Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2017

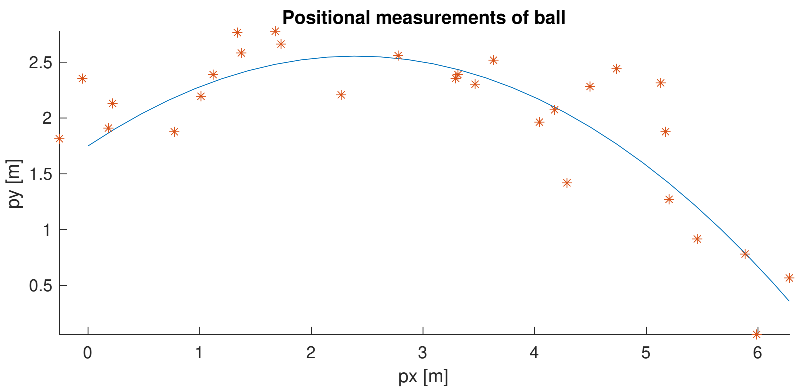
Exercise 11: Nonlinear least squares estimation

Prof. Dr. Moritz Diehl, Robin Verschueren, Alexander Resch

In this exercise you will use nonlinear least squares estimation for a parametric ODE. This exercise ties together a few concepts learned in all of the course.

Computer exercise (10 points)

Consider the following situation: you are standing on a field and you throw a ball away from you with an initial velocity $(v_0^x\,[\mathrm{m}], v_0^y\,[\mathrm{m}])$, as described in the following sketch:



The position $(p_0^x[\mathrm{m}], p_0^y[\mathrm{m}])$ is where you release the ball. The velocity of the ball can be modeled by the following set of ordinary differential equations:

$$\dot{v}^x = -\beta v^x$$
$$\dot{v}^y = -g - \beta v^y,$$

in the x and y direction respectively. Parameter $\beta\,[1/\mathrm{s}]$ is a constant air friction coefficient. You have noisy measurements (with a sampling time of 0.04 s) of the position of the ball (assuming only additive output error), obtained from your brand new DGPS (differential GPS) device. Your task is to estimate the gravitational acceleration $g[\mathrm{m/s}^2]$, the initial velocity v_0^x, v_0^y and initial position p_0^x, p_0^y. We can gather all of these in one parameter vector:

$$\theta = \begin{bmatrix} p_0^x \\ p_0^y \\ v_0^x \\ v_0^y \\ g \\ \beta \end{bmatrix}.$$

1. Find an *analytical* expression for the position and velocity of the ball, as a function of time t and initial state $x_0 = [p_0^x, p_0^y, v_0^x, v_0^y]^\top$. *Hint: use your knowledge from linear ODE theory.*

2. Write down the optimization problem that you want to solve in order to find an estimate for θ, in the form:

$$\hat{\theta} = \arg\min_{\theta} \frac{1}{2} \|R(\theta)\|_2^2$$

3. Solve the optimization problem with `lsqnonlin`.

4. Plot the trajectory together with the measurements.

5. Compute the Jacobian $J(\theta) = \frac{\partial R}{\partial \theta}(\theta)$.

6. Calculate an estimate of the covariance of the parameters $\Sigma_{\hat{\theta}}$. *Hint: see section 5.5 of the script.*

7. Write down the estimates in the form:

$$g = \hat{g} \pm \sqrt{\sigma_{\hat{g}}^2},$$
$$p_0^x = \dots$$

8. Same as the previous question, but express the interval in percentages relative to the estimate.

9. Add, in the plot, a 1-σ confidence ellipsoid for p_0.

10. Is your estimated gravitational acceleration close to the true one? What can you say about the other estimates for the parameters?