Two-Stage Robust Integer Programming

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Two-Stage Robust Integer Programs

Two-Stage Robust Integer Program:

t

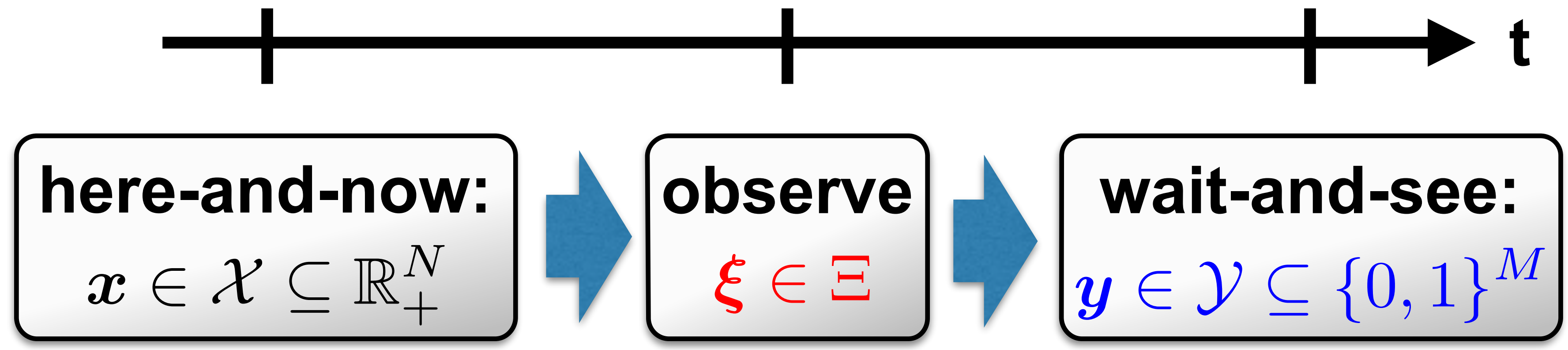
here-and-now: $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}_+^N$

observe $\boldsymbol{\xi} \in \Xi$

wait-and-see: $\boldsymbol{y} \in \mathcal{Y} \subseteq \{0,1\}^M$

Two-Stage Robust Integer Programs

Two-Stage Robust Integer Program:



Mathematical Formulation:

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{\boldsymbol{y} \in \mathcal{Y}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y} \, : \, \boldsymbol{T} \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y} \leq \boldsymbol{H} \boldsymbol{\xi} \right\} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$

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Applications:

operations mgmt.

investment planning

game theory

Literature Review

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{\boldsymbol{\xi}\in\Xi}\left[\boldsymbol{\xi}^\top \boldsymbol{C}\,\boldsymbol{x} + \min_{\boldsymbol{y}\in\mathcal{Y}}\left\{\boldsymbol{\xi}^\top \boldsymbol{Q}\,\boldsymbol{y} \;:\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \le \boldsymbol{H}\boldsymbol{\xi}\right\}\right] \\ \text{subject to} & \boldsymbol{x}\in\mathcal{X} \end{array}$$

Literature Review

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1. **Exact Approaches:** not available

Literature Review

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\text{minimize} & \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{\boldsymbol{y} \in \mathcal{Y}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y} \, : \, \boldsymbol{T} \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y} \leq \boldsymbol{H} \boldsymbol{\xi} \right\} \right] \\
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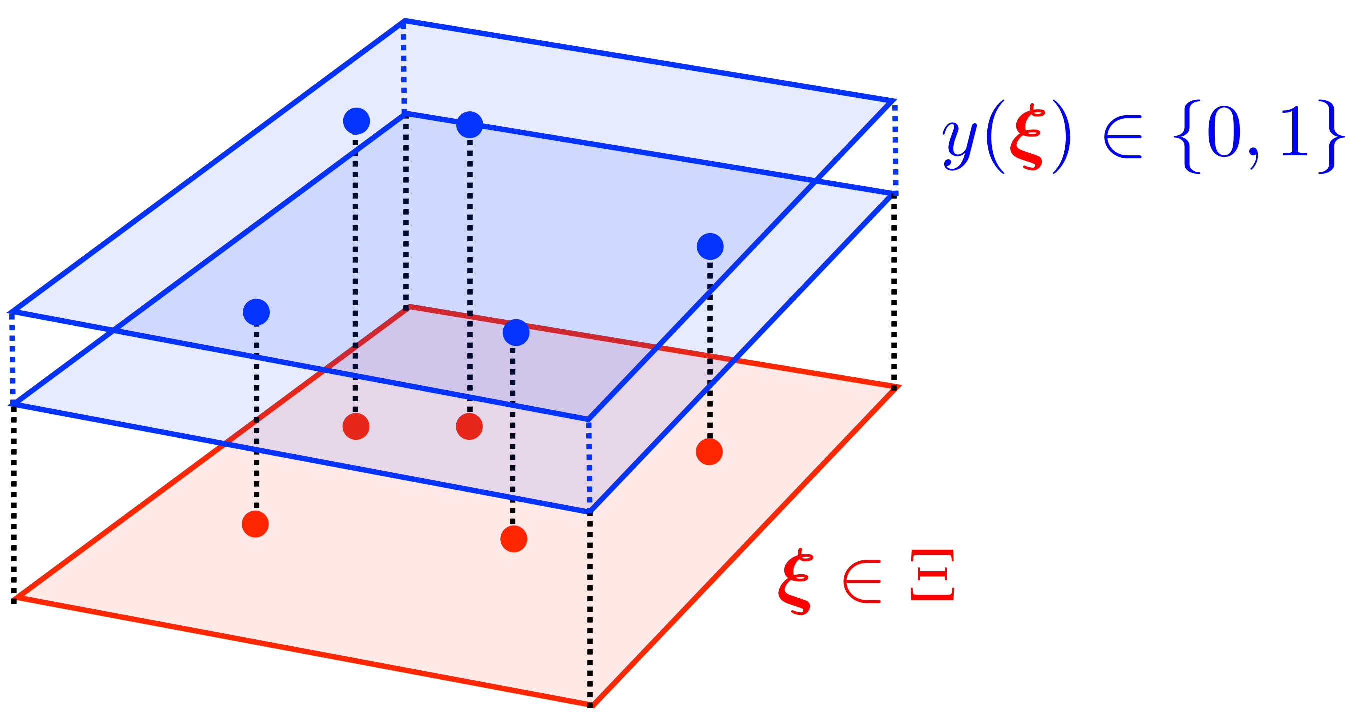
\end{array}
$$

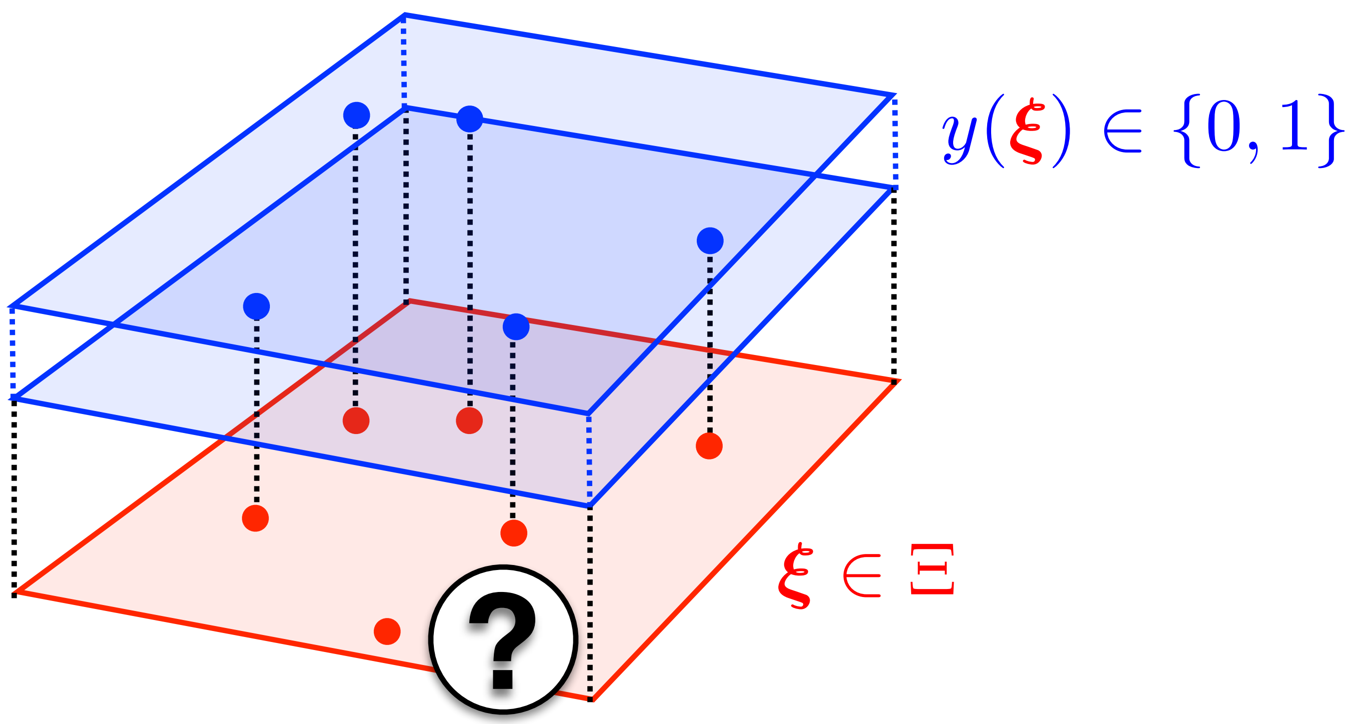
1. **Exact Approaches:** not available
2. **Sampling-Based Approximations:**



Literature Review

$$\begin{array}{ll} \text{minimize} & \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{\boldsymbol{y} \in \mathcal{Y}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y} \,:\, \boldsymbol{T} \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y} \leq \boldsymbol{H} \boldsymbol{\xi} \right\} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$

1. **Exact Approaches:** not available
2. **Sampling-Based Approximations:** progressive approximation



Literature Review

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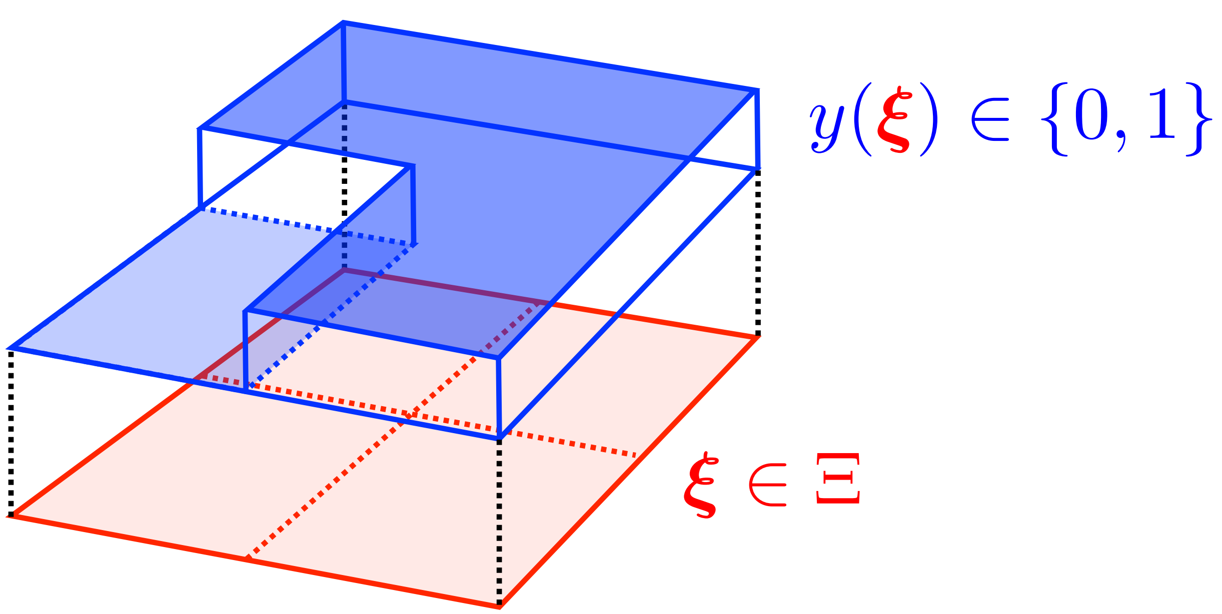
1. **Exact Approaches:** not available
2. **Sampling-Based Approximations:** progressive approximation
3. **Space-Partitioning Approximations:**



