Exercises for Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2022-2023

Exercise 2: Statistics + Parameter Estimation

(to be returned on November 7th, 8:00)

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In this exercise you get to know some matrix properties. In addition, you investigate some important facts from statistics in numerical experiments. Pen-and-paper exercises can be uploaded on the Ilias course page as a digitally created PDF or handed in during the lecture.

Exercise Tasks

1. PAPER: The covariance matrix of a vector-valued random variable $X \in \mathbb{R}^n$ with mean $\mathbb{E}\{X\} = \mu_X$ is defined by

$$\mathrm{cov}\,(X) := \mathbb{E}\left\{(X - \mu_X)\,(X - \mu_X)^\top\right\}.$$

Prove that the covariance matrix of a vector-valued variable $Y = AX + b$ with constant $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ is given by

$$\mathrm{cov}\,(Y) = A\,\mathrm{cov}\,(X)\,A^\top.$$

(2 points)

2. PAPER: Let $X \in \mathbb{R}^n$ be a vector-valued random variable with mean $\mu \in \mathbb{R}^n$. Show that the covariance matrix $\mathrm{cov}(X)$ can also be calculated by

$$\mathrm{cov}(X) = \mathbb{E}\left\{XX^\top\right\} - \mu\mu^\top$$

(2 points)

3. PAPER: Suppose we are measuring a constant $x_0 \in \mathbb{R}$ perturbed by random independent noise ϵ with mean $\mu_\epsilon = 0$ and variance $\sigma_\epsilon^2 > 0$, i.e. we have

$$x = x_0 + \epsilon.$$

   (a) State the mean μ_x and the variance σ_x^2 of the random variable x. (1 point)

   (b) Let $x(n) = (x_1, \ldots, x_n)$ denote a sample of n observations of x. The sample mean is given by $\bar{x}(n) = \frac{1}{n}\sum_{i=1}^{n} x_i$ and it is an unbiased estimator of the mean μ_x.
   What is the variance of $\bar{x}(n)$? (1 point)

   (c) Prove that the Least Squares (LS) estimate for x_0 is the sample mean $\bar{x}(n)$. State the minimization problem explicitly. Is it convex? (2 bonus points)

4. Consider the following experimental setup, where we measure the temperature-dependent expansion of a steel bar. Here L_0 [cm] is the length of the bar at the beginning of the experiment and $L(T)$ [cm] represents the length of the bar at temperature T [K]. The following relationship holds, between the length of the bar at temperature T_0 [K]: $L_0 = L(T_0)$. We define $\Delta T := T - T_0$ as the independent variable. Furthermore, we define $A := \alpha \cdot L_0$ [cm/K], where α [1/K] is the specific expansion coefficient. Then the model is given by

$$m(\Delta T(k); A, L_0) = A \cdot \Delta T(k) + L_0. \tag{1}$$

Below, you find the datapoints. Using the data, you will compute estimates for the parameters A and L_0.

k	1	2	3	4
$\Delta T(k)$ [K]	5	15	35	60
$L(k)$ [cm]	6.55	9.63	17.24	29.64

(a) CODE: Plot the $\Delta T(k)$, $L(k)$ relation using 'x' markers. (0.5 points)

(b) PAPER: Using the model from above, calculate the experimental values for the parameters A and L_0 by minimizing the sum of squared distances, i.e.

$$A^*, L_0^* = \underset{A,\, L_0}{\arg\min} \quad \sum_{k=1}^{4} d_k(A, L_0)^2, \tag{2}$$

where the distance d_k is given by

$$d_k(A, L_0) = L(k) - m(\Delta T(k); A, L_0).$$

CODE: Plot the fit $m(\Delta T; A^*, L_0^*) = A^* \Delta T + L_0^*$ over the range $[0, 100]$ in the same figure as before.

Hint: Compute the solution by setting the gradient of the objective function with respect to the parameters (A, L_0) to zero, i.e. $\nabla_{(A,L_0)} \sum_k d_k^2 = 0$. This will give you a 2×2 linear system. Check if the objective function is convex! (2 points)

(c) CODE: Now, use a third order polynomial and fit it to the data using `np.polyfit`. Again minimize the sum of squared distances to find optimal values for the coefficients of your model equation. Plot the fit in the same figure as before. (0.5 point)

(d) CODE: You take another measurement: at $\Delta T = 70$K you measure a length of $L = 32.89$ cm. You can use this additional datapoint to validate your fit. Therefore plot it in the existing plot.
PAPER: Which fit looks more reasonable to you?
Hint: The phenomenon of fitting a model to a data set which then does not pass validation is called 'overfitting'. (1 point)

This sheet gives in total 10 points and 2 bonus points.