$h_2(x) := 3 - x_1 + x_2 \geq 0$$

constrained minimum:

$$\nabla f(x^*) = \mu_1 \nabla h_1(x^*)$$

Lagrange multiplier

# Ball on a spring with two active constraints



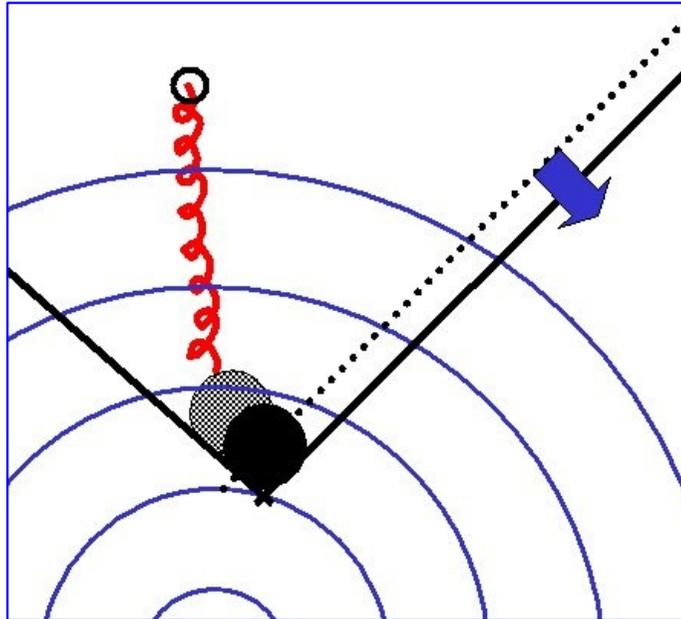
$$\begin{aligned} \min f(x) \\ h_1(x) := 1 + x_1 + x_2 &\geq 0 \\ h_2(x) := 3 - x_1 + x_2 &\geq 0 \end{aligned}$$

„equilibrium of forces“

$$\nabla f(x^*) = \mu_1 \nabla h_1(x^*) + \mu_2 \nabla h_2(x^*) \quad \mu_1, \mu_2 \geq 0$$

„constraint forces“

# Multipliers as „shadow prices“



old constraint:  $h(x) \geq 0$

new constraint:  $h(x) + \varepsilon \geq 0$

What happens if we relax a constraint?  
Feasible set becomes bigger,  
so new minimum  $f(x_\varepsilon^*)$  becomes smaller.  
How much would we gain?

$$f(x_\varepsilon^*) \approx f(x^*) - \mu\varepsilon$$

*Multipliers show the hidden cost of constraints.*

# The Lagrangian Function

For constrained problems, introduce modification of objective function:

$$L(x, \lambda, \mu) := f(x^*) - \sum \lambda_i g_i(x) - \sum \mu_i h_i(x)$$

- equality multipliers  $\lambda_i$  may have both signs in a solution
- inequality multipliers  $\mu_i$  cannot be negative (cf. shadow prices)
- for inactive constraints, multipliers  $\mu_i$  are zero

# Optimality conditions (constrained)

*Karush-Kuhn-Tucker necessary conditions (KKT-conditions):*

- $x^*$  feasible
- there exist  $\lambda^*$ ,  $\mu^*$  such that

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0$$

$$(\Leftrightarrow \text{"Equilibrium"} \nabla f = \sum \lambda_i \nabla g_i + \sum \mu_i \nabla h_i )$$

- $\mu^* \geq 0$  holds
- and it holds the complementarity condition

$$\mu^{*T} h(x^*) = 0$$

i.e.  $\mu_j^* = 0$  or  $h_j(x^*) = 0$  for each  $i$

# Sequential Quadratic Programming (SQP)

Constrained problem:

$$\begin{aligned} \min f(x) \\ g(x) &= 0 \\ h(x) &\geq 0 \end{aligned}$$

*SQP Idea:* Consider successively quadratic approximations of the problem:

$$\begin{aligned} \min_{\Delta x} (\nabla f^k)^T \Delta x + \frac{1}{2} \Delta x^T H^k \Delta x \\ g(x^k) + \nabla g(x^k)^T \Delta x = 0 \\ h(x^k) + \nabla h(x^k)^T \Delta x \geq 0 \end{aligned}$$

# SQP method

- if we use the exact hessian of the Lagrangian

$$H = \nabla^2 L(x, \lambda, \mu)$$

this leads to a newton-method for the optimality conditions and feasibility.

- with update-formulas for  $H^k$ , we obtain quasi-Newton SQP-methods.
- if we use appropriate update-formulas, we can have superlinear convergence.
- global convergence can be achieved by using a stepsize strategy.

# SQP algorithm

0. Start with  $k=0$ , start value  $x^0$  and  $H^0=I$
1. Compute  $f(x^k)$ ,  $g(x^k)$ ,  $h(x^k)$ ,  $\nabla f(x^k)$ ,  $\nabla g(x^k)$ ,  $\nabla h(x^k)$
2. If  $x^k$  feasible and

$$\|\nabla L(x, \lambda, \mu)\| < \varepsilon$$

then *stop*  $\rightarrow$  convergence achieved

3. Solve quadratic problem and get  $\Delta x^k$
4. Perform line search and get stepsize  $t^k$
5. Iterate

$$x^{k+1} = x^k + t^k \Delta x^k$$

6. Update hessian
7.  $k=k+1$ , goto step 1

# Summary

- Lagrangian function plays important role in constrained optimization
- Lagrange multipliers of inequalities have positive sign
- KKT conditions are necessary optimality conditions
- Look at SQP again in the following presentation...