Literature Review

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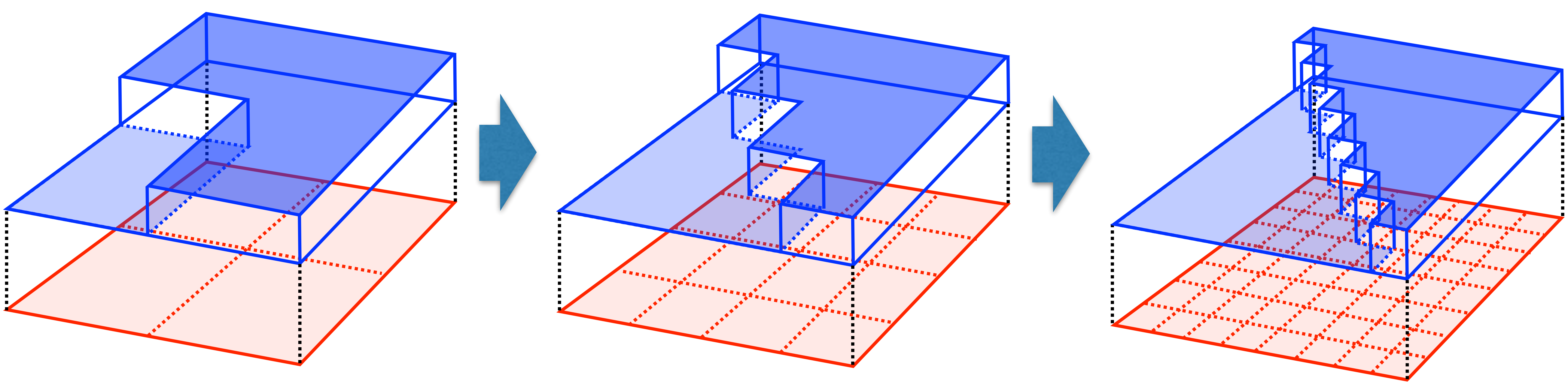
\text{minimize} & \displaystyle\max_{\boldsymbol{\xi}\in\Xi}\left[\boldsymbol{\xi}^\top \boldsymbol{C}\,\boldsymbol{x} + \min_{\boldsymbol{y}\in\mathcal{Y}}\left\{\boldsymbol{\xi}^\top \boldsymbol{Q}\,\boldsymbol{y} \;:\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \leq \boldsymbol{H}\boldsymbol{\xi}\right\}\right] \\
\text{subject to} & \boldsymbol{x} \in \mathcal{X}
\end{array}
$$

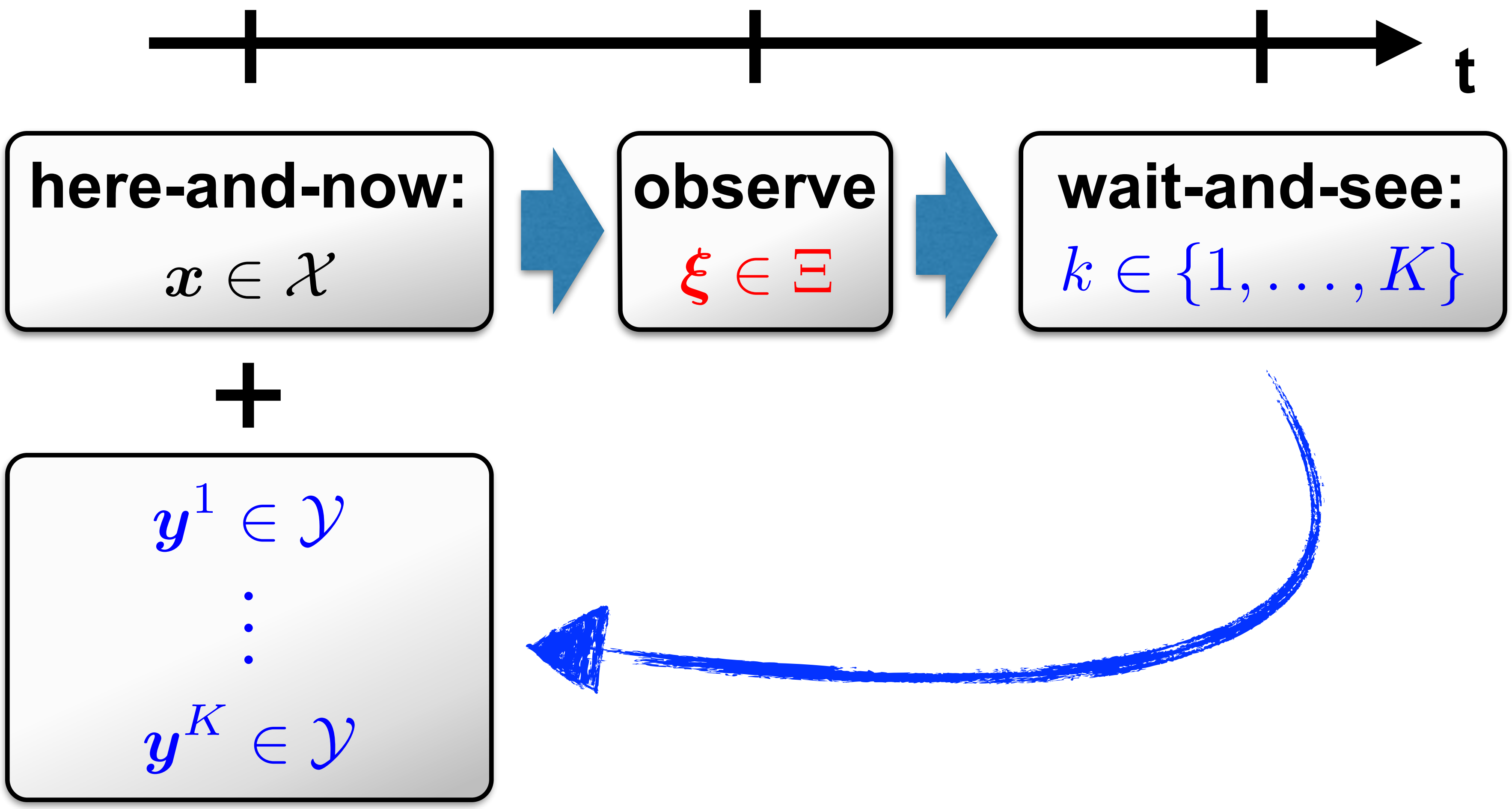
1. **Exact Approaches:** not available
2. **Sampling-Based Approximations:** progressive approximation
3. **Space-Partitioning Approximations:** exponential growth

$$
\max_{\boldsymbol{\xi}\in[0,1]^2}\;\min_{y\in\{0,1\}}\;\left\{y - \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \;:\; y \geq \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 - 1\right\}
$$



The *K*-Adaptability Problem

***K*-Adaptability Problem:** Bertsimas and Caramanis (2010)



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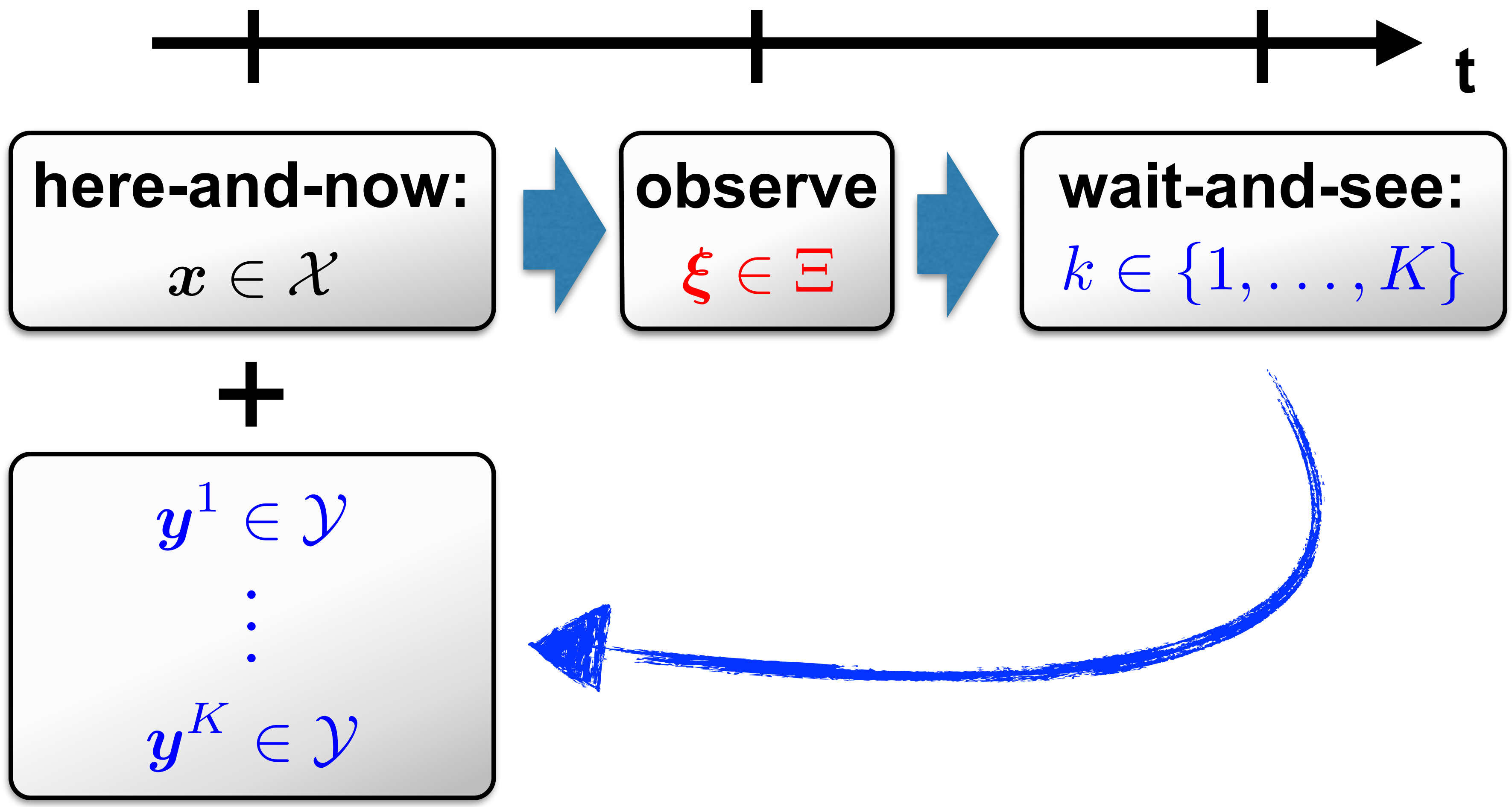

$$\begin{array}{ll} \text{minimize} & \displaystyle \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C}\, \boldsymbol{x} + \min_{k \in \mathcal{K}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q}\, \boldsymbol{y}^k \;:\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y},\, k \in \mathcal{K} \end{array}$$

The *K*-Adaptability Problem

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t

here-and-now:
$\boldsymbol{x} \in \mathcal{X}$

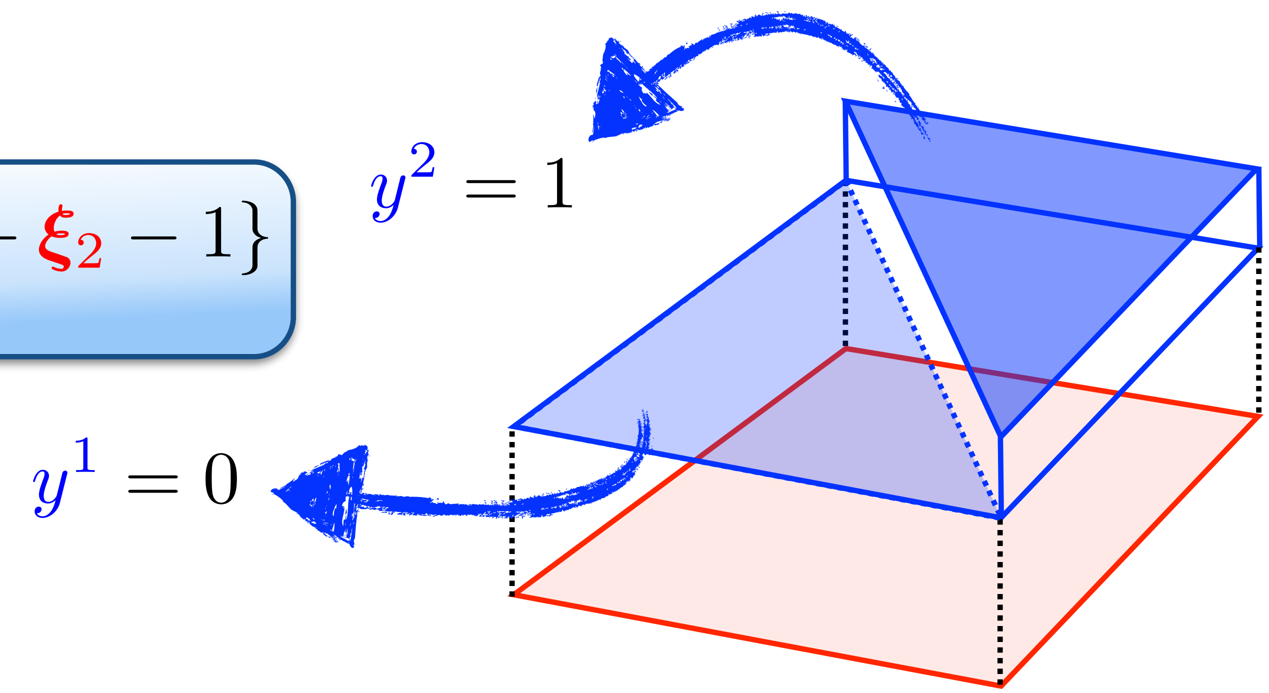
observe
$\boldsymbol{\xi} \in \Xi$

wait-and-see:
$k \in \{1, \ldots, K\}$

+

$\boldsymbol{y}^1 \in \mathcal{Y}$
$\vdots$
$\boldsymbol{y}^K \in \mathcal{Y}$

$$\max_{\boldsymbol{\xi} \in [0,1]^2} \min_{y \in \{0,1\}} \{y - \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \, : \, y \geq \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 - 1\}$$



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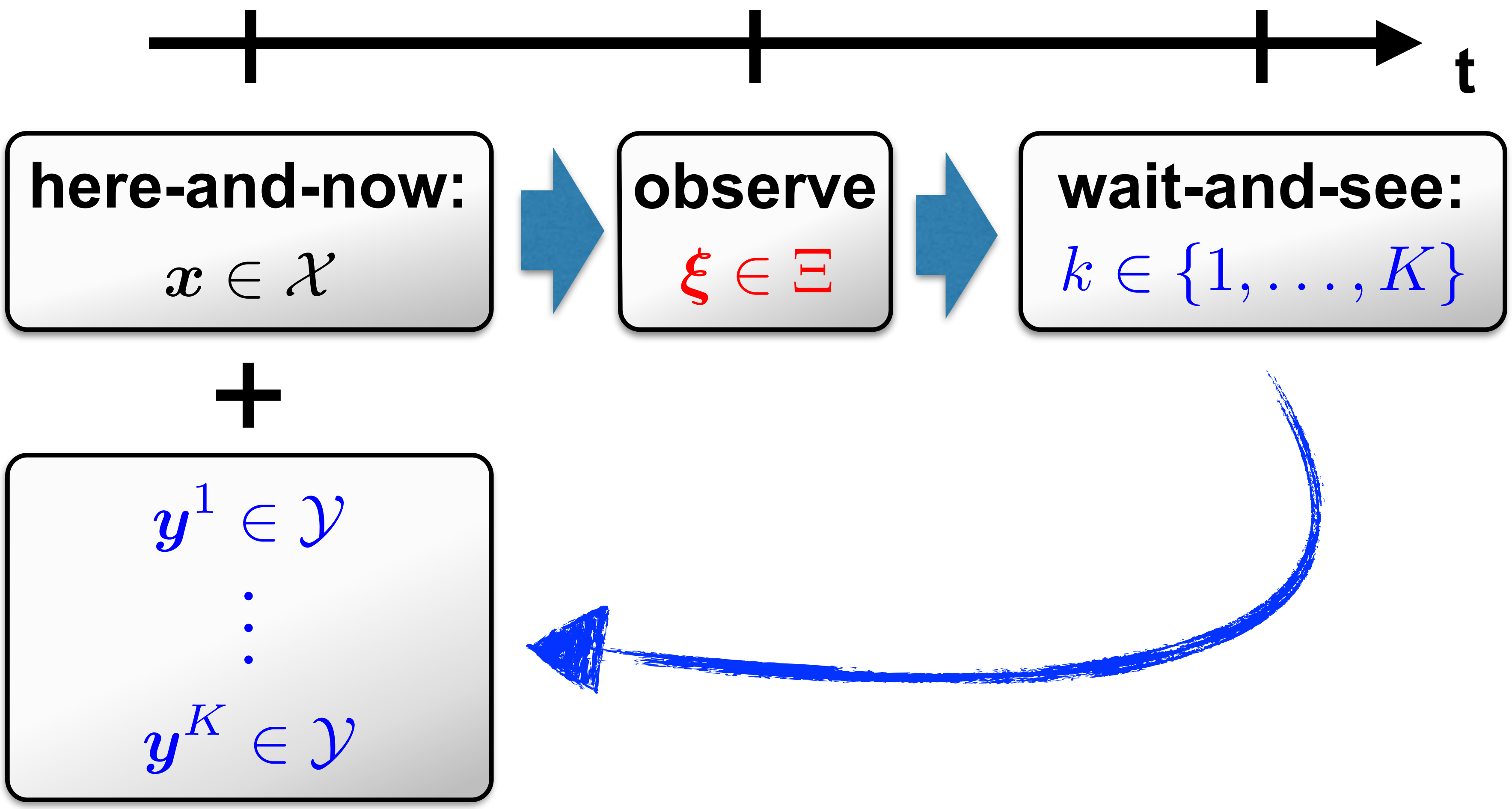

$$
\begin{array}{ll}
\text{minimize} & \displaystyle \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C}\, \boldsymbol{x} + \min_{k \in \mathcal{K}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q}\, \boldsymbol{y}^k \ : \ \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right] \\
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$$

How good is the approximation?

The *K*-Adaptability Problem

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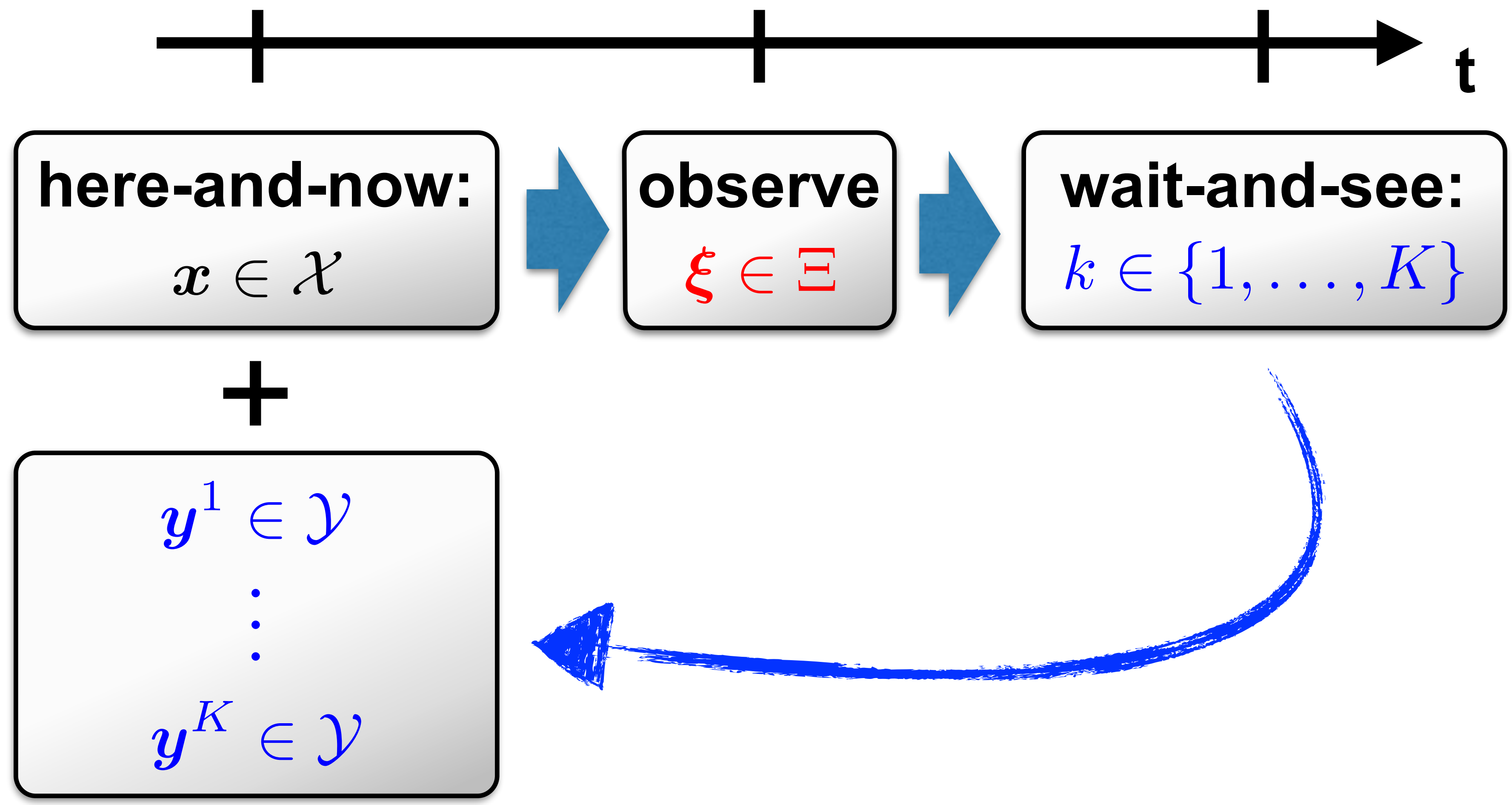

$$
\begin{array}{ll}
\text{minimize} & \max_{\xi \in \Xi} \left[\xi^\top C x + \min_{k \in \mathcal{K}} \left\{ \xi^\top Q y^k \; : \; T x + W y^k \leq H \xi \right\} \right] \\
\text{subject to} & x \in \mathcal{X}, \;\; y^k \in \mathcal{Y}, \; k \in \mathcal{K}
\end{array}
$$

How good is the approximation, and can we solve it?

The K-Adaptability Problem:

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{k \in \mathcal{K}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \, : \, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right] \\
\text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y},\ k \in \mathcal{K}
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Objective Uncertainty

The *K*-Adaptability Problem with Objective Uncertainty:

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\max_{\boldsymbol{\xi}\in\Xi}\left[\boldsymbol{\xi}^\top \boldsymbol{C}\,\boldsymbol{x} + \min_{k\in\mathcal{K}}\left\{\boldsymbol{\xi}^\top \boldsymbol{Q}\,\boldsymbol{y}^k \,:\, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}\right\}\right] \\
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\end{array}
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The *K*-Adaptability Problem with Objective Uncertainty:

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Objective Uncertainty: Approximation Quality

The *K*-Adaptability Problem with Objective Uncertainty:

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How good is the approximation?

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How good is the approximation?

Theorem (Objective Uncertainty): The *K*-Adaptability Problem attains the same objective value as the Two-Stage Robust Integer Program whenever $K \geq \min\{\dim \mathcal{Y}, \dim \Xi\} + 1$.

Objective Uncertainty: Approximation Quality

Theorem (Objective Uncertainty): The K-Adaptability Problem attains the same objective value as the Two-Stage Robust Integer Program whenever $K \geq \min\{\dim \mathcal{Y}, \dim \Xi\} + 1$.

Proof Outline: for $K \geq \dim \mathcal{Y} + 1$

Consider the $\overline{K}$-Adaptability Problem with $\overline{\mathcal{K}} = \{1, \ldots, \overline{K} = |\mathcal{Y}|\}$:

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C}\, \boldsymbol{x} + \min_{k \in \overline{\mathcal{K}}} \boldsymbol{\xi}^\top \boldsymbol{Q}\, \boldsymbol{y}^k \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \; k \in \overline{\mathcal{K}} \end{array}$$

$\overline{K}$-Adaptability Problem

Objective Uncertainty: Approximation Quality

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$$\min_{k \in \overline{\mathcal{K}}} \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \;=\; \min_{\boldsymbol{\lambda} \in \Delta_{\overline{K}}} \sum_{k \in \overline{\mathcal{K}}} \boldsymbol{\lambda}_k \cdot \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k$$

where $\Delta_{\overline{K}}$ denotes the unit simplex in $\mathbb{R}^{\overline{K}}$.

$\overline{K}$-Adaptability Problem

Objective Uncertainty: Approximation Quality

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$\overline{K}$-Adaptability Problem $\xrightarrow{1}$ $\Delta_{\overline{K}}$-Reformulation

Objective Uncertainty: Approximation Quality

Theorem (Objective Uncertainty): The K-Adaptability Problem attains the same objective value as the Two-Stage Robust Integer Program whenever $K \geq \min\{\dim \mathcal{Y}, \dim \Xi\} + 1$.

Proof Outline: for $K \geq \dim \mathcal{Y} + 1$

Use Minimax Theorem to exchange order of max and min:

$$\begin{array}{ll} \text{minimize} & \displaystyle \min_{\boldsymbol{\lambda} \in \Delta_{\overline{K}}} \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C}\, \boldsymbol{x} + \sum_{k \in \overline{\mathcal{K}}} \boldsymbol{\lambda}_k \cdot \boldsymbol{\xi}^\top \boldsymbol{Q}\, \boldsymbol{y}^k \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \; k \in \overline{\mathcal{K}} \end{array}$$

$\overline{K}$-Adaptability Problem $\xrightarrow{1}$ $\Delta_{\overline{K}}$-Reformulation $\xrightarrow{2}$ Minimax Theorem

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Proof Outline: for $K \geq \dim \mathcal{Y} + 1$

Combine minimization problems:

$$
\begin{array}{ll}
\text{minimize} & \displaystyle \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C}\, \boldsymbol{x} + \sum_{k \in \overline{\mathcal{K}}} \boldsymbol{\lambda}_k \cdot \boldsymbol{\xi}^\top \boldsymbol{Q}\, \boldsymbol{y}^k \right] \\
\text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \; k \in \overline{\mathcal{K}}, \\
& \boldsymbol{\lambda} \in \Delta_{\overline{K}}
\end{array}
$$

$\overline{K}$-Adaptability Problem $\xrightarrow{1}$ $\Delta_{\overline{K}}$-Reformulation $\xrightarrow{2}$ Minimax Theorem

Objective Uncertainty: Approximation Quality

Theorem (Objective Uncertainty): The *K*-Adaptability Problem attains the same objective value as the Two-Stage Robust Integer Program whenever $K \geq \min\{\dim \mathcal{Y}, \dim \Xi\} + 1$.

Proof Outline: for $K \geq \dim \mathcal{Y} + 1$

Reformulate objective function:

$$\begin{array}{ll} \text{minimize} & \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \boldsymbol{\xi}^\top \boldsymbol{Q} \cdot \sum_{k \in \overline{\mathcal{K}}} \boldsymbol{\lambda}_k \boldsymbol{y}^k \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \ \ \boldsymbol{y}^k \in \mathcal{Y}, \ \ \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \ k \in \overline{\mathcal{K}}, \\ & \boldsymbol{\lambda} \in \Delta_{\overline{K}} \end{array}$$

$\overline{K}$-Adaptability Problem $\xrightarrow{1}$ $\Delta_{\overline{K}}$-Reformulation $\xrightarrow{2}$ Minimax Theorem

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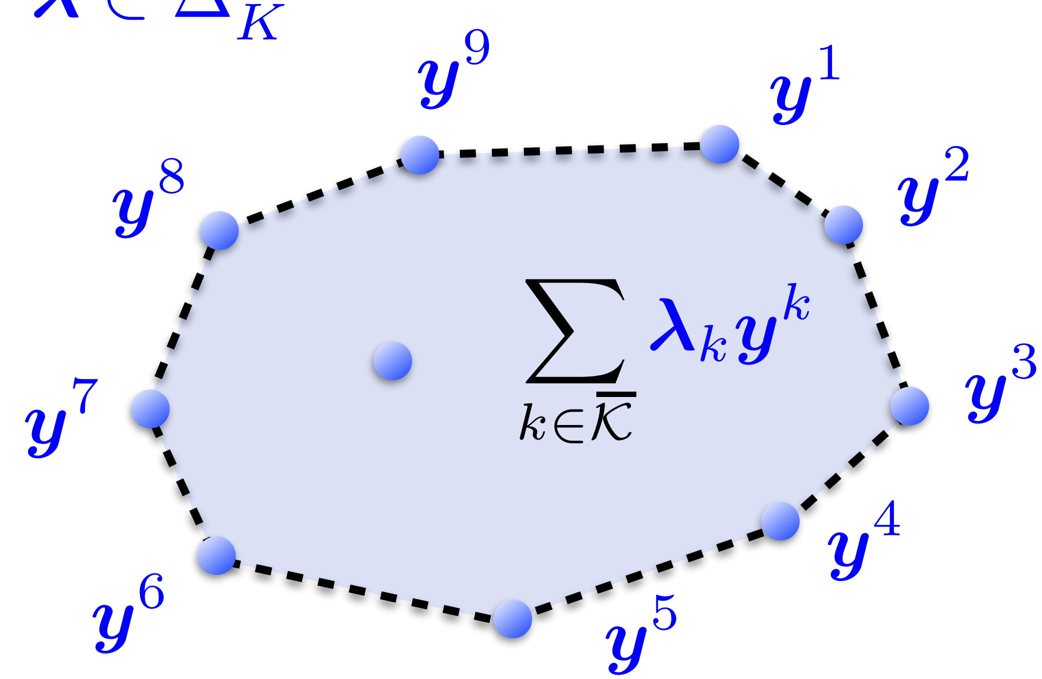
Proof Outline: for $K \geq \dim \mathcal{Y} + 1$

Apply Carathéodory's Theorem:

$$\begin{array}{ll} \text{minimize} & \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \boldsymbol{\xi}^\top \boldsymbol{Q} \cdot \boxed{\sum_{k \in \overline{\mathcal{K}}} \boldsymbol{\lambda}_k \boldsymbol{y}^k} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y}, \quad \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \ k \in \overline{\mathcal{K}}, \\ & \boldsymbol{\lambda} \in \Delta_{\overline{K}} \end{array}$$



Objective Uncertainty: Approximation Quality

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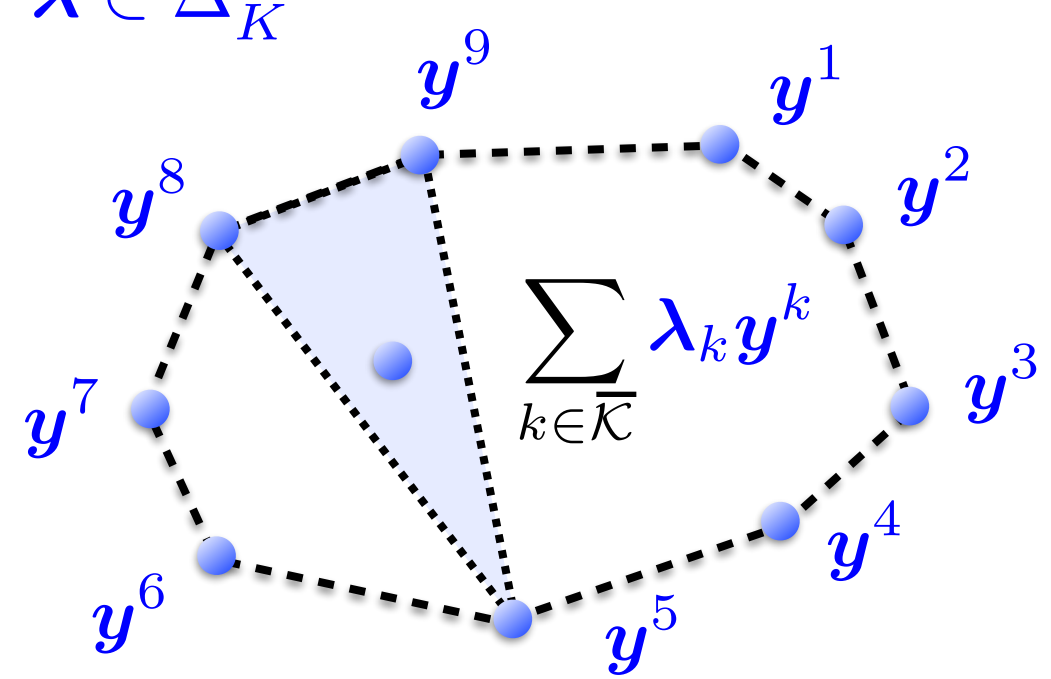

Objective Uncertainty: Approximation Quality

Theorem (Objective Uncertainty): The K-Adaptability Problem attains the same objective value as the Two-Stage Robust Integer Program whenever $K \geq \min\{\dim \mathcal{Y}, \dim \Xi\} + 1$.

Proof Outline: for $K \geq \dim \mathcal{Y} + 1$

Apply Carathéodory's Theorem:

$$\begin{array}{ll} \text{minimize} & \max\limits_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \boldsymbol{\xi}^\top \boldsymbol{Q} \cdot \sum\limits_{\boxed{k \in \mathcal{K}}} \boldsymbol{\lambda}_k \boldsymbol{y}^k \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \boxed{k \in \mathcal{K},} \\ & \boldsymbol{\lambda} \in \Delta_{\boxed{K}} \end{array}$$

where $\mathcal{K} = \{1, \ldots, K = \dim \mathcal{Y} + 1\}$.

$\overline{K}$-Adaptability Problem $\xrightarrow{1}$ $\Delta_{\overline{K}}$-Reformulation $\xrightarrow{2}$ Minimax Theorem $\xrightarrow{3}$ Carathéodory's Theorem

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Theorem (Objective Uncertainty): The K-Adaptability Problem attains the same objective value as the Two-Stage Robust Integer Program whenever $K \geq \min\{\dim \mathcal{Y}, \dim \Xi\} + 1$.

Proof Outline: for $K \geq \dim \mathcal{Y} + 1$

Separate the two minimizations:

$$\begin{array}{ll} \text{minimize} & \displaystyle\min_{\boldsymbol{\lambda} \in \Delta_K} \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \sum_{k \in \mathcal{K}} \boldsymbol{\lambda}_k \cdot \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \; k \in \mathcal{K} \end{array}$$

$\overline{K}$-Adaptability Problem $\xrightarrow{1}$ $\Delta_{\overline{K}}$-Reformulation $\xrightarrow{2}$ Minimax Theorem $\xrightarrow{3}$ Carathéodory's Theorem

Objective Uncertainty: Approximation Quality

Theorem (Objective Uncertainty): The K-Adaptability Problem attains the same objective value as the Two-Stage Robust Integer Program whenever $K \geq \min\{\dim \mathcal{Y}, \dim \Xi\} + 1$.

Proof Outline: for $K \geq \dim \mathcal{Y} + 1$

Use Minimax Theorem to exchange order of min and max:

$$\begin{array}{ll} \text{minimize} & \max\limits_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min\limits_{\boldsymbol{\lambda} \in \Delta_K} \sum\limits_{k \in \mathcal{K}} \boldsymbol{\lambda}_k \cdot \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \; k \in \mathcal{K} \end{array}$$

$$\boxed{\overline{K}\text{-Adaptability Problem}} \xrightarrow{1} \boxed{\Delta_{\overline{K}}\text{-Reformulation}} \xrightarrow{2} \boxed{\text{Minimax Theorem}} \underset{4}{\overset{3}{\rightleftarrows}} \boxed{\text{Carathéodory's Theorem}}$$

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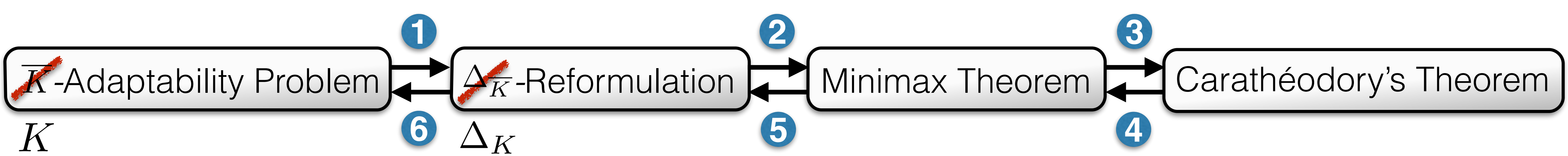
Theorem (Objective Uncertainty): The *K*-Adaptability Problem attains the same objective value as the Two-Stage Robust Integer Program whenever $K \geq \min\{\dim \mathcal{Y}, \dim \Xi\} + 1$.

Proof Outline: for $K \geq \dim \mathcal{Y} + 1$

We recover the *K*-Adaptability Problem:

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{k \in \mathcal{K}} \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \; k \in \mathcal{K} \end{array}$$



Objective Uncertainty: Tractability

The *K*-Adaptability Problem with Objective Uncertainty:

$$\begin{array}{ll} \text{minimize} & \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{k \in \mathcal{K}} \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \; k \in \mathcal{K} \end{array}$$

Can we solve the approximation?

Objective Uncertainty: Tractability

The *K*-Adaptability Problem with Objective Uncertainty:

$$\begin{array}{ll} \text{minimize} & \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \, \boldsymbol{x} + \min_{k \in \mathcal{K}} \boldsymbol{\xi}^\top \boldsymbol{Q} \, \boldsymbol{y}^k \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y}, \quad \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \; k \in \mathcal{K} \end{array}$$

Can we solve the approximation?

Theorem (Objective Uncertainty): The *K*-Adaptability Problem has equivalent MILP reformulation whose size scales *polynomially* in the problem data.

Objective Uncertainty: Tractability

Theorem (Objective Uncertainty): The *K*-Adaptability Problem has equivalent MILP reformulation whose size scales *polynomially* in the problem data.

Proof Outline:

Consider the *K*-Adaptability Problem:

$$\begin{array}{ll} \text{minimize} & \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \, \boldsymbol{x} + \min_{k \in \mathcal{K}} \boldsymbol{\xi}^\top \boldsymbol{Q} \, \boldsymbol{y}^k \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \; k \in \mathcal{K} \end{array}$$

Objective Uncertainty: Tractability

Theorem (Objective Uncertainty): The *K*-Adaptability Problem has equivalent MILP reformulation whose size scales *polynomially* in the problem data.

Proof Outline:

Epigraph reformulation of inner min:

$$\begin{array}{ll} \text{minimize} & \max\limits_{\substack{\boldsymbol{\xi}\in\Xi,\\ \tau\in\mathbb{R}}} \left[\boldsymbol{\xi}^\top \boldsymbol{C}\,\boldsymbol{x} + \tau \;:\; \tau \leq \boldsymbol{\xi}^\top \boldsymbol{Q}\,\boldsymbol{y}^k \;\; \forall k \in \mathcal{K}\right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X},\;\; \boldsymbol{y}^k \in \mathcal{Y},\;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h},\; k \in \mathcal{K} \end{array}$$

1 Epigraph Reformulation

Objective Uncertainty: Tractability

Theorem (Objective Uncertainty): The *K*-Adaptability Problem has equivalent MILP reformulation whose size scales *polynomially* in the problem data.

Proof Outline:

Strong LP duality:

$$\begin{array}{ll} \text{minimize} & \displaystyle\min_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^R_+,\\ \boldsymbol{\beta}\in\mathbb{R}^K_+}} \left[\boldsymbol{b}^\top \boldsymbol{\alpha} \;:\; \boldsymbol{A}^\top \boldsymbol{\alpha} = \boldsymbol{C}\boldsymbol{x} + \sum_{k\in\mathcal{K}} \boldsymbol{\beta}_k \boldsymbol{Q}\boldsymbol{y}^k, \;\; \mathbf{e}^\top \boldsymbol{\beta} = 1 \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \le \boldsymbol{h}, \; k \in \mathcal{K} \end{array}$$

1 Epigraph Reformulation → 2 Dualize Inner Problem

Objective Uncertainty: Tractability

Theorem (Objective Uncertainty): The K-Adaptability Problem has equivalent MILP reformulation whose size scales *polynomially* in the problem data.

Proof Outline:

Strong LP duality:

$$
\begin{aligned}
&\text{minimize} && \boldsymbol{b}^\top \boldsymbol{\alpha} \\
&\text{subject to} && \boldsymbol{x} \in \mathcal{X},\;\; \boldsymbol{y}^k \in \mathcal{Y},\;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h},\; k \in \mathcal{K} \\
& && \boldsymbol{\alpha} \in \mathbb{R}^R_+,\;\; \boldsymbol{\beta} \in \mathbb{R}^K_+,\;\; \boldsymbol{A}^\top \boldsymbol{\alpha} = \boldsymbol{C}\boldsymbol{x} + \sum_{k \in \mathcal{K}} \boldsymbol{\beta}_k \boldsymbol{Q}\boldsymbol{y}^k,\;\; \mathbf{e}^\top \boldsymbol{\beta} = 1
\end{aligned}
$$

1 Epigraph Reformulation → 2 Dualize Inner Problem

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Proof Outline:

Strong LP duality:

$$\begin{array}{ll} \text{minimize} & \boldsymbol{b}^\top \boldsymbol{\alpha} \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \;\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{h}, \; k \in \mathcal{K} \\ & \boldsymbol{\alpha} \in \mathbb{R}^R_+, \;\; \boldsymbol{\beta} \in \mathbb{R}^K_+, \;\; \boldsymbol{A}^\top \boldsymbol{\alpha} = \boldsymbol{C}\boldsymbol{x} + \sum_{k \in \mathcal{K}} \boldsymbol{\beta}_k \boldsymbol{Q}\boldsymbol{y}^k, \;\; \mathbf{e}^\top \boldsymbol{\beta} = 1 \end{array}$$

Linearize bilinear terms: via auxiliary variables $\boldsymbol{z}^k \in \mathbb{R}^M_+$, $k \in \mathcal{K}$

$$\boldsymbol{z}^k = \boldsymbol{\beta}_k \boldsymbol{y}^k \quad \Longleftrightarrow \quad \boldsymbol{z}^k \leq \boldsymbol{y}^k, \;\; \boldsymbol{z}^k \leq \boldsymbol{\beta}_k \mathbf{e}, \;\; \boldsymbol{z}^k \geq (\boldsymbol{\beta}_k - 1)\mathbf{e} + \boldsymbol{y}^k$$

1 Epigraph Reformulation → 2 Dualize Inner Problem → 3 Exact Linearization

Summary

The *K*-Adaptability Problem:

$$\begin{array}{ll} \text{minimize} & \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{k \in \mathcal{K}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \, : \, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \ \ \boldsymbol{y}^k \in \mathcal{Y}, \, k \in \mathcal{K} \end{array}$$

Objective Uncertainty:

- ⋆ *strong* approximation guarantees
- ⋆ MILP reformulation that scales *polynomially*

Constraint Uncertainty

The *K*-Adaptability Problem with Constraint Uncertainty:

$$\begin{array}{ll} \text{minimize} & \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{k \in \mathcal{K}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \, : \, \boldsymbol{T} \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}^k \leq \boldsymbol{H} \boldsymbol{\xi} \right\} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y}, \, k \in \mathcal{K} \end{array}$$

Constraint Uncertainty: Approximation Quality

The *K*-Adaptability Problem with Constraint Uncertainty:

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{\boldsymbol{\xi}\in\Xi}\left[\boldsymbol{\xi}^\top \boldsymbol{C}\,\boldsymbol{x} + \min_{k\in\mathcal{K}}\left\{\boldsymbol{\xi}^\top \boldsymbol{Q}\,\boldsymbol{y}^k \;:\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi}\right\}\right] \\ \text{subject to} & \boldsymbol{x}\in\mathcal{X},\;\; \boldsymbol{y}^k\in\mathcal{Y},\, k\in\mathcal{K} \end{array}$$

How good is the approximation?

Constraint Uncertainty: Approximation Quality

The *K*-Adaptability Problem with Constraint Uncertainty:

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{\boldsymbol{\xi}\in\Xi}\left[\boldsymbol{\xi}^\top \boldsymbol{C}\,\boldsymbol{x} + \min_{k\in\mathcal{K}}\left\{\boldsymbol{\xi}^\top \boldsymbol{Q}\,\boldsymbol{y}^k \;:\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi}\right\}\right] \\ \text{subject to} & \boldsymbol{x}\in\mathcal{X},\quad \boldsymbol{y}^k\in\mathcal{Y},\ k\in\mathcal{K} \end{array}$$

How good is the approximation?

Theorem (Constraint Uncertainty): The *K*-Adaptability Problem can attain a *strictly larger objective value* than the Two-Stage Robust Integer Program whenever $K < |\mathcal{Y}|$.

Constraint Uncertainty: Tractability

The *K*-Adaptability Problem with Constraint Uncertainty:

$$\begin{array}{ll} \text{minimize} & \max\limits_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C}\, \boldsymbol{x} + \min\limits_{k \in \mathcal{K}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q}\, \boldsymbol{y}^k \, : \, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y},\, k \in \mathcal{K} \end{array}$$

Can we solve the approximation?

Constraint Uncertainty: Tractability

The *K*-Adaptability Problem with Constraint Uncertainty:

$$\begin{array}{ll} \text{minimize} & \max\limits_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min\limits_{k \in \mathcal{K}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \; : \; \boldsymbol{T} \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}^k \leq \boldsymbol{H} \boldsymbol{\xi} \right\} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y}, \; k \in \mathcal{K} \end{array}$$

Can we solve the approximation?

Theorem (Constraint Uncertainty): The *K*-Adaptability Problem has an equivalent MILP reformulation that scales *exponentially* in the number of policies *K* (but *polynomially* in other problem data).

Constraint Uncertainty: Tractability

Theorem (Constraint Uncertainty): The *K*-Adaptability Problem has an equivalent MILP reformulation that scales *exponentially* in the number of policies *K* (but *polynomially* in other problem data).

Proof Outline: Consider the *2*-Adaptability Problem:

$$\begin{array}{ll} \text{minimize} & \max\limits_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min\limits_{k \in \{1,2\}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \, : \, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \, k \in \{1, 2\} \end{array}$$

Constraint Uncertainty: Tractability

Theorem (Constraint Uncertainty): The *K*-Adaptability Problem has an equivalent MILP reformulation that scales *exponentially* in the number of policies *K* (but *polynomially* in other problem data).

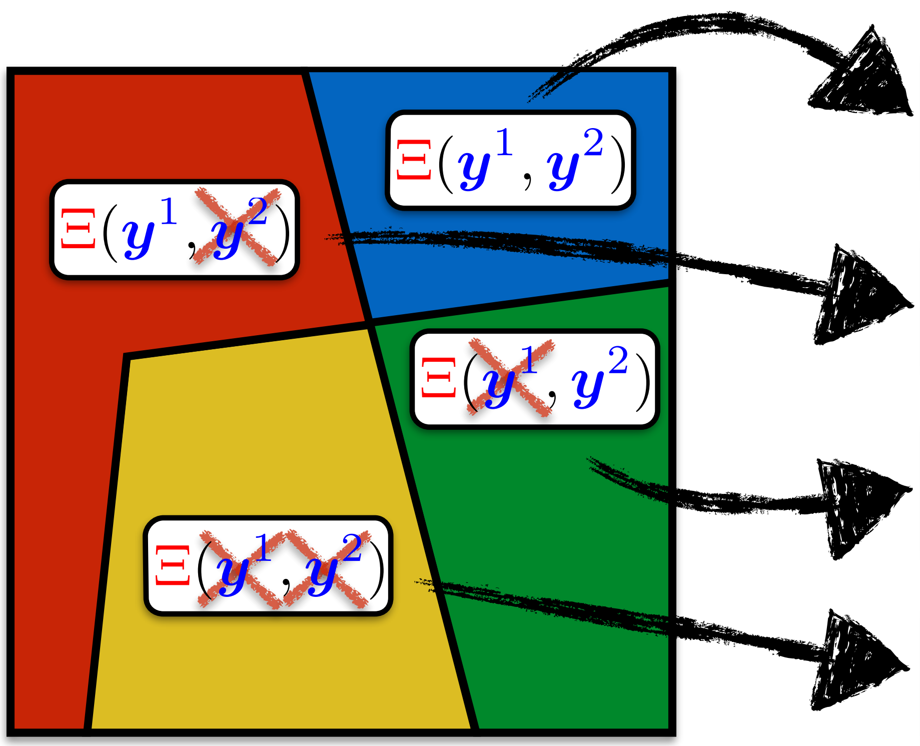
Proof Outline: Move second-stage constraints to uncertainty set:

$$\begin{array}{ll} \text{minimize} & \max\limits_{\boxed{\boldsymbol{\xi} \in \Xi}} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min\limits_{k \in \{1,2\}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \; : \boxed{\boldsymbol{T}\boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}^k \leq \boldsymbol{H} \boldsymbol{\xi} \}} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y}, \; k \in \{1,2\} \end{array}$$

Constraint Uncertainty: Tractability

Theorem (Constraint Uncertainty): The *K*-Adaptability Problem has an equivalent MILP reformulation that scales *exponentially* in the number of policies *K* (but *polynomially* in other problem data).

Proof Outline: Move second-stage constraints to uncertainty set:



$$\Xi(\boldsymbol{y}^1, \boldsymbol{y}^2) = \left\{ \boldsymbol{\xi} \in \Xi : \begin{array}{l} \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^1 \leq \boldsymbol{H}\boldsymbol{\xi} \\ \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^2 \leq \boldsymbol{H}\boldsymbol{\xi} \end{array} \right\}$$

$$\Xi(\boldsymbol{y}^1, \cancel{\boldsymbol{y}^2}) = \left\{ \boldsymbol{\xi} \in \Xi : \begin{array}{l} \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^1 \leq \boldsymbol{H}\boldsymbol{\xi} \\ \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^2 \not\leq \boldsymbol{H}\boldsymbol{\xi} \end{array} \right\}$$

$$\Xi(\cancel{\boldsymbol{y}^1}, \boldsymbol{y}^2) = \left\{ \boldsymbol{\xi} \in \Xi : \begin{array}{l} \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^1 \not\leq \boldsymbol{H}\boldsymbol{\xi} \\ \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^2 \leq \boldsymbol{H}\boldsymbol{\xi} \end{array} \right\}$$

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$$\Xi = \Xi(\boldsymbol{y}^1, \boldsymbol{y}^2) \cup \Xi(\cancel{\boldsymbol{y}^1}, \boldsymbol{y}^2) \cup \Xi(\boldsymbol{y}^1, \cancel{\boldsymbol{y}^2}) \cup \Xi(\cancel{\boldsymbol{y}^1}, \cancel{\boldsymbol{y}^2})$$

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$$= \max \left\{ \begin{array}{l} \displaystyle\max_{\boldsymbol{\xi} \in \Xi(\boldsymbol{y}^1, \boldsymbol{y}^2)} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{k \in \{1,2\}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \, : \, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right], \\ \displaystyle\max_{\boldsymbol{\xi} \in \Xi(\boldsymbol{y}^1, \cancel{\boldsymbol{y}^2})} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{k \in \{1,2\}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \, : \, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right], \\ \displaystyle\max_{\boldsymbol{\xi} \in \Xi(\cancel{\boldsymbol{y}^1}, \boldsymbol{y}^2)} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{k \in \{1,2\}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \, : \, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right], \\ \displaystyle\max_{\boldsymbol{\xi} \in \Xi(\cancel{\boldsymbol{y}^1}, \cancel{\boldsymbol{y}^2})} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{k \in \{1,2\}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \, : \, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right], \end{array} \right\}$$

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$$= \max \left\{ \begin{array}{l} \displaystyle\max_{\boldsymbol{\xi}\in\Xi(\boldsymbol{y}^1,\boldsymbol{y}^2)}\left[\boldsymbol{\xi}^\top \boldsymbol{C}\boldsymbol{x} + \min_{k\in\{1,2\}}\left\{\boldsymbol{\xi}^\top \boldsymbol{Q}\boldsymbol{y}^k\right\}\right], \\ \displaystyle\max_{\boldsymbol{\xi}\in\Xi(\boldsymbol{y}^1,\cancel{\boldsymbol{y}^2})}\left[\boldsymbol{\xi}^\top \boldsymbol{C}\boldsymbol{x} + \boldsymbol{\xi}^\top \boldsymbol{Q}\boldsymbol{y}^1\right], \\ \displaystyle\max_{\boldsymbol{\xi}\in\Xi(\cancel{\boldsymbol{y}^1},\boldsymbol{y}^2)}\left[\boldsymbol{\xi}^\top \boldsymbol{C}\boldsymbol{x} + \min_{k\in\{1,2\}}\left\{\boldsymbol{\xi}^\top \boldsymbol{Q}\boldsymbol{y}^k \,:\, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \le \boldsymbol{H}\boldsymbol{\xi}\right\}\right], \\ \displaystyle\max_{\boldsymbol{\xi}\in\Xi(\cancel{\boldsymbol{y}^1},\cancel{\boldsymbol{y}^2})}\left[\boldsymbol{\xi}^\top \boldsymbol{C}\boldsymbol{x} + \min_{k\in\{1,2\}}\left\{\boldsymbol{\xi}^\top \boldsymbol{Q}\boldsymbol{y}^k \,:\, \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \le \boldsymbol{H}\boldsymbol{\xi}\right\}\right], \end{array} \right\}$$

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$$= \max \left\{ \begin{array}{l} \displaystyle \max_{\boldsymbol{\xi} \in \Xi(\boldsymbol{y}^1, \boldsymbol{y}^2)} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \min_{k \in \{1,2\}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^k \right\} \right], \\ \displaystyle \max_{\boldsymbol{\xi} \in \Xi(\boldsymbol{y}^1, \cancel{\boldsymbol{y}^2})} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^1 \right], \\ \displaystyle \max_{\boldsymbol{\xi} \in \Xi(\cancel{\boldsymbol{y}^1}, \boldsymbol{y}^2)} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \boldsymbol{\xi}^\top \boldsymbol{Q} \boldsymbol{y}^2 \right], \\ \displaystyle \max_{\boldsymbol{\xi} \in \Xi(\cancel{\boldsymbol{y}^1}, \cancel{\boldsymbol{y}^2})} \left[\boldsymbol{\xi}^\top \boldsymbol{C} \boldsymbol{x} + \infty \right], \end{array} \right\}$$

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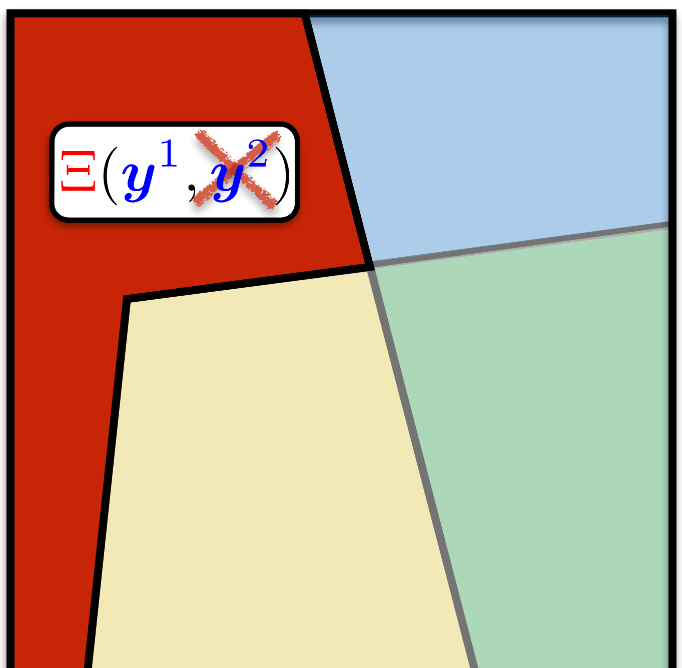
Proof Outline: Uncertainty sets have become non-convex:



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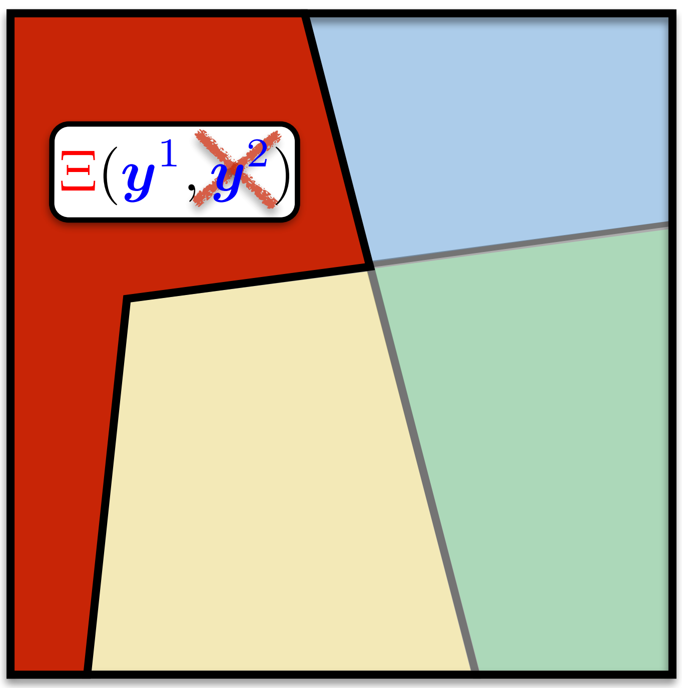

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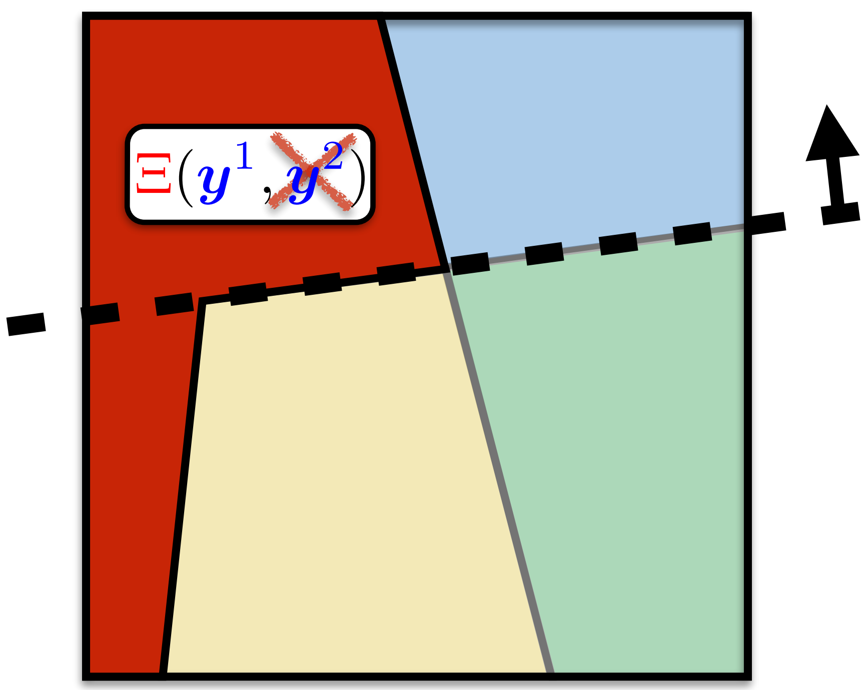

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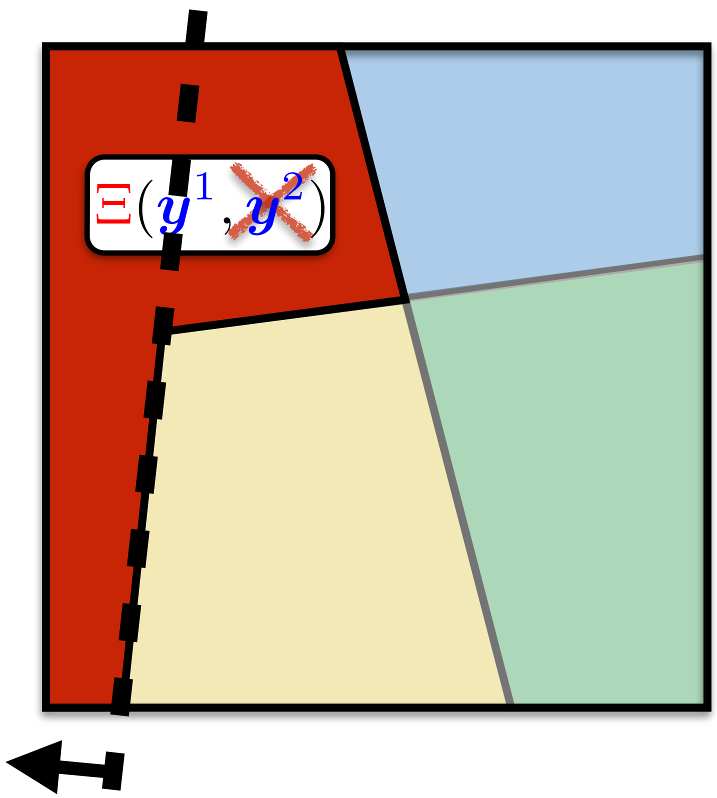

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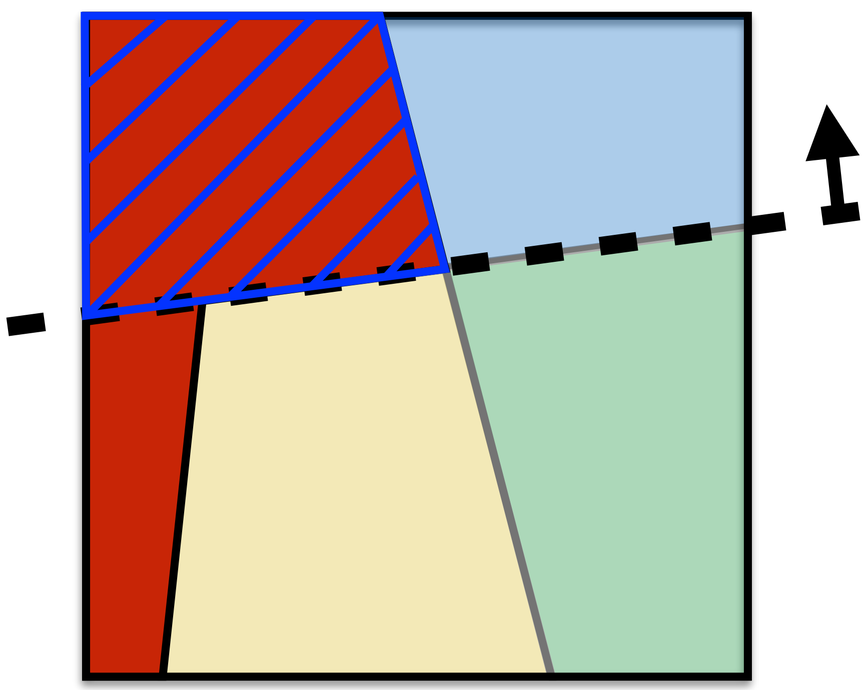

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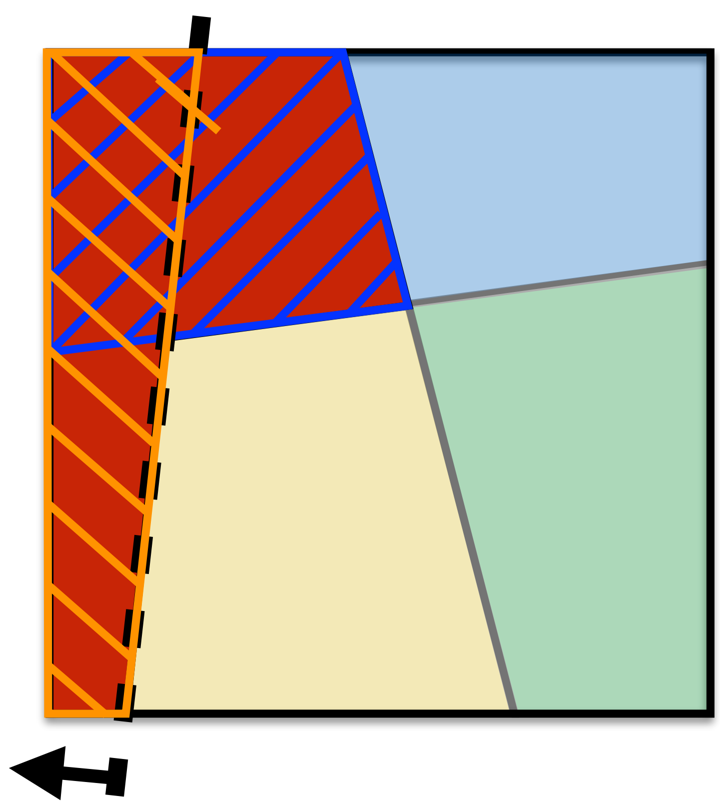

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Summary

The *K*-Adaptability Problem:

$$\begin{array}{ll} \text{minimize} & \max_{\boldsymbol{\xi} \in \Xi} \left[\boldsymbol{\xi}^\top \boldsymbol{C}\,\boldsymbol{x} + \min_{k \in \mathcal{K}} \left\{ \boldsymbol{\xi}^\top \boldsymbol{Q}\,\boldsymbol{y}^k \;:\; \boldsymbol{T}\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \leq \boldsymbol{H}\boldsymbol{\xi} \right\} \right] \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y},\; k \in \mathcal{K} \end{array}$$

Objective Uncertainty:

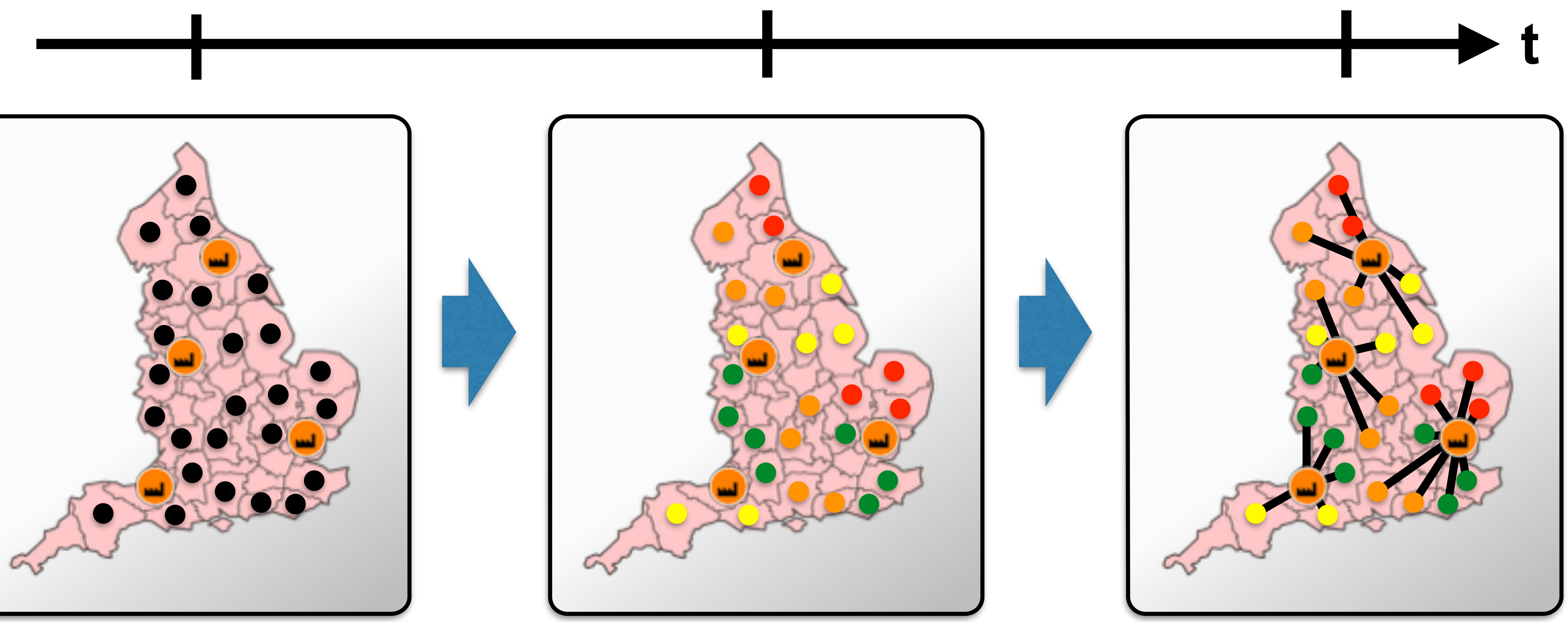
- *strong* approximation guarantees
- MILP reformulation that scales *polynomially*

Constraint Uncertainty:

- *weak* approximation guarantees
- MILP reformulation that scales
  - *exponentially* in *K*
  - *polynomially* in rest

Numerical Experiments

Supply Chain Design:



here-and-now:
build facilities

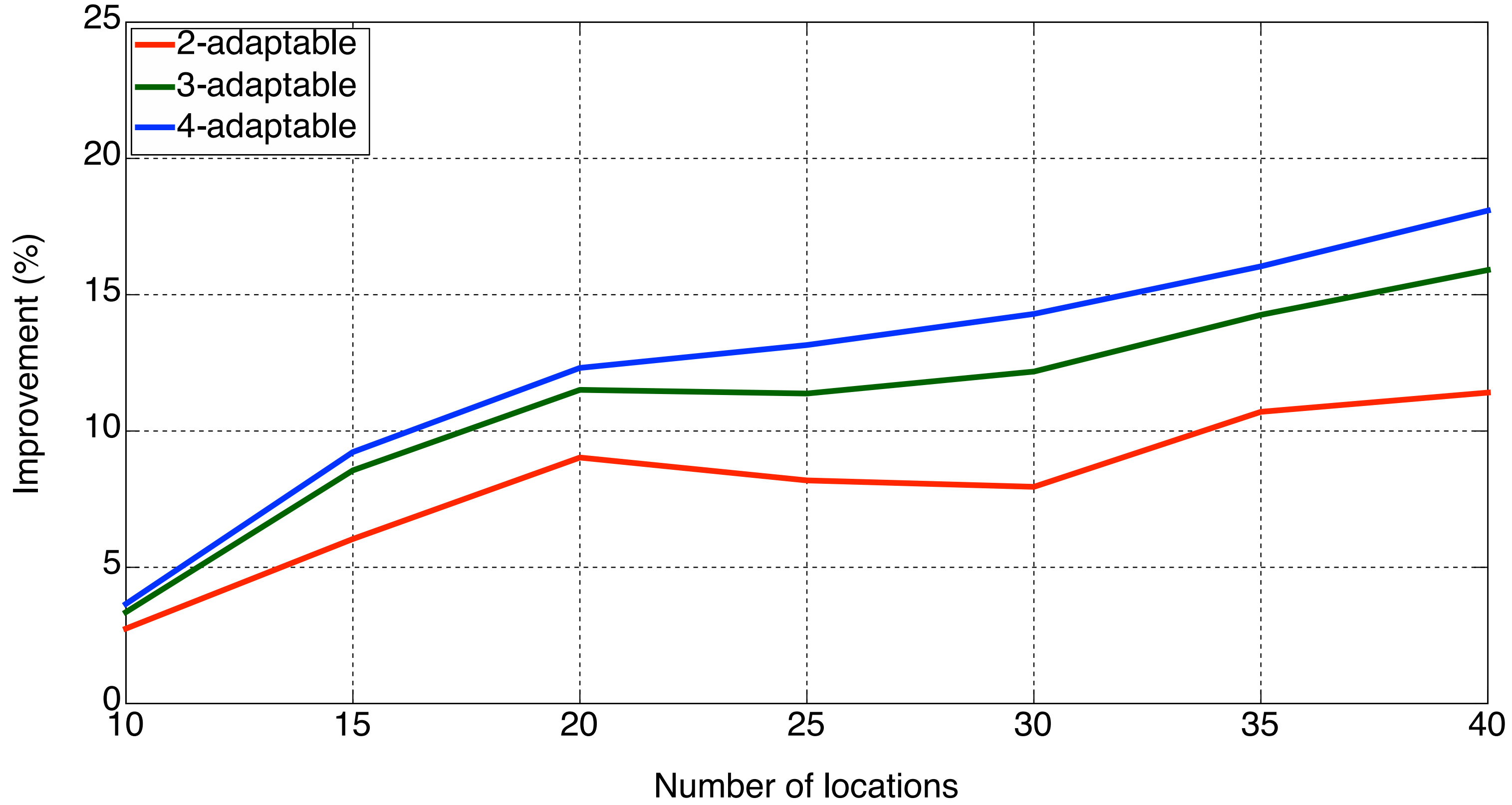
observe:
customer demands

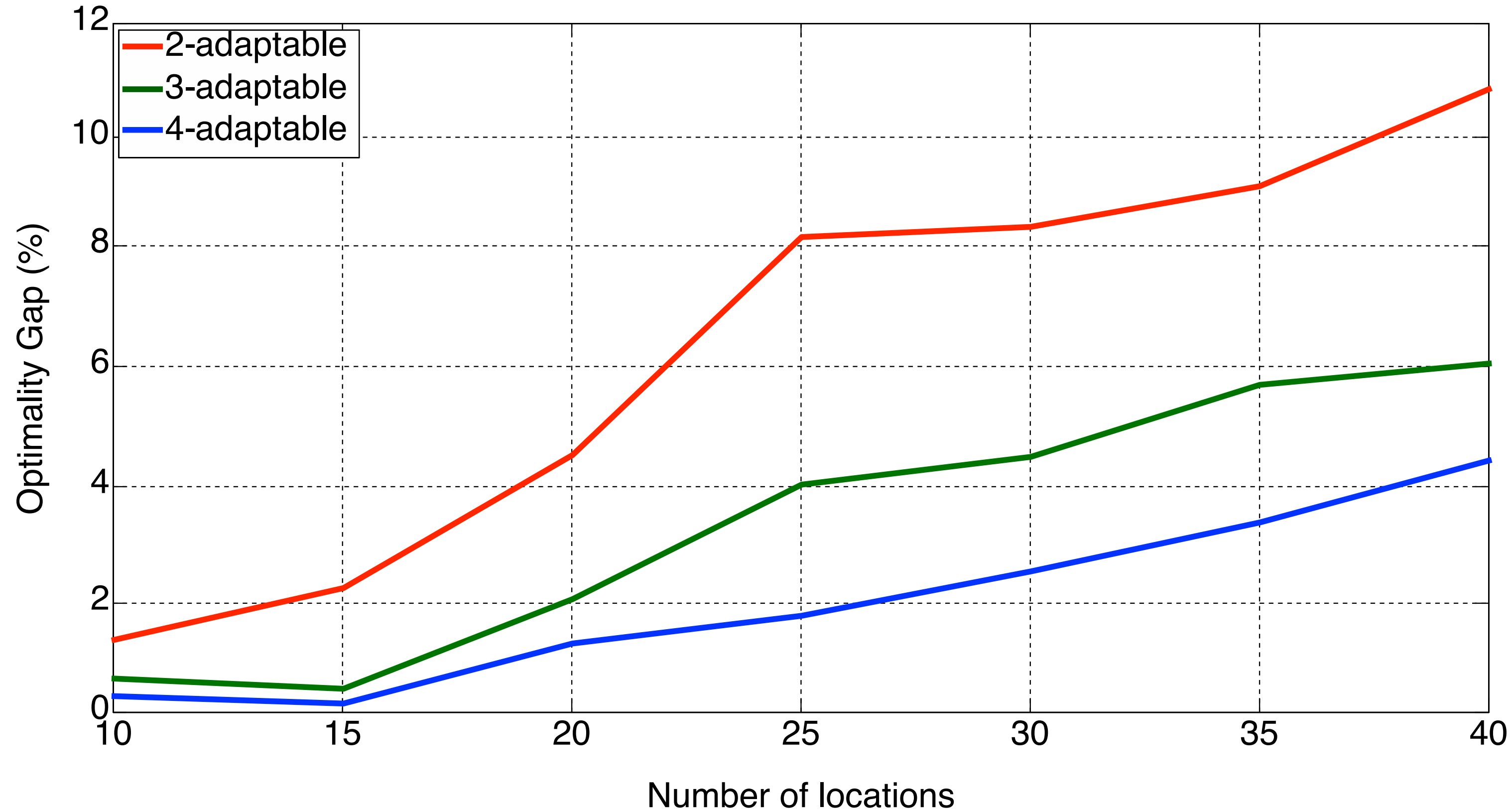
wait-and-see:
product delivery

*Can be modeled as two-stage robust integer program with **objective uncertainty**!*

Numerical Experiments

Supply Chain Design:





Numerical Experiments

Investment Planning:

here-and-now:
early-stage investment
(first mover advantage)

observe:
risk factors

wait-and-see:
late-stage investment
(no advantage)

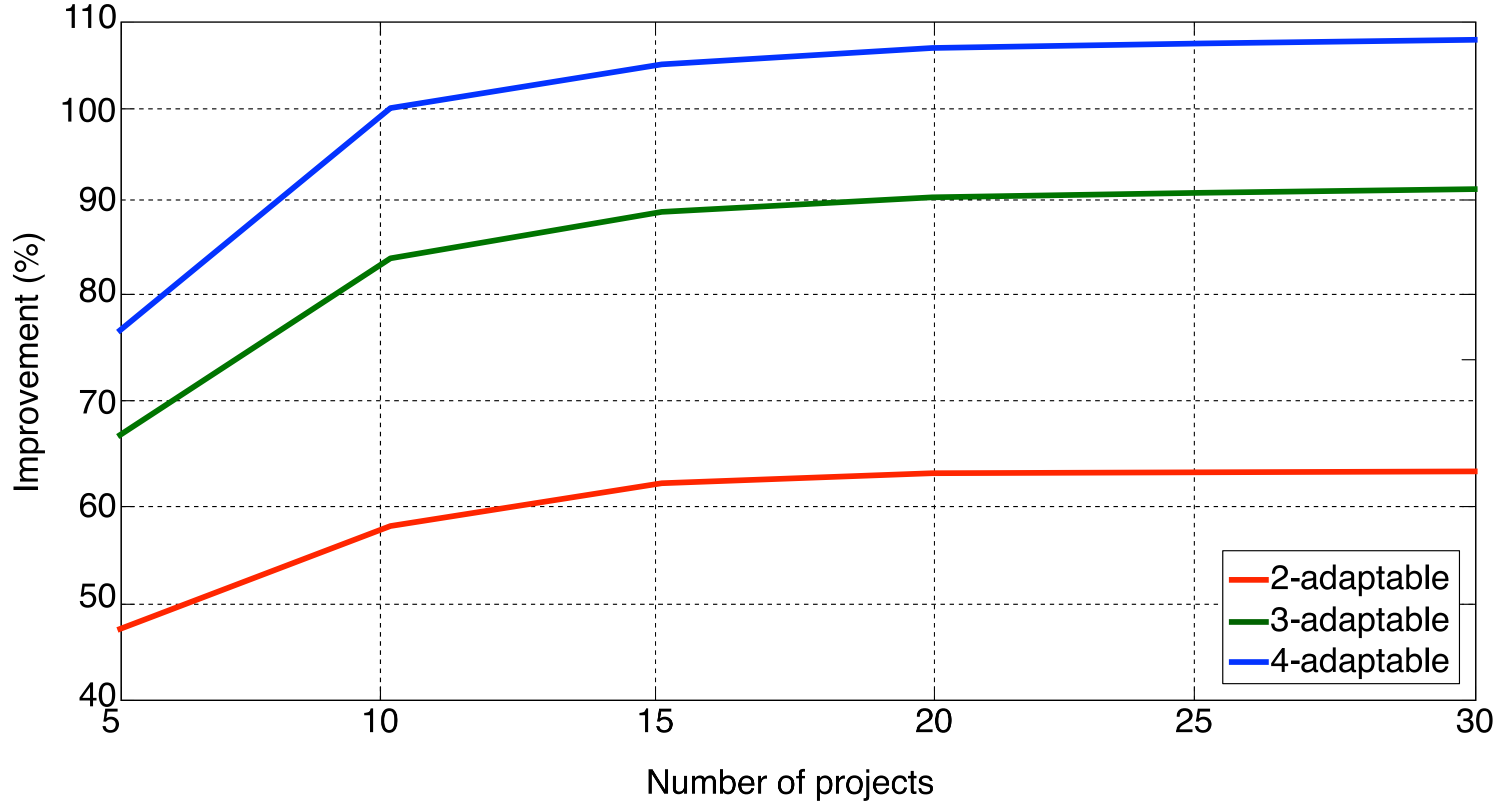
costs: $\boldsymbol{c}(\boldsymbol{\xi})^\top \boldsymbol{x}$
profits: $\boldsymbol{r}(\boldsymbol{\xi})^\top \boldsymbol{x}$

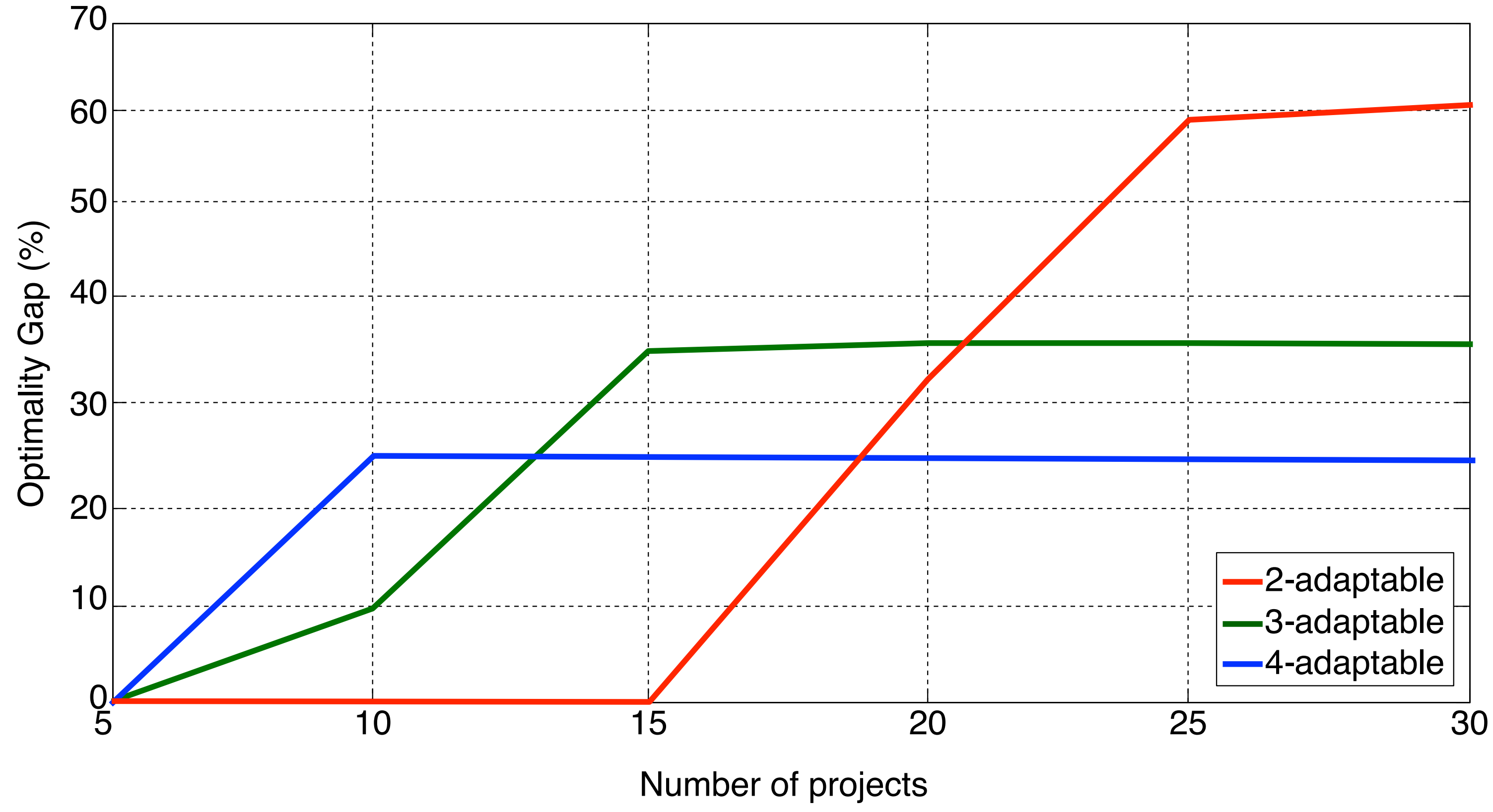
costs: $\boldsymbol{c}(\boldsymbol{\xi})^\top \boldsymbol{y}(\boldsymbol{\xi})$
profits: $0.8 \cdot \boldsymbol{r}(\boldsymbol{\xi})^\top \boldsymbol{y}(\boldsymbol{\xi})$

Can be modeled as two-stage robust integer program with ***constraint uncertainty****!*

Numerical Experiments

Investment Planning:





